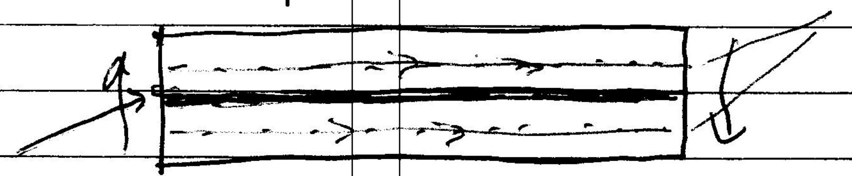
Foliations and orientation reversal

<u>An example</u> The Klein bottle K can be viewed as $S^1 \times S^1/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts freely by $T(z,w) = (-z, \overline{w})$. Note that K fibers over $S^1 \cong S^1/\mathbb{Z}_2$ via the map

$$K \longrightarrow S^1/\mathbb{Z}_2 \cong S^1 \qquad [\ ] = \text{EQUIV. CLASS}$$
$$[z,w] \longrightarrow [z]$$

Consider the nowhere zero vector field on $S^1 \times S^1$ defined by $\frac{\partial}{\partial\theta}$ where $\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}$ generate the vector fields on $S^1 \times S^1$. Its integral curves have the form $S^1 \times \{c\}$ for some $c \in S^1$, and these also define a codimension 1 foliation of $S^1 \times S^1$. Look at the image of this foliation in K. It defines a codimension 1 foliation of K which is transverse to the fibers of the circle bundle $K \longrightarrow S^1$. If $c \neq \pm 1$, then $S^1 \times \{c\}$ maps 1-1 onto a leaf L such that $L \longrightarrow S^1$ has degree 2, and if $c = \pm 1$ then $S^1 \times \{c\}$ maps 2-1 onto a leaf L s.t. $L \longrightarrow S^1$ has degree 1. Visually, this is two copies of the usual foliation of the Möbius strip



joined along their boundaries.

<u>QUESTION:</u> Does this foliation extend to the solid Klein bottle $S^1 \times_{\mathbb{Z}_2} D^2 \cong [S^1 \times D^2$ modulo $(z, w) \sim (-z, \bar{w})]$?

<u>CLAIMS</u>

1. Answer is no.

2. The negative answer implies that if $\mathbb{Z}_{2k}$ acts orientation reversingly and linearly on S^3, then

there is no codim 1 foliation compatible with the action. — Note that if $k > 1$ then the fixed point set of the group is S^0, while if $k = 1$ then it is S^2 or S^0. Since the case of S^2 is excluded by other considerations it is only necessary to consider the S^0 case here.

Nonexistence result

The normal bundle to the foliation on the Klein bottle can be expressed as the image of the subbundle of $T(S^1 \times D^2) \cong S^1 \times D^2 \times \mathbb{R}^3$ as follows:

$$\begin{array}{ll} \overset{\shortparallel}{S^1 \times D^2 \times \mathbb{R} \times \mathbb{C}} & \text{all } (z, w; 0, -iwt). \\ & t \in \mathbb{R}. \end{array}$$

Check that the latter is invariant under the corresponding identification to T:

$T(z, w; 0, tiwt)$

T' on $T(S^1 \times D^2) \cong S^1 \times D^2 \times \mathbb{R} \times \mathbb{C}$

maps $(z, w; s, \zeta)$ to

$$(-z, \overline{w}; s, \overline{\zeta})$$

so $s = 0$ and $\zeta = -iwt \Rightarrow$ Since $\overline{t} = t$, $\overline{i} = -i$

$\overline{\zeta} = i\overline{w}t = -iwt.$ ←

Hence the normal bundle to the foliation is just $S^1 \times S^1 \times \mathbb{R}$ modulo the identification

$(z, w, t) \sim (-z, \overline{w}, -t)$. Note that this bundle extends to $S^1 \times_{\mathbb{Z}_2} D^2$

We need to verify that this line bundle embedding does not extend to a line bundle embedding over $S^1 \times_{\mathbb{Z}_2} D^2$.

