Singularity Analysis of Evolving Curves using the Distance Comparison Principle

David L. Johnson and Murugiah Muraleetharan

Abstract

In this paper, we extend Grayson's theorem [Gra89] on curvature flow of embedded curves in a compact Riemannian surface. The main result is a new proof of a theorem of X. Zhu that, if a singularity develops in finite time, then the curve converges to a round point in a C^∞ sense. The proof will extend Huisken's distance comparison technique for curvature flow of embedded curves in the plane [Hui98].

KEYWORDS: Curvature flow, singularities

2000 A. M. S. SUBJECT CLASSIFICATION: 53C44

1 Introduction

Let γ be a closed embedded curve evolving under the curvature flow in a compact surface M. If a singularity develops in finite time, then the curve shrinks to a point [Gra89]. So when t is close enough to the blow-up time ω, we may assume that the curve is contained in a small neighborhood of the collapsing point on the surface. Using a local conformal diffeomorphism $\phi : U(\subseteq M) \to U' \subseteq \mathbb{R}^2$ between compact neighborhoods, we get a corresponding flow in the plane which satisfies the following equation:

$$\frac{\partial \gamma'}{\partial t} = \left(\frac{k'}{J^2} - \frac{\nabla_N J}{J^2} \right) N' \tag{1}$$

where $\gamma'(p,t) = \phi(\gamma(p,t))$, k' is the curvature of γ' in U', N' is the unit normal vector, and the conformal factor J is smooth, bounded and bounded away from 0.

We define the extrinsic and intrinsic distance functions

$$d, l : \Gamma \times \Gamma \times [0,T] \to \mathbb{R}$$

by

$$d(p,q,t) = |\gamma(p,t) - \gamma(q,t)|_{\mathbb{R}^2} \quad \text{and} \quad l(p,q,t) = \int_p^q ds_t = s_t(q) - s_t(p)$$

where Γ is either S^1 or an interval.

We also define the smooth function

$$\psi : S^1 \times S^1 \times [0,T] \to \mathbb{R}$$

by

$$\psi : (p,q,t) := \frac{L(t)}{\pi} \sin\left(\frac{l(p,q,t)\pi}{L(t)}\right).$$

We use the distance comparison $\frac{d}{l}$ and $\frac{d}{\psi}$ to prove the following theorem.

Main Theorem. *Let γ be a closed embedded curve evolving by curvature flow on a smooth compact Riemannian surface. If a singularity develops in finite time, then the curve converges to a round point in the C^∞ sense.*

This extends Huisken's distance comparison technique for curvature flow of embedded curves in the plane [Hui98]. Hamilton used isoperimetic estimates techniques to prove that when a closed embedded curve in the plane evolves by curvature flow the curve converges to a round point [Ham95b], and Zhu used Hamilton's isoperimetric estimates techniques for asymptotic behavior of anisotropic curves flows [Zhu98].

2 Evolving Closed Curves in a Surface

Grayson [Gra89] and Gage [Gag90], generalized the study of curvature flow of closed curves in the plane to that in surfaces. The *curvature flow* is a gradient flow for the length functional on the space of immersed curves in the surface M^2 with Riemannian metric g.

Let (M,g) be a smooth compact oriented 2-dimensional Riemannian manifold with bounded scalar curvature. Let $\gamma_0 : S^1 \to M$ be a smooth embedded curve in M and let $\gamma : S^1 \times [0,\omega) \to M$ be a one-parameter smooth family of embedded curves satisfying $\gamma(\cdot,0) = \gamma_0$. If γ evolves by curvature flow, then

$$\frac{\partial\gamma}{\partial t}(p,t) = k(p,t)N(p,t), \quad (p,t) \in S^1 \times [0,\omega), \tag{2}$$

where k is the geodesic curvature of γ and N is its unit normal.
Arclength is given by

$$s(p,t) = \int_0^p \left|\frac{\partial\gamma}{\partial q}(q,t)\right| dq.$$

Differentiating,

$$\frac{\partial s}{\partial p}(p,t) = \left|\frac{\partial\gamma}{\partial p}(p,t)\right| = v(p,t)$$

$$\Rightarrow \frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial p}, \quad \text{and} \quad ds = v dp.$$

From the Frenet formulas, we have

$$\nabla_s T = kN \quad \text{and} \quad \nabla_s N = -kT.$$

Now we recall some standard results for the evolution [Gra89].

Lemma 2.1. *For the curvature flow:*

1. *The speed v evolves according to $\frac{\partial v}{\partial t} = -k^2 v$.*
2. $[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = k^2 \frac{\partial}{\partial s}$.

3. $\nabla_t T = \frac{\partial k}{\partial s} N$ *and* $\nabla_t N = -\frac{\partial k}{\partial s} T$.

4. *The arclength* L *of the curve evolves according to* $\frac{dL}{dt} = -\int_{\gamma_t} k^2 ds$.

5. $\nabla_t \nabla_s = \nabla_s \nabla_t + k^2 \nabla_s - kR(T, N)$.

6. *The curvature* k *of the curve evolves according to* $\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3 + K\, k$, *where* $K = \langle R(N,T)T, N \rangle$ *is the Gaussian curvature of* M *restricted to* $\gamma(\cdot, t)$.

Theorem 2.1. *[Gra89] A closed embedded curve moving on a smooth compact Riemannian surface by curvature flow must either collapse to a point in finite time or else converge to a simple closed geodesic as* $t \to \infty$.

Grayson proof was rather delicate, requiring separate analyses of what may happen under various geometric configurations, and special arguments for each cases. First he showed that the solution remains smooth and embedded as long as its curvature remains bounded. He then proved that if a singularity develops in finite time, then the curvature remains bounded until the entire curve shrinks to a point. Finally, he proved that if the length of the curve does not converge to zero, then its curvature must converge to zero in the C^∞ norm and that the curve approaches a geodesic in the C^∞ sense.

In this paper, the proof has been simplified using distance comparison techniques to rule out certain kinds of singularity and we extend Grayson's theorem [Gra89] by showing that if the curve shrinks to a point, then it shrinks to a round point in a C^∞ sense. Since the curve does shrink to a point, we can transform the curvature flow in surfaces to a corresponding flow in the plane (more general than the curvature flow in the plane per se). In a series of papers, Angenent [Ang90, Ang91b, Ang91a] developed a more general theory of parabolic equations for curves on surfaces. We now summarize some of the important results of Angenent that we will need.

2.1 Parabolic Equations for Curves on Surfaces

Consider a closed curve evolving by an arbitrary uniformly parabolic equation,

$$\frac{\partial \gamma}{\partial t} = V(T, k) N, \tag{3}$$

on a smooth oriented 2-dimensional Riemannian manifold M, and denote its unit tangent bundle by $S^1(M) = \{\xi \in T(M) : g(\xi, \xi) = 1\}$. Then the normal velocity is

$$v^\perp(p,t) = V(T,k)(p,t) \equiv V(T_{\gamma(p,t)}, k_{\gamma(p,t)}),$$

for some function $V : S^1(M) \times \mathbb{R} \to \mathbb{R}$ which satisfies:

(V_1) $V(T,k)$ is $C^{2,1}$,

(V_2) $\lambda^{-1} \leq \dfrac{\partial V}{\partial k} \leq \lambda$,

(V_3) $|V(T,0)| \leq \mu$ for all $T \in S^1(M)$,

(V_4) $|\nabla^h V| + |k \nabla^v V| \leq \nu(1 + k^2)$,

(V_5) $V(-T,-k) = -V(T,k)$,

for positive constants λ, μ, and ν.

