<u>Given</u> Open book decomp of M^3

Binding C, pages P, fiber bundle is

$$P \to X \to S^1$$

$$\partial X \to S^1$$
$$\cong$$
$$C \times S^1$$

$$(M \cong C \times D^2 \cup_\partial X)$$

Also given a link L in M^3 s.t. L is transverse to the pages of X.

$$L \cap C = \phi$$
$$(\text{so } L \subseteq \mathrm{Int}\, X)$$

<u>General</u> L has a fiberwise tubular nbhd.

$$\begin{tikzcd}
 & L \times D^2 \arrow[dl, "proj"'] \arrow[dr, "q"] & \subseteq & \mathrm{Int}\, X \arrow[dl, "proj"] \\
L \arrow[rr, bend right, "q'"'] & & S^1 &
\end{tikzcd}$$

q is a submersion.

q' is too. Note that L = union of circles and q' on each component is a covering space proj.

<u>Follow Thurston–Winkelnkemper</u>

$u = r\cos\theta$, $v = r\sin\theta$

On D^2 we have the 1-form

$$\underset{\text{CART.}}{\frac{1}{2}(u\,dv - v\,du)} = \underset{\text{POLAR}}{\frac{1}{2} r^2 d\theta} \quad \text{Apply } d \text{; then}$$

we get $du\,dv = r\,dr\,d\theta$, the standard volume form on D^2.

In TW's notation, $X = W(h)$.
Let $W_0 = W - \mathrm{Int}(L\times D^2)$, so $W_0 = W$ with some disks removed. If $X_0 = X - \mathrm{Int}(L\times D^2)$, then, as bundles, we have

$$\begin{array}{c} X = X_0 \cup_{L\times S^1} L\times D^2. \\ \downarrow \\ S^1 \end{array}$$

One crucial step in T-W is the construction of a 1-form on X with certain properties. We need to refine this construction so that the restriction of this 1-form to $L\times D^2 \subseteq X$ will be the pullback of $\frac{1}{2}(u\,dv - v\,du)$. This reduces to finding a suitable 1-form on X_0 whose restriction to a collar $L\times S^1\times[1,\varepsilon)$ is given by $\frac{1}{2}(u\,dv - v\,du)$. In order to avoid conflicts with T-W's notation we should (shall) replace θ by ψ henceforth.
[Note: the form on $L\times D^2$ and $L\times S^1\times[1,\varepsilon)$ does not involve the L-variable!]

so it's a pullback of a form on $\mathbb{R}^2$

One can now construct the form α over X_0, such that its restriction to $\partial X_0 = \partial X \sqcup L \times S^1$ has the standard forms, as in the 3rd paragraph on p. 346 of T-W. Furthermore, we can proceed as in the remainder of the paper to construct a contact form ω on M s.t. its restriction to a neighborhood of $L \times D^2$ ~~has~~ is given by $\alpha + K q'^* d\phi$, where ϕ and K are given as in T-W.

Now suppose we have a regular cyclic branched covering $\pi : N \to M$ of degree $p > 1$ with branch set equal to L; to simplify the discussion, assume that p is prime (the case of a general orbit space projection requires more elaborate notation). Composing with an automorphism of $\mathbb{Z}_p$ if necessary, we can arrange things so that

$\mathbb{Z}_p$ acts on ~~L×D² [illegible]~~

$\mathbb{Z}_p$ acts here by, say, $\xi(x,z) = (x, \xi z)$ where $\xi^p = 1$

$$\begin{tikzcd} N \arrow[r, "\pi"] & M \\ L\times D^2 \arrow[u, "\subseteq"] \arrow[r] & L\times D^2 \arrow[u, "\subseteq"'] \\ (x,z) \arrow[r, squiggly] & (x, z^p) \end{tikzcd}$$

commutes. $\boxed{\text{POLAR FORM: } (r,\psi) \rightsquigarrow (r^p, p\psi)}$

Since $\pi \mid N - L\times\{0\}$ is a smooth submersion, if ω is a contact form on M then $\pi^*\omega$ will be a contact form on $N - L\times\{0\}$. Thus it remains to find a contact form on $L\times D^2$ which extends $\pi^*\omega$ near $L\times S^1$. in N

An obvious first step is to write out $\pi^*\omega \mid L\times D^2$ explicitly:

$$\pi^*\omega \mid L\times D^2 = \frac{p}{2} r^p\, d\psi + K \boxed{q'^* d\phi}$$

" $d\phi_1$

($p > 1$ prime)

~~Note that this form is 0 on $L\times\{0\}$ and hence is not a contact form on~~

Note $\pi^*\omega$ is a contact form near $\pi^{-1}[C]$, so we don't need to modify the form near $\pi^{-1}[C]$.

Note that $\pi^*\omega \wedge d\pi^*\omega = 0$ on $L \times \{0\}$ and hence $\pi^*\omega$ is not a contact form. We need to modify it so that the restriction to a ~~neighborhood~~ neighborhood of $L \times \{0\}$ looks like $\frac{1}{2} r^2 d\psi + K d\phi_1$, which is a $\mathbb{Z}_p$-invariant contact form on this nbhd.

More precisely, we need a function $F(r)$ such that $F'(r) > 0$ if $r > 0$

The (~~or~~ one) contruction for F has been outlined.

$$F(r) = \frac{1}{2} r^2 \text{ ~~near~~ on } 0 \le r < \varepsilon$$

some $\varepsilon > 0$ small

Note that this is $\mathbb{Z}_p$-invariant where $\mathbb{Z}_p$ acts on $L \times D^2$ via 2 coord.

$$F(r) = \frac{p}{2} r^p \text{ ~~near~~ on } 1 - \varepsilon < r$$

Given F, let $\mu = F(r)\, d\psi + K d\phi_1$.

μ = standard near $L \times \{0\}$

Then $d\mu = K \cdot F'(r)\, dr \wedge d\psi$ and

$\mu \wedge d\mu = K \cdot F'(r)\, dr \wedge d\psi \wedge d\phi_1 \Rightarrow \mu$ is again a contact form on $L \times D^2 - \{0\}$ with $\mu = \pi^*\omega$ near $L \times S^1$. Since μ is standard near $L \times \{0\}$, μ is also a contact form near $L \times \{0\}$, and we are done. $\blacksquare$

Reminders

Given $(\mathbb{Z}_p, N)$, we take $M = N/\mathbb{Z}_p$, and we denote $N^{\mathbb{Z}_p}$ and its image in M by L; L is a link in M. If we take an open book decomposition of M, then we can isotop it so that L is disjoint from the binding and transverse to the pages

(One isotops L w.r.t. the given binding and uses the isotopy to construct a perturbed open book structure on M).