CONTACT STRUCTURES & ORIENTATION REVERSAL

Assume all group actions are effective

<u>CLAIM</u> M^3 oriented, $\mathbb{Z}_{2k}$ acts smoothly but not orientation preservingly on $M^3 \Rightarrow$ there is no contact form on M^3 which is compatible with the group action.

<u>NOTE</u> In contrast, one <u>can</u> have foliations (codim 1). For examples, let L & W be non-trivial reps of $\mathbb{Z}_{2k}$ with dimensions 1 and 2 respectively, then $W \oplus L$ ($\cong \mathbb{R}^3$ as a manifold) has the product foliation $W \times \{v\}$ where $v \in L$.

<u>OUTLINE OF APPROACH</u> Assume M^3 connected.

① Reduce to the free case (why is the union of free orbits open and dense? look first at linear actions)

② Show that the orbit manifold $M^* = M^3/\mathbb{Z}_{2k}$ is nonorientable (Hints: [cont'd next page]

Orientability $\Leftrightarrow$ nowhere zero 3-form. If $\Omega \in \Lambda^3(M^*)$ is nowhere zero, so is its pullback $p^*\Omega \in \Lambda^3(M)$. Why is the latter fixed under g^* for all $g \in \mathbb{Z}_{2k}$? Since $\mathbb{Z}_{2k}$ does not act orientation preservingly, there is a (another?) nowhere zero form $\Omega' \in \Lambda^3(M)$ s.t. $g^*\Omega' = -\Omega'$ if $g \in \mathbb{Z}_{2k}$ generates (why?). Why can we choose normalization [unit length] of $p^*\Omega$ and Ω' ~~that~~ are $\pm$ each other? Why does this yield a contradiction?)

3 Suppose we have a compatible contact structure defined by a 1-form θ on M^3 s.t. $g^*\theta = \varepsilon(g)\theta$ for some hom. $\varepsilon : \mathbb{Z}_{2k} \to \{\pm 1\}$.

Show $g^*(\theta \wedge d\theta) = \theta \wedge d\theta$, and using the latter show that $\theta \wedge d\theta$ is the pullback of a nowhere ~~nonzero~~ zero form in $\Lambda^3(M^*)$. (see footnote on next page)

Extensions

$\boxed{A}$ This works in all dim $\equiv 3\ (4)$.

$\boxed{B}$ In dimensions $\equiv 1(4)$, if $\mathbb{Z}_{2k}$ acts orientation preservingly and θ is a compatible contact form, then $g^*\theta = \theta$ for all g.

$\boxed{C}$ What happens if $\mathbb{Z}_{2k}$ does <u>not</u> preserve orientation?

Footnote. If $q : N' \longrightarrow N$ is a regular covering space projection in the smooth category ($\Rightarrow$ transitive group of covering transformations; call it Γ), then

$$q^* : \Lambda^*(N) \longrightarrow \Lambda^*(N')$$ is

1–1 and its image is the subcomplex $\Lambda^*(N')^{\Gamma}$ of forms invariant under the action of Γ. (Why?)