The tangent bundle to $S^1(M)$ splits into the Whitney sum of the bundle of horizontal vectors and bundle of vertical vectors. $\nabla^v V$ and $\nabla^h V$ denote the vertical and horizontal components of $\nabla(V)$ (holding the second argument of V fixed).

These assumptions on V are necessary to make the set of allowable initial curves as large as possible, and necessary for the short-time existence of the solutions. The way in which maximal classical solutions can become singular (limit curves) is based on these assumptions on V and the initial curves. Our application of this theory will be for the flow given by $V(T,k) = \left(\frac{k}{J^2} - \frac{\nabla_N J}{J^2}\right)$, where $J(x,y)$ is a smooth bounded function that is also bounded away from 0. We will see that the curvature flow in a surface corresponds to the flow with this normal velocity in a plane.

Lemma 2.2. *For the flow (3):*

1. *The speed v evolves according to $\frac{\partial v}{\partial t} = -kVv$.*
2. $[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = kV\frac{\partial}{\partial s}$.
3. $\nabla_t T = \frac{\partial V}{\partial s}N$, *and* $\nabla_t N = -\frac{\partial V}{\partial s}T$.
4. *The arclength L of the curve evolves according to $\frac{dL}{dt} = -\int_{\gamma_t} kV ds$.*
5. $\nabla_t \nabla_s = \nabla_s \nabla_t + kV\nabla_s - VR(T,N)$.
6. *The curvature k of the curve evolves according to $\frac{\partial k}{\partial t} = \frac{\partial^2 V}{\partial s^2} + k^2 V + KV$, where $K = \langle R(N,T)T, N\rangle$ is the Gaussian curvature of M restricted to $\gamma(\cdot,t)$.*
7. *The enclosed area A of the curve evolves according to $\frac{dA}{dt} = -\int_{\gamma_t} V ds$.*

We now state the main result from [Ang90] and [Ang91b].

Theorem 2.2. *Let V satisfy $(V_1) - (V_5)$, and let $\gamma : S^1 \times [0,\omega) \to M$ be a maximal classical solution of (3) which becomes singular in finite time. Then the limit curve γ^* of the $\gamma(\cdot,t)$ either has fewer self-intersections than any of the $\gamma(\cdot,t)$'s, or else the total absolute curvature of the limit curve drops by at least π.*

Oaks [Oak94] improved Theorem 2.2 by showing that the latter case never occurs. So if the initial curve is embedded, and the singularity develops in finite time, then the curve shrinks to a point. So when t is close enough to the blow-up time ω, we may assume that the curve is contained in a small neighborhood of the collapsing point on the surface.

Now from the following theorem, it is enough to work locally in $\mathbb{R}^2$.

Theorem 2.3. *[Oak94] Let $\phi : U(\subseteq M) \to U' \subseteq \mathbb{R}^2$ be a conformal diffeomorphism between compact neighborhoods. If $V : S^1(M) \times \mathbb{R} \to \mathbb{R}$ satisfies $(V_1) - (V_5)$, then there is a function $V' : S^1(U') \times \mathbb{R} \to \mathbb{R}$ which satisfies $(V_1) - (V_5)$ such that whenever $\gamma(p,t)$ is a curve in U evolving by (3), $\gamma'(p,t) = \phi(\gamma(p,t))$ satisfies $\frac{\partial \gamma'}{\partial t} = V'(T',k')N'$, where T' and N' are the unit tangent and normal vectors, and k' is the curvature of γ' in U'.*

Moreover, $V(T,k) = J(p)V'(T',k')$ and $ds = J(p)ds'$, where $J(p) > 0$ is smooth, bounded, and bounded away from 0.

The metric in U can be written as

$$g = J^2(x,y)(dx^2 + dy^2),$$

where the coordinates in U are obtained by ϕ^{-1}. Because U' is compact, $J(x,y)$ is both bounded and bounded away from 0.

Let $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ be the coordinate vector fields on U, and let $X = \frac{1}{J}\frac{\partial}{\partial x}$, and $Y = \frac{1}{J}\frac{\partial}{\partial y}$. Then X and Y are unit vectors. Since ϕ is conformal, $\phi_*(N) = \frac{1}{J}N'$. So γ' evolves by the equation:

$$\frac{\partial \gamma'}{\partial t} = (\frac{1}{J}V)N'.$$

Therefore, $V' = \frac{1}{J}V$.

We next show that $k' = kJ + \nabla_N J$. First, we need the following lemma.

Lemma 2.3.

$$\nabla_X X = -\frac{\nabla_Y J}{J}Y \qquad \nabla_X Y = \frac{\nabla_Y J}{J}X$$
$$\nabla_Y X = \frac{\nabla_X J}{J}Y \qquad \nabla_Y Y = -\frac{\nabla_X J}{J}X.$$

PROOF

Since $0 = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = [JX, JY]$, we have $\nabla_{JX}JY = \nabla_{JY}JX$. Therefore, $J\nabla_X Y + (\nabla_X J)Y = J\nabla_Y X + (\nabla_Y J)X$. Since $\nabla_X Y \perp Y$ and $\nabla_Y X \perp X$, we get $\nabla_X Y = \frac{\nabla_Y J}{J}X$ and $\nabla_Y X = \frac{\nabla_X J}{J}Y$. The other two formulas follow from differentiating $\langle X, Y\rangle = 0$ with respect to X and Y. $\square$

Let θ be the angle T makes with X in U. Then

$$T = \cos\theta X + \sin\theta Y, \quad N = -\sin\theta X + \cos\theta Y.$$

Thus,

$$\begin{aligned}\nabla_T X &= -\cos\theta\,\frac{\nabla_Y J}{J}Y + \sin\theta\,\frac{\nabla_X J}{J}Y \\ &= \left(-\frac{\nabla_Y J}{J}\cos\theta + \frac{\nabla_X J}{J}\sin\theta\right)Y,\end{aligned}$$

and

$$\nabla_T Y = \left(\frac{\nabla_Y J}{J}\cos\theta - \frac{\nabla_X J}{J}\sin\theta\right)X.$$

We have $\nabla_T\theta = \frac{1}{J}k'$. Then

$$\begin{aligned}kN = \gamma'' = \nabla_T T &= \nabla_T(\cos\theta\, X + \sin\theta\, Y) \\ &= -\sin\theta(\frac{k'}{J})X + \cos\theta\left(-\frac{\nabla_Y J}{J}\cos\theta + \frac{\nabla_X J}{J}\sin\theta\right)Y + \cos\theta(\frac{k'}{J})Y \\ &\quad + \sin\theta\left(\frac{\nabla_Y J}{J}\cos\theta - \frac{\nabla_X J}{J}\sin\theta\right)X \\ &= \left(\frac{k'}{J} + \frac{\nabla_X J}{J}\sin\theta - \frac{\nabla_Y J}{J}\cos\theta\right)N,\end{aligned}$$

and thus,

$$\begin{aligned} k &= \left(\frac{k'}{J} + \frac{\nabla_X J}{J} \sin\theta - \frac{\nabla_Y J}{J} \cos\theta \right) \\ &= \frac{k'}{J} - \frac{1}{J} \nabla_N J. \end{aligned}$$

That is,

$$k' = kJ + \nabla_N J. \tag{4}$$

J is bounded away from 0 and both J and $\nabla_N J$ are bounded. So $\lim_{t\to\omega} |k(p,t)|$ is unbounded if and only if $\lim_{t\to\omega} |k'(p,t)|$ is also unbounded.

