# PUTNAM PRACTICE SET # 2

(1) An old woman went to the market and a horse stepped on her basket and smashed her eggs. The rider offered to pay for the eggs and asked her how many there were. She did not remember the exact number, but when she had taken them two at a time there was one egg left, and the same happened when she took three, four, five, and six at a time. But when she took them seven at a time, they came out even. What is the smallest number of eggs she could have had?

(2) Show that for any positive integers $a$ and $b$, the product $(36a+b)(a+36b)$ cannot be a power of 2.

(3) Prove that any subset of 55 numbers chosen from the set $\{1,2,3,...,100\}$ must contain two numbers differing by 9.

(4) Prove that there is no polynomial with integer coefficients $P(x)$ with the property that $P(7)=5$ and $P(15)=9$.

(5) Find all functions $f:\{0,1,2,...\}\to\{0,1,2,...\}$ with the property that for every $m,n\geq 0$,

$$2f(m^2+n^2)=(f(m))^2+(f(n))^2.$$

(6) Show that the equation $ax+by=c$, $a,b,c$ integers, has a solution in integers $x$ and $y$ if and only if $gcd(a,b)$ divides $c$. Moreover, if $(x_0,y_0)$ is an integer solution, then for each integer $k$, the values

$$x'=x_0+bk/d, \quad y'=y_0-ak/d,$$

$d=gcd(a,b)$, are also a solution, and all integer solutions are of this form.

(7) Suppose that $a_1,...,a_n$ are integers whose greatest common divisor is 1. Let $S$ be a set of inetgers with the following properties:
   (a) For $i=1,...,n$, $a_i\in S$
   (b) For $i,j=1,...,n$ (not necessarily distinct), $a_i-a_j\in S$
   (c) For any integers $x,y\in S$, if $x+y\in S$ then $x-y\in S$.

   Prove that $S$ must equal the set of all integers.