When $V = k$, i.e., for the curvature flow in a surface M, we have $V' = \frac{1}{J} V = \frac{k}{J} = \frac{k'}{J^2} - \frac{\nabla_N J}{J^2}$. So the curvature flow in a surface corresponds to the following flow in $\mathbb{R}^2$:

$$\frac{\partial \gamma'}{\partial t} = \left(\frac{k'}{J^2} - \frac{\nabla_N J}{J^2} \right) N'. \tag{1}$$

3 Distance Comparison Principles for Evolving Curves

Huisken [Hui98] showed that the curvature flow of an embedded curve in a plane converges smoothly to a round point using his distance comparison principles to eliminate type-II singularities for the curvature flow and also measure the deviation of the evolving curve from a round circle.

From [Gra89] when a closed embedded curve evolves under the curvature flow in a surface, the solution remains smooth and embedded as long as its curvature remains bounded. If a singularity develops in finite time then the curve shrinks to a point. So when t is close enough to the blow-up time ω, we may assume that the curve is contained in a small neighborhood of the collapsing point on the surface. Now by theorem 2.3, using a local conformal diffeomorphism $\phi : U(\subseteq M) \to U' \subseteq \mathbb{R}^2$ between compact neighborhoods, we get a corresponding flow in the plane which satisfies the following equation:

$$\frac{\partial \gamma'}{\partial t} = \left(\frac{k'}{J^2} - \frac{\nabla_N J}{J^2} \right) N', \tag{1}$$

where $\gamma'(p,t) = \phi(\gamma(p,t))$, k' is the curvature of γ' in U', and N' is the unit normal vector.

In this section, we will apply Huisken's techniques to the flow (1) in $\mathbb{R}^2$ which corresponds to the curvature flow in a surface.

3.1 Comparison between Extrinsic Distance and Intrinsic Distance

We define the extrinsic and intrinsic distance functions

$$d, l : \Gamma \times \Gamma \times [0,T] \to \mathbb{R}$$

by

$$d(p,q,t) = |\gamma(p,t) - \gamma(q,t)|_{\mathbb{R}^2},$$

and

$$l(p,q,t) = \int_p^q ds_t = s_t(q) - s_t(p),$$

where Γ is either S^1 or an interval. Notice that $0 < \frac{d}{l} \leq 1$, with equality on the diagonal of $\Gamma \times \Gamma$ or if γ is a straight line. The ratio $\frac{d}{l}$ can be considered as a measure for the straightness of an embedded curve.

We now prove that under the parabolic flow (1), the ratio $\frac{d}{l}$ improves at a local minimum. This proves that embedded curves stay embedded for this parabolic flow.

Lemma A. *Let $\gamma : I \times [0,T] \to \mathbb{R}^2$ be a smooth embedded solution of the flow (1), where I is an interval such that l is smoothly defined on $I \times I$. Suppose $\frac{d}{l}$ attains a local minimum at (p_0, q_0) in the interior of $I \times I$ at time $t_0 \in [0,T]$. Then*

$$\frac{d}{dt}\left(\frac{d}{l}\right)(p_0,q_0,t_0) \geq 0,$$

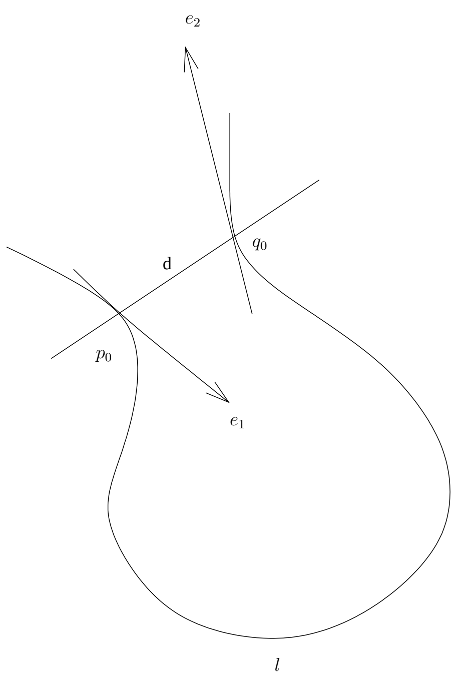
with equality if and only if γ is a straight line.

Proof

We may assume, without loss of generality, that $p_0 \neq q_0$, and $s(q_0,t_0) > s(p_0,t_0)$. Since $\frac{d}{l}$ attains a local minimum at (p_0,q_0), we have

$$\delta(\xi)\left(\frac{d}{l}\right)(p_0,q_0,t_0) = 0, \quad \text{and} \quad \delta^2(\xi)\left(\frac{d}{l}\right)(p_0,q_0,t_0) \geq 0, \tag{5}$$

where $\delta(\xi)$ and $\delta^2(\xi)$ denote the first and second variation with regard to the variation vector $\xi = v_1 \oplus v_2 \in T_{p_0}\gamma_{t_0} \oplus T_{q_0}\gamma_{t_0}$.



We have

$$\delta(\xi)\left(\frac{d}{l}\right)(p_0,q_0,t_0) := \left.\frac{d}{d\tau}\right|_{\tau=0}\left(\frac{d}{l}\right)(\alpha_p(\tau),\alpha_q(\tau),t_0),$$

where

$$\alpha_p(0)=\gamma(p_0,t_0) \quad \text{and} \quad \alpha_p'(0)=v_1\in T_{p_0}\gamma_{t_0},$$
$$\alpha_q(0)=\gamma(q_0,t_0) \quad \text{and} \quad \alpha_q'(0)=v_2\in T_{q_0}\gamma_{t_0}.$$

Also,

$$\delta^2(\xi)\left(\frac{d}{l}\right)(p_0,q_0,t_0) := \left.\frac{d^2}{d\tau^2}\right|_{\tau=0}\left(\frac{d}{l}\right)(\alpha_p(\tau),\alpha_q(\tau),t_0).$$

Let

$$e_1=\frac{\partial\gamma}{\partial s}(p_0,t_0), \quad e_2=\frac{\partial\gamma}{\partial s}(q_0,t_0), \quad \text{and} \quad \omega=\frac{\gamma(q_0,t_0)-\gamma(p_0,t_0)}{d(p_0,q_0,t_0)}.$$

Then using (5), we will show that $\langle\omega,e_1\rangle=\langle\omega,e_2\rangle=\frac{d}{l}$.
First, calculate $\delta(e_1\oplus e_2)d(p_0,q_0,t_0)$:

$$\begin{aligned}\frac{d}{d\tau}d(\alpha_p(\tau),\alpha_q(\tau),t_0) &= \frac{d}{d\tau}\sqrt{\langle\alpha_p(\tau)-\alpha_q(\tau),\,\alpha_p(\tau)-\alpha_q(\tau)\rangle}\\ &= \frac{\left\langle\alpha_p'(\tau)-\alpha_q'(\tau),\,\alpha_p(\tau)-\alpha_q(\tau)\right\rangle}{\sqrt{\langle\alpha_p(\tau)-\alpha_q(\tau),\,\alpha_p(\tau)-\alpha_q(\tau)\rangle}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\delta(e_1\oplus e_2)d(p_0,q_0,t_0) &:= \left.\frac{d}{d\tau}\right|_{\tau=0} d(\alpha_p(\tau),\alpha_q(\tau),t_0)\\ &= \frac{\left\langle\alpha_p'(0)-\alpha_q'(0),\,\alpha_p(0)-\alpha_q(0)\right\rangle}{d(p_0,q_0,t_0)}\\ &= \langle e_1-e_2,-\omega\rangle\,. \end{aligned} \tag{6}$$

Assume $\xi=e_1\oplus 0$:
Then using $\delta(e_1\oplus 0)d(p_0,q_0,t_0)=\langle e_1,-\omega\rangle$ and
$\delta(e_1\oplus 0)l(p_0,q_0,t_0)=\frac{d}{d\tau}|_{\tau=0}(l(p_0,q_0,t_0)-\tau)=-1$, we get

$$\begin{aligned}0 &= \delta(e_1\oplus 0)\left(\frac{d}{l}\right)(p_0,q_0,t_0)\\ &= \frac{l(p_o,q_0,t_0)\delta(e_1\oplus 0)d(p_0,q_0,t_0)-d(p_0,q_0,t_0)\delta(e_1\oplus 0)l(p_0,q_0,t_0)}{l^2}\\ &= \frac{l\,\langle e_1,-\omega\rangle-d(-1)}{l^2}.\end{aligned}$$

Hence,

$$\langle\omega,e_1\rangle=\frac{d}{l}. \tag{7}$$

Assume $\xi=0\oplus e_2$:

Then using $\delta(0\oplus e_2)d(p_0,q_0,t_0)=\langle -e_2,-\omega\rangle$ and $\delta(0\oplus e_2)l(p_0,q_0,t_0)=1$, we get

$$\begin{aligned}0&=\delta(0\oplus e_2)\left(\frac{d}{l}\right)(p_0,q_0,t_0)\\&=\frac{l\langle e_2,\omega\rangle-d(1)}{l^2},\end{aligned}$$

and hence,

$$\langle \omega, e_2\rangle=\frac{d}{l}. \tag{8}$$

Then either $e_1=e_2$ or $e_1\neq e_2$. Notice that in the later case e_1+e_2 is parallel to ω.
Case 1: $e_1=e_2$. In this case, we choose $\xi=e_1\oplus e_2$.
Since $\frac{d}{d\tau}l(\alpha_p(\tau),\alpha_q(\tau),t_0)=0$, we have $\delta(\xi)(l)=0$. Hence,

$$\begin{aligned}0\leq\delta^2(\xi)&\left(\frac{d}{l}\right)(p_0,q_0,t_0)=\frac{d^2}{d\tau^2}\Big|_{\tau=0}\left(\frac{d}{l}\right)(\alpha_p(\tau),\alpha_q(\tau),t_0)\\&=\frac{d}{d\tau}\Big|_{\tau=0}\frac{l\frac{d}{d\tau}d-d\frac{d}{d\tau}l}{l^2}=\frac{d}{d\tau}\Big|_{\tau=0}\frac{1}{l}\frac{d}{d\tau}d\\&=\frac{d}{d\tau}\Big|_{\tau=0}\left[\frac{1}{ld}\left\langle\alpha_p'(\tau)-\alpha_q'(\tau),\,\alpha_p(\tau)-\alpha_q(\tau)\right\rangle\right]\\&=\Bigg[\frac{ld}{(ld)^2}\left(\left\langle\alpha_p'(\tau)-\alpha_q'(\tau),\,\alpha_p'(\tau)-\alpha_q'(\tau)\right\rangle+\left\langle\alpha_p''(\tau)-\alpha_q''(\tau),\,\alpha_p(\tau)-\alpha_q(\tau)\right\rangle\right)\\&\qquad+\frac{1}{(ld)^2}\left\langle\alpha_p'(\tau)-\alpha_q'(\tau),\,\alpha_p(\tau)-\alpha_q(\tau)\right\rangle\frac{d}{d\tau}(ld)\Bigg]_{\tau=0}\\&=\frac{1}{l}\left\langle\overrightarrow{k}(p_0,t_0)-\overrightarrow{k}(q_0,t_0),\,-\omega\right\rangle\quad\text{Since}\quad\alpha_p'(0)-\alpha_q'(0)=0.\end{aligned}$$

Thus,

$$\left\langle\omega,\,\overrightarrow{k}(q_0,t_0)-\overrightarrow{k}(p_0,t_0)\right\rangle\geq 0. \tag{9}$$

Case 2: $e_1\neq e_2$. In this case, we choose $\xi=e_1\ominus e_2$.

Since $\frac{d}{d\tau}l(\alpha_p(\tau),\alpha_q(\tau),t_0) = \frac{d}{d\tau}(l(p_0,q_0,t_0)-2\tau) = -2$, we have $\delta(\xi)(l) = -2$. Thus,

$$
\begin{aligned}
0 \le \delta^2(\xi)\left(\frac{d}{l}\right)(p_0,q_0,t_0) = {} & \frac{d^2}{d\tau^2}\bigg|_{\tau=0}\left(\frac{d}{l}\right)(\alpha_p(\tau),\alpha_q(\tau),t_0) \\
= {} & \frac{d}{d\tau}\bigg|_{\tau=0} \frac{l\frac{d}{d\tau}d - d\frac{d}{d\tau}l}{l^2} \\
= {} & \frac{d}{d\tau}\bigg|_{\tau=0}\left[\frac{1}{ld}\left\langle \alpha_p'(\tau)-\alpha_q'(\tau),\, \alpha_p(\tau)-\alpha_q(\tau)\right\rangle + \frac{2d}{l^2}\right] \\
= {} & \left[\frac{ld}{(ld)^2}\left(\left\langle \alpha_p'(\tau)-\alpha_q'(\tau),\, \alpha_p'(\tau)-\alpha_q'(\tau)\right\rangle + \left\langle \alpha_p''(\tau)-\alpha_q''(\tau),\, \alpha_p(\tau)-\alpha_q(\tau)\right\rangle\right)\right. \\
& \left. -\frac{1}{(ld)^2}\left\langle \alpha_p'(\tau)-\alpha_q'(\tau),\, \alpha_p(\tau)-\alpha_q(\tau)\right\rangle\left(l\frac{d}{d\tau}d + d\frac{d}{d\tau}l\right) + \left(\frac{2l^2\frac{d}{d\tau}d - 2d.2l.\frac{d}{d\tau}l}{l^4}\right)\right]_{\tau=0} \\
= {} & \frac{1}{l}\left\langle \overrightarrow{k}(p_0,t_0) - \overrightarrow{k}(q_0,t_0),\, -\omega\right\rangle + \frac{1}{ld}|e_1+e_2|^2 + \frac{1}{l^2 d}\left\langle e_1+e_2,\, \omega\right\rangle \\
& (l\left\langle e_1+e_2, -\omega\right\rangle - 2d) + \frac{2}{l^2}\left\langle e_1+e_2,\, -\omega\right\rangle - \frac{4d}{l^3}(-2) \\
= {} & \frac{1}{l}\left\langle \omega,\, \overrightarrow{k}(q_0,t_0) - \overrightarrow{k}(p_0,t_0)\right\rangle + \frac{1}{ld}|e_1+e_2|^2 - \frac{1}{ld}\left\langle e_1+e_2,\, \omega\right\rangle^2 \\
& - \frac{4}{l^2}\left\langle e_1+e_2,\, \omega\right\rangle + \frac{8d}{l^3} \\
= {} & \frac{1}{l}\left\langle \omega,\, \overrightarrow{k}(q_0,t_0) - \overrightarrow{k}(p_0,t_0)\right\rangle.
\end{aligned}
$$

The last line follows from $\langle \omega,\ e_1+e_2\rangle = \frac{2d}{l}$, which implies that $-\frac{4}{l^2}\langle e_1+e_2,\, \omega\rangle = -\frac{8d}{l^3}$. Also $\omega \parallel e_1+e_2$ gives $\langle e_1+e_2,\, \omega\rangle^2 = |e_1+e_2|^2$, and so $\frac{1}{ld}\langle e_1+e_2,\, \omega\rangle^2 = \frac{1}{ld}|e_1+e_2|^2$.
Hence,

$$
\left\langle \omega,\, \overrightarrow{k}(q_0,t_0) - \overrightarrow{k}(p_0,t_0)\right\rangle \ \ge 0. \tag{10}
$$

We now use the evolution equation (1) to compute $\frac{d}{dt}\left(\frac{d}{l}\right)(p_0,q_0,t_0)$.

$$
\frac{d}{dt}\left(\frac{d}{l}\right)(p_0,q_0,t_0) = \frac{l\frac{d}{dt}d - d\frac{d}{dt}l}{l^2} = \frac{1}{l}\frac{d}{dt}d - \frac{d}{l^2}\frac{d}{dt}l.
$$

we have

$$
\begin{aligned}
\frac{d}{dt}d(p_o,q_0,t_0) = {} & \frac{d}{dt}\sqrt{\left\langle \gamma(p_0,t_0)-\gamma_(q_0,t_0), \gamma(p_0,t_0)-\gamma_(q_0,t_0)\right\rangle} \\
& \frac{\left\langle \frac{\partial\gamma}{\partial t}(p_0,t_0) - \frac{\partial\gamma}{\partial t}(q_0,t_0), \gamma(p_0,t_0)-\gamma_(q_0,t_0)\right\rangle}{d(p_0,q_0,t_0)} \\
& = \left\langle \frac{\partial\gamma}{\partial t}(p_0,t_0) - \frac{\partial\gamma}{\partial t}(q_0,t_0), -\omega\right\rangle \\
& = \left\langle \left(\frac{k}{J^2} - \frac{\nabla_N J}{J^2}\right)N(p_0,t_0) - \left(\frac{k}{J^2} - \frac{\nabla_N J}{J^2}\right)N(q_0,t_0), -\omega\right\rangle,
\end{aligned}
$$

and

$$\frac{d}{dt}l = \frac{d}{dt}\int_p^q ds_t = -\int_p^q k\left(\frac{k}{J^2} - \frac{\nabla_N J}{J^2}\right) ds_t.$$

Since $\langle \omega, e_1\rangle = \langle \omega, e_2\rangle = \frac{d}{l}$, let $\alpha = \angle(\omega, e_1) = \angle(\omega, e_2)$ with $0 < \alpha < \pi/2$. Then $\langle \omega,\, N(q_0, t_0)\rangle = -\sin\alpha$ and $\langle \omega,\, N(p_0, t_0)\rangle = \sin\alpha$. Since $\left\langle \omega,\, \overrightarrow{k}(q_0, t_0) - \overrightarrow{k}(p_0, t_0)\right\rangle \geq 0$, we have $-\sin\alpha(k(p_0, t_0) + k(q_0, t_0)) \geq 0$. Therefore,

$$\frac{d}{dt}d(p_o, q_0, t_0) = -\left(\frac{k}{J^2}(p_0, t_0) + \frac{k}{J^2}(q_0, t_0)\right)\sin\alpha + \left(\frac{\nabla_N J}{J^2}(p_0, t_0) + \frac{\nabla_N J}{J^2}(q_0, t_0)\right)\sin\alpha.$$

Hence,

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{d}{l}\right)(p_0, q_0, t_0) \\
&= \frac{1}{l}\frac{d}{dt}d - \frac{d}{l^2}\frac{d}{dt}l \\
&= \frac{1}{l}\left(-\sin\alpha\left(\frac{k}{J^2}(p_0, t_0) + \frac{k}{J^2}(q_0, t_0)\right) + \left(\frac{\nabla_N J}{J^2}(p_0, t_0) + \frac{\nabla_N J}{J^2}(q_0, t_0)\right)\sin\alpha\right) \\
&\qquad + \frac{d}{l^2}\int_p^q k\left(\frac{k}{J^2} - \frac{\nabla_N J}{J^2}\right) ds_t \\
&\geq \frac{1}{l}\left[-\sin\alpha(k(p_0, t_0) + k(q_0, t_0))C_1 - 2C_2\sin\alpha\right] + \frac{d}{l^2}C_1\int_p^q k^2\, ds - \frac{d}{l^2}C_2\int_p^q k\, ds \\
&= \frac{1}{l}\left\{\left(-\sin\alpha(k(p_0, t_0) + k(q_0, t_0)) + \frac{d}{l}\int_p^q k^2\, ds\right)C_1 - \left(2\sin\alpha + \frac{d}{l}\int_p^q k\, ds\right)C_2\right\},
\end{aligned}$$

where

$$C_1 = \min_U\left(\frac{1}{J^2}\right), \quad C_2 = \max_U\left(\frac{|\nabla_N J|}{J^2}\right).$$

If t_0 is close enough to the blow-up time ω, we can make C_1 approach 1 and C_2 approach 0. By the Hölder inequality,

$$l\int_p^q k^2\, ds \geq \left(\int_p^q |k|\, ds\right)^2 \geq 4\alpha^2 > 0.$$

The last inequality is true since $\cos\alpha = \langle \omega, e_1\rangle = \langle \omega, e_2\rangle = \frac{d}{l} \Rightarrow \cos\alpha < 1 \Rightarrow \alpha > 0$. So we have $-\sin\alpha(k(p_0, t_0) + k(q_0, t_0)) \geq 0$, $2\sin\alpha + \frac{d}{l}\int_p^q k\, ds$ is bounded, and $\frac{d}{l}\int_p^q k^2\, ds$ is bounded away from 0, hence $\frac{d}{dt}\left(\frac{d}{l}\right)(p_0, q_0, t_0) > 0$. □

3.2 Deviation of the Evolving Curve from a Circle

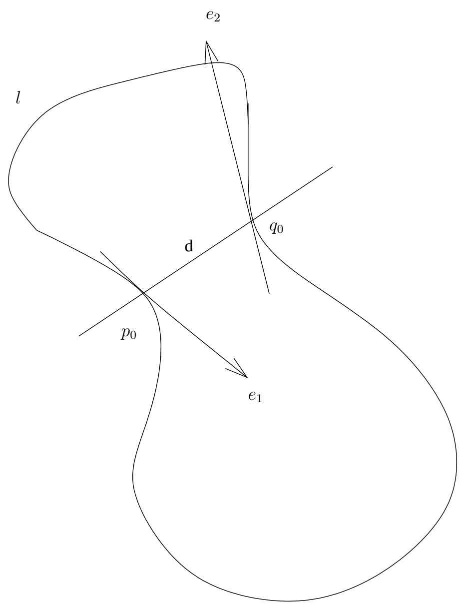
Now let $\gamma : S^1 \times [0, T] \to \mathbb{R}^2$ be a closed smooth embedded curve moving by the flow (1). Let $L(t)$ be the total length of the curve. The intrinsic distance function l is now only smoothly defined for $0 \leq l < \frac{L}{2}$. We define the smooth function

$$\psi : S^1 \times S^1 \times [0, T] \to \mathbb{R}$$

by

$$\psi : (p,q,t) := \frac{L(t)}{\pi} \sin\left(\frac{l(p,q,t)\pi}{L(t)}\right).$$

So the isoperimetric ratio $\frac{d}{\psi} = \frac{d}{l}\left(\frac{\frac{l\pi}{L}}{\sin(\frac{l\pi}{L})}\right) \to 1$ on the diagonal of $S^1 \times S^1$ and $\frac{d}{\psi} \equiv 1$ on any circle. We now prove that the ratio $\frac{d}{\psi}$ improves at a local minimum under the parabolic flow (1). Therefore, it plays the role of an improving isoperimetric ratio that measures the deviation of the evolving curve from a circle.



Lemma B. *Let $\gamma : S^1 \times [0,T] \to \mathbb{R}^2$ be a smooth embedded solution of the flow (1). Suppose $\frac{d}{\psi}$ attains a local minimum $(\frac{d}{\psi})(p_0,q_0,t_0) < 1$ at some point $(p_0,q_0) \in S^1 \times S^1$ at time $t_0 \in [0,T]$. Then*

$$\frac{d}{dt}\left(\frac{d}{\psi}\right)(p_0,q_0,t_0) \geq 0,$$

with equality if and only if $\frac{d}{\psi} \equiv 1$ or $\gamma(S^1,\cdot)$ is a circle.
Proof

We may assume, without loss of generality, that $0 = s(p_0,t_0) < s(q_0,t_0) < \frac{L(t_0)}{2}$, such that $l(p_0,q_0,t_0) = s(q_0,t_0) - s(p_0,t_0)$. Since $\frac{d}{\psi}$ attains a local minimum at (p_0,q_0), we have

$$\delta(\xi)\left(\frac{d}{\psi}\right)(p_0,q_0,t_0) = 0, \quad \text{and} \quad \delta^2(\xi)\left(\frac{d}{\psi}\right)(p_0,q_0,t_0) \geq 0, \tag{11}$$

where $\delta(\xi)$ and $\delta^2(\xi)$ denote the first and second variation with regard to the variation vector $\xi = v_1 \oplus v_2 \in T_{p_0}\gamma_{t_0} \oplus T_{q_0}\gamma_{t_0}$. Let

$$e_1 = \frac{\partial\gamma}{\partial s}(p_0,t_0), \quad e_2 = \frac{\partial\gamma}{\partial s}(q_0,t_0), \quad \text{and} \quad \omega = \frac{\gamma(q_0,t_0)-\gamma(p_0,t_0)}{d(p_0,q_0,t_0)}.$$

Then using (11), we first show that $\langle \omega, e_1 \rangle = \langle \omega, e_2 \rangle = \frac{d}{\psi}\cos(\frac{l\pi}{L})$.
Now from (6), we have

$$\delta(e_1 \oplus e_2) d(p_0, q_0, t_0) = \langle e_1 - e_2, -\omega \rangle ,$$

and

$$\begin{aligned}\frac{d}{d\tau}\psi(\alpha_p(\tau), \alpha_q(\tau), t_0) &= \frac{L}{\pi}\cos\left(\frac{l\pi}{L}\right)\frac{\pi}{L}\frac{d}{d\tau}(l) \\ &= \cos\left(\frac{l\pi}{L}\right)\frac{d}{d\tau}(l).\end{aligned}$$

Assume $\xi = e_1 \oplus 0$: Then

$$\begin{aligned}0 &= \delta(e_1 \oplus 0)\left(\frac{d}{\psi}\right)(p_0, q_0, t_0) \\ &= \frac{\psi(p_o, q_0, t_0)\delta(e_1 \oplus 0)d(p_0, q_0, t_0) - d(p_0, q_0, t_0)\delta(e_1 \oplus 0)\psi(p_0, q_0, t_0)}{l^2} \\ &= \frac{\psi\langle e_1, -\omega\rangle - d\cos(\frac{l\pi}{L})(-1)}{\psi^2},\end{aligned}$$

and hence,

$$\langle \omega, e_1 \rangle = \frac{d}{\psi}\cos\left(\frac{l\pi}{L}\right). \tag{12}$$

Assume $\xi = 0 \oplus e_2$:

$$\begin{aligned}0 &= \delta(0 \oplus e_2)\left(\frac{d}{\psi}\right)(p_0, q_0, t_0) \\ &= \frac{\psi\langle e_2, \omega\rangle - d\cos(\frac{l\pi}{L})(1)}{l^2},\end{aligned}$$

and hence,

$$\langle \omega, e_2 \rangle = \frac{d}{\psi}\cos\left(\frac{l\pi}{L}\right). \tag{13}$$

Then either $e_1 = e_2$ or $e_1 \neq e_2$. Notice that, in the later case, $e_1 + e_2$ is parallel to ω.
Case 1: $e_1 = e_2$. In this case, we choose $\xi = e_1 \oplus e_2$.
Since $\frac{d}{d\tau}\psi(\alpha_p(\tau), \alpha_q(\tau), t_0) = 0$ we have $\delta(\xi)(\psi) = 0$. Then

$$\begin{aligned}0 \le \delta^2(\xi)\left(\frac{d}{\psi}\right)(p_0, q_0, t_0) &= \left.\frac{d^2}{d\tau^2}\right|_{\tau=0}\left(\frac{d}{\psi}\right)(\alpha_p(\tau), \alpha_q(\tau), t_0) \\ &= \left.\frac{d}{d\tau}\right|_{\tau=0}\frac{\psi\frac{d}{d\tau}d - d\frac{d}{d\tau}\psi}{\psi^2} = \left.\frac{d}{d\tau}\right|_{\tau=0}\frac{1}{\psi}\frac{d}{d\tau}d \\ &= \left.\frac{d}{d\tau}\right|_{\tau=0}\left[\frac{1}{\psi d}\left\langle \alpha_p'(\tau) - \alpha_q'(\tau),\, \alpha_p(\tau) - \alpha_q(\tau)\right\rangle\right] \\ &= \left[\frac{\psi d}{(\psi d)^2}\left(\left\langle \alpha_p'(\tau) - \alpha_q'(\tau),\, \alpha_p'(\tau) - \alpha_q'(\tau)\right\rangle + \left\langle \alpha_p''(\tau) - \alpha_q''(\tau),\, \alpha_p(\tau) - \alpha_q(\tau)\right\rangle\right)\right. \\ &\quad \left. + \frac{1}{(\psi d)^2}\left\langle \alpha_p'(\tau) - \alpha_q'(\tau),\, \alpha_p(\tau) - \alpha_q(\tau)\right\rangle \frac{d}{d\tau}(\psi d)\right]_{\tau=0} \\ &= \frac{1}{\psi}\left\langle \overrightarrow{k}(p_0, t_0) - \overrightarrow{k}(q_0, t_0),\, -\omega\right\rangle \quad \text{Since} \quad \alpha_p'(0) - \alpha_q'(0) = 0.\end{aligned}$$

Thus,

$$\left\langle \omega,\ \overrightarrow{k}(q_0,t_0) - \overrightarrow{k}(p_0,t_0) \right\rangle \ \geq 0. \tag{14}$$

Case 2: $e_1 \neq e_2$. In this case, we choose $\xi = e_1 \ominus e_2$.
Since $\frac{d}{d\tau}\psi(\alpha_p(\tau),\alpha_q(\tau),t_0) = \cos(\frac{l\pi}{L})\frac{d}{d\tau}(l(p_0,q_0,t_0)-2\tau) = \cos(\frac{l\pi}{L})(-2)$, we have $\delta(\xi)(\psi) = -2\cos(\frac{l\pi}{L})$. Then,

$$\begin{aligned}
0 \leq \delta^2(\xi)\left(\frac{d}{\psi}\right)(p_0,q_0,t_0) &= \frac{d^2}{d\tau^2}\bigg|_{\tau=0}\left(\frac{d}{\psi}\right)(\alpha_p(\tau),\alpha_q(\tau),t_0) \\
&= \frac{d}{d\tau}\bigg|_{\tau=0}\frac{\psi\frac{d}{d\tau}d - d\frac{d}{d\tau}\psi}{\psi^2} \\
&= \frac{d}{d\tau}\bigg|_{\tau=0}\left[\frac{1}{\psi d}\left\langle \alpha_p'(\tau)-\alpha_q'(\tau),\, \alpha_p(\tau)-\alpha_q(\tau)\right\rangle + \frac{2d}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\right] \\
&= \Bigg[\frac{\psi d}{(\psi d)^2}\left(\left\langle \alpha_p'(\tau)-\alpha_q'(\tau),\, \alpha_p'(\tau)-\alpha_q'(\tau)\right\rangle + \left\langle \alpha_p''(\tau)-\alpha_q''(\tau),\, \alpha_p(\tau)-\alpha_q(\tau)\right\rangle\right) \\
&\quad -\frac{1}{(\psi d)^2}\left\langle \alpha_p'(\tau)-\alpha_q'(\tau),\, \alpha_p(\tau)-\alpha_q(\tau)\right\rangle\left(\psi\frac{d}{d\tau}d + d\frac{d}{d\tau}\psi\right) \\
&\quad +\frac{2d}{\psi^2}\left(-\sin\left(\frac{l\pi}{L}\right)\left(\frac{\pi}{L}\right)(-2)\right) + \cos\left(\frac{l\pi}{L}\right)\left(\frac{2\psi^2\frac{d}{d\tau}d - 2d.2\psi.\frac{d}{d\tau}\psi}{\psi^4}\right)\Bigg]_{\tau=0} \\
&= \frac{1}{\psi}\left\langle \overrightarrow{k}(p_0,t_0) - \overrightarrow{k}(q_0,t_0),\, -\omega \right\rangle + \frac{1}{\psi d}|e_1+e_2|^2 + \frac{1}{\psi^2 d}\left\langle e_1+e_2,\, \omega\right\rangle \\
&\quad\left(\psi\left\langle e_1+e_2, -\omega\right\rangle - 2d\cos\left(\frac{l\pi}{L}\right)\right) + \frac{4d\pi}{\psi^2 L}\sin\left(\frac{l\pi}{L}\right) + \frac{2}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\left\langle e_1+e_2,\, -\omega\right\rangle \\
&\quad -\frac{4d}{\psi^3}(-2)\cos^2\left(\frac{l\pi}{L}\right) \\
&= \frac{1}{\psi}\left\langle \omega,\ \overrightarrow{k}(q_0,t_0) - \overrightarrow{k}(p_0,t_0)\right\rangle + \frac{1}{\psi d}|e_1+e_2|^2 - \frac{1}{\psi d}\left\langle e_1+e_2,\, \omega\right\rangle^2 \\
&\quad -\frac{4}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\left\langle e_1+e_2,\, \omega\right\rangle + \frac{4d\pi^2}{\psi^2 L^2}\left(\frac{L}{\pi}\sin\left(\frac{l\pi}{L}\right)\right) + \frac{8d}{\psi^3}\cos^2\left(\frac{l\pi}{L}\right) \\
&= \frac{1}{\psi}\left\langle \omega,\ \overrightarrow{k}(q_0,t_0) - \overrightarrow{k}(p_0,t_0)\right\rangle + \frac{4\pi^2 d}{L^2\psi}.
\end{aligned}$$

The last line follows from $\langle \omega,\ e_1+e_2\rangle = \frac{2d}{\psi}\cos\left(\frac{l\pi}{L}\right)$, which implies that $-\frac{4}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\langle e_1+e_2,\, \omega\rangle = -\frac{8d}{\psi^3}\cos^2\left(\frac{l\pi}{L}\right)$. Then $\omega \parallel e_1+e_2$ gives $\langle e_1+e_2,\, \omega\rangle^2 = |e_1+e_2|^2$, and so $\frac{1}{\psi d}\langle e_1+e_2,\, \omega\rangle^2 = \frac{1}{\psi d}|e_1+e_2|^2$.
Hence,

$$\frac{1}{\psi}\left\langle \omega,\ \overrightarrow{k}(q_0,t_0) - \overrightarrow{k}(p_0,t_0)\right\rangle + \frac{4\pi^2 d}{L^2\psi} \ \geq 0. \tag{15}$$

We now use the evolution equation (1) to compute $\frac{d}{dt}\left(\frac{d}{\psi}\right)(p_0,q_0,t_0)$.

$$\frac{d}{dt}\left(\frac{d}{\psi}\right)(p_0,q_0,t_0) = \frac{\psi\frac{d}{dt}d - d\frac{d}{dt}\psi}{\psi^2} = \frac{1}{\psi}\frac{d}{dt}d - \frac{d}{\psi^2}\frac{d}{dt}\psi.$$

We have

$$
\begin{aligned}
\frac{d}{dt}d(p_o,q_0,t_0) &= \frac{d}{dt}\sqrt{\left\langle \gamma(p_0,t_0)-\gamma_(q_0,t_0),\gamma(p_0,t_0)-\gamma_(q_0,t_0)\right\rangle} \\
& \frac{\left\langle \frac{\partial\gamma}{\partial t}(p_0,t_0)-\frac{\partial\gamma}{\partial t}(q_0,t_0),\gamma(p_0,t_0)-\gamma(q_0,t_0)\right\rangle}{d(p_0,q_0,t_0)} \\
&= \left\langle \frac{\partial\gamma}{\partial t}(p_0,t_0)-\frac{\partial\gamma}{\partial t}(q_0,t_0),-\omega\right\rangle \\
&= \left\langle \left(\frac{k}{J^2}-\frac{\nabla_N J}{J^2}\right)N(p_0,t_0)-\left(\frac{k}{J^2}-\frac{\nabla_N J}{J^2}\right)N(q_0,t_0),-\omega\right\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d\psi}{dt} &= \frac{L}{\pi}\cos\left(\frac{l\pi}{L}\right)\pi\frac{d}{dt}\left(\frac{l}{L}\right)+\frac{dL}{dt}\frac{1}{\pi}\sin\left(\frac{l\pi}{L}\right) \\
&= L\,\cos\left(\frac{l\pi}{L}\right)\left(\frac{L\frac{d}{dt}l-l\frac{d}{dt}L}{L^2}\right)+\frac{1}{\pi}\sin\left(\frac{l\pi}{L}\right)\frac{dL}{dt} \\
&= \cos\left(\frac{l\pi}{L}\right)\frac{d(l)}{dt}+\left(\frac{1}{\pi}\sin\left(\frac{l\pi}{L}\right)-\frac{l}{L}\cos\left(\frac{l\pi}{L}\right)\right)\frac{dL}{dt} \\
&= -\cos\left(\frac{l\pi}{L}\right)\int_p^q k\left(\frac{k}{J^2}-\frac{\nabla_N J}{J^2}\right)ds_t \\
&\quad -\left(\frac{1}{\pi}\sin\left(\frac{l\pi}{L}\right)-\frac{l}{L}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1} k\left(\frac{k}{J^2}-\frac{\nabla_N J}{J^2}\right)ds_t.
\end{aligned}
$$

Since $\langle \omega, e_1 \rangle = \langle \omega, e_2 \rangle = \frac{d}{l}\cos(\frac{l\pi}{L})$, let $\alpha = \angle(\omega, e_1) = \angle(\omega, e_2)$ with $0 < \alpha < \pi/2$. Then $\langle \omega,\ N(q_0, t_0)\rangle = -\sin\alpha$, and $\langle \omega,\ N(p_0, t_0)\rangle = \sin\alpha$. Therefore we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{d}{\psi}\right)(p_0, q_0, t_0) \\
&= \frac{1}{\psi}\frac{d}{dt}d - \frac{d}{\psi^2}\frac{d}{dt}\psi \\
&= -\frac{1}{\psi}\left(\left(\frac{k}{J^2}(p_0, t_0) + \frac{k}{J^2}(q_0, t_0)\right) - \left(\frac{\nabla_N J}{J^2}(p_0, t_0) + \frac{\nabla_N J}{J^2}(q_0, t_0)\right)\right)\sin\alpha \\
&\quad + \frac{d}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\int_p^q k\left(\frac{k}{J^2} - \frac{\nabla_N J}{J^2}\right) ds_t \\
&\quad + \frac{d}{\psi^2}\left(\frac{1}{\pi}\sin\left(\frac{l\pi}{L}\right) - \frac{l}{L}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1} k\left(\frac{k}{J^2} - \frac{\nabla_N J}{J^2}\right) ds_t \\
&\geq -\frac{1}{\psi}[(k(p_0, t_0) + k(q_0, t_0))C_1 - 2C_2]\sin\alpha \\
&\quad + \frac{d}{\psi^2}\cos\left(\frac{l\pi}{L}\right) C_1 \int_p^q k^2\, ds - \frac{d}{\psi^2}\cos\left(\frac{l\pi}{L}\right) C_2 \int_p^q k\, ds \\
&\quad + \frac{d}{\psi L}\left(1 - \frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\left[C_1\int_{S^1} k^2\, ds - C_2\int_{S^1} k\, ds\right] \\
&= \frac{1}{\psi}\left[-\sin\alpha(k(p_0, t_0) + k(q_0, t_0)) + \frac{d}{\psi}\cos\left(\frac{l\pi}{L}\right)\int_p^q k^2\, ds\right. \\
&\quad \left. + \frac{d}{L}\left(1 - \frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1} k^2\, ds\right] C_1 \\
&\quad - \frac{1}{\psi}\left[2\sin\alpha + \frac{d}{\psi}\cos\left(\frac{l\pi}{L}\right)\int_p^q k\, ds + \frac{d}{L}\left(1 - \frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1} k\, ds\right] C_2,
\end{aligned}
$$

where

$$C_1 = \min_U\left(\frac{1}{J^2}\right), \quad C_2 = \max_U\left(\frac{|\nabla_N J|}{J^2}\right).$$

Since

$$\frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right) = \frac{\frac{l\pi}{L}}{\tan(\frac{l\pi}{L})} < 1,$$

we have

$$1 - \frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right) > 0.$$

If t_0 is close enough to the blow-up time ω, we can make C_1 approach 1 and C_2 approach 0. We also have $\int k^2\, ds > 0$. We now consider case 1 and case 2 separately to show $\frac{d}{dt}\left(\frac{d}{\psi}\right)(p_0, q_0, t_0) > 0$.

Case 1: $e_1 = e_2$. Since $\left\langle \omega,\ \overrightarrow{k}(q_0, t_0) - \overrightarrow{k}(p_0, t_0)\right\rangle \geq 0$, we have $-\sin\alpha(k(p_0, t_0) + k(q_0, t_0)) \geq 0$. Hence, $\frac{d}{dt}\left(\frac{d}{\psi}\right)(p_0, q_0, t_0) > 0$.

Case 2: $e_1 \neq e_2$. Since $\frac{1}{\psi}\left\langle \omega,\ \overrightarrow{k}(q_0, t_0) - \overrightarrow{k}(p_0, t_0)\right\rangle + \frac{4\pi^2 d}{L^2\psi} \geq 0$, we have $-\frac{\sin\alpha}{\psi}(k(p_0, t_0) + k(q_0, t_0)) \geq -\frac{4\pi^2 d}{L^2\psi}$.

Claim:
$$\frac{d}{\psi L}\left(1-\frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1} k^2\,ds-\frac{4\pi^2 d}{L^2\psi}\ge -\frac{4\pi^2 dl}{\psi^2 L^2}\cos\left(\frac{l\pi}{L}\right).$$

Using $-\frac{\sin\alpha}{\psi}(k(p_0,t_0)+k(q_0,t_0))\ge -\frac{4\pi^2 d}{L^2\psi}$ and then the claim, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{d}{\psi}\right)(p_0,q_0,t_0)\\
&\ge\left[-\frac{4\pi^2 d}{L^2\psi}+\frac{d}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\int_p^q k^2\,ds+\frac{d}{\psi L}\left(1-\frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1}k^2\,ds\right]C_1\\
&\quad-\frac{1}{\psi}\left[2\sin\alpha+\frac{d}{\psi}\cos\left(\frac{l\pi}{L}\right)\int_p^q k\,ds+\frac{d}{L}\left(1-\frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1}k\,ds\right]C_2\\
&\ge\left[\frac{d}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\int_p^q k^2\,ds-\frac{4\pi^2 dl}{\psi^2L^2}\cos\left(\frac{l\pi}{L}\right)\right]C_1\\
&\quad-\frac{1}{\psi}\left[2\sin\alpha+\frac{d}{\psi}\cos\left(\frac{l\pi}{L}\right)\int_p^q k\,ds+\frac{d}{L}\left(1-\frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1}k\,ds\right]C_2\\
&\ge\left[\frac{d}{\psi^2 l}\cos\left(\frac{l\pi}{L}\right)\left(l\int_p^q k^2\,ds-\frac{4\pi^2l^2}{L^2}\right)\right]C_1\\
&\quad-\frac{1}{\psi}\left[2\sin\alpha+\frac{d}{\psi}\cos\left(\frac{l\pi}{L}\right)\int_p^q k\,ds+\frac{d}{L}\left(1-\frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1}k\,ds\right]C_2.
\end{aligned}$$

By the Hölder inequality,

$$l\int_p^q k^2\,ds\ge\left(\int_p^q|k|\,ds\right)^2\ge 4\alpha^2>\frac{4\pi^2l^2}{L^2}.$$

The last inequality is true since $\cos\alpha=\langle\omega,e_1\rangle=\langle\omega,e_2\rangle=\frac{d}{\psi}\cos(\frac{l\pi}{L})\Rightarrow\cos\alpha<\cos(\frac{l\pi}{L})\Rightarrow\alpha>\frac{l\pi}{L}$. Hence, $\frac{d}{dt}\left(\frac{d}{\psi}\right)(p_0,q_0,t_0)>0$.

We now prove the claim:

$$\begin{aligned}
\frac{d}{\psi L}\left(1-\frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1}k^2\,ds-\frac{4\pi^2 d}{L^2\psi} &\geq \frac{d}{\psi L}\left(1-\frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\left(\frac{4\pi^2}{l}\right)-\frac{4\pi^2 d}{L^2\psi}\\
&= \frac{4\pi^2 dl}{\psi^2L^2}\left[\frac{\psi L}{l^2}\left(1-\frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)-\frac{\psi}{l}\right]\\
&= \frac{4\pi^2 dl}{\psi^2L^2}\left[\frac{\psi}{l}\left(\frac{L}{l}-1\right)-\frac{L}{l}\cos\left(\frac{l\pi}{L}\right)\right]\\
&> \frac{4\pi^2 dl}{\psi^2L^2}\left[\frac{\psi}{l}-\frac{L}{l}\cos\left(\frac{l\pi}{L}\right)\right]\\
&= \frac{4\pi^2 dl}{\psi^2L^2}\left[\frac{\sin(\frac{l\pi}{L})}{(\frac{l\pi}{L})}-\frac{L}{l}\cos\left(\frac{l\pi}{L}\right)\right]\\
&\geq \frac{4\pi^2 dl}{\psi^2L^2}\left[\cos\left(\frac{l\pi}{L}\right)-\frac{L}{l}\cos\left(\frac{l\pi}{L}\right)\right]\\
&= -\frac{4\pi^2 dl}{\psi^2L^2}\cos\left(\frac{l\pi}{L}\right)\left(\frac{L}{l}-1\right)\\
&= -\frac{4\pi^2 dl}{\psi^2L^2}\cos\left(\frac{l\pi}{L}\right).
\end{aligned}$$

□

The distance comparison principles thus established immediately rule out slowly forming (type-II) singularities for the flow, where the ratios estimated above tend to zero. Thus, using only the known classification of possible singularities, we have proved the main theorem using distance comparison principles.

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Department of mathematics
Lehigh University
Bethlehem, PA 18015
USA.
E-mail address: david.johnson@lehigh.edu

Department of Mathematics
University of California
Riverside, CA 92521
USA.
E-mail addresses: muralee@math.ucr.edu