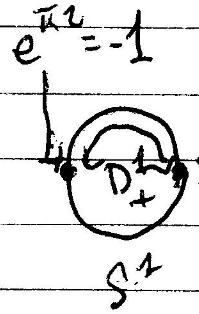
(Note: Since $H^1(S^1) \to H^1(K)$ is 1-1 with $\mathbb{Z}_2$ coeffs and the bundle over K extends to $S^1 \times_{\mathbb{Z}_2} D^2$, it follows that there is a unique extension to a line bundle over the latter.)

Equivalently, we need to check that the $\mathbb{Z}_2$ equivariant vector bundle embedding of

$$S^1\times S^1\times\mathbb{R}\longrightarrow S^1\times D^2\times\mathbb{R}\times\mathbb{C}$$

does not extend to a $\mathbb{Z}_2$ equivariant vector bundle embedding of

$$S^1\times D^2\times\mathbb{R} \quad\text{into}\quad S^1\times D^2\times\mathbb{R}\times\mathbb{C}.$$



Suppose such an embedding exists, and look at its restriction to $D^1_+\times D^2\times\mathbb{R}\cong [0,\pi]\times D^2\times\mathbb{R}$. We then get

$$[0,\pi]\times D^2\times\mathbb{R}\xrightarrow{q}\mathbb{R}\times\mathbb{C}$$

by projecting onto the last coords.

Also, the restriction to $[0,\pi]\times D^2\times\{1\}$ is nonzero, and the restrictions to $\{0\}\times D^2\times\{1\}$ and $\{\pi\}\times D^2\times\{1\}$ are related by the identification condition:

Linearity on each fiber implies that

$$\varphi(s,w,t) = t\,\underset{\varphi_0(s,w)}{\underset{\shortparallel}{\varphi(s,w,1)}} \quad \text{with } \varphi_0(s,w) \neq 0.$$

Since ~~$\varphi_0(0,w)t \neq t \in \mathbb{R}$~~

$(0,w,t) \sim (\pi, \bar{w}, -t)$ we have

$$\varphi_0(0,w)t = -\varphi_0(\pi,\bar{w})t \quad \text{all } w,t$$

On the other hand, equivariance implies that we have a commutative diagram

$$\begin{tikzcd}
\{0\}\times D^2 \times \mathbb{R} \arrow[r, "\varphi"] \arrow[d, "\text{const} \times \text{conj} \times -1"] & \mathbb{R}\times\mathbb{C} \arrow[d, "\text{id}\times\text{conj}"] \\
\{\pi\}\times D^2 \times \mathbb{R} \arrow[r, "\varphi"] & \mathbb{R}\times\mathbb{C}
\end{tikzcd}$$

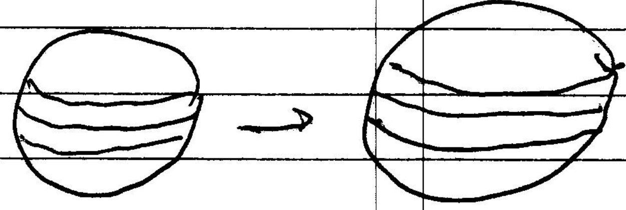
so that $-\varphi_0(\pi,\bar{w})t = A\,\varphi_0(0,w)t$

where $A : \mathbb{R}\times\mathbb{C} \longrightarrow \mathbb{R}\times\mathbb{C}$
$(\theta, w) \longrightarrow (\theta, \bar{w})$.

Thus $\varphi_0(\pi, w) = -A\varphi_0(0, \bar{w})$.

Let's look at the restriction of φ_0 to $S^2 \cong \partial([0,\pi] \times D^2)$ more closely. This is essentially a map from S^2 to S^2, and since φ_0 extends to $[0,\pi] \times D^2$ its degree must be zero.

On $[0,\pi] \times S^1$ the map sends (s, w) to $(s, -iw)$, so we can deform it so that it sends $[\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta] \times S^1$ into a belt around the equator in S^2 in the obvious diffeomorphic fashion. So we have



tropical zone goes diffeomorphically

The maps on the top and bottom disks are equivalent by the formula on the first line of page 7. In other words, we have diffeos α and β of D^2 and S^2 such that

$$g = \beta\alpha f.$$

~~Approximate to and Alex~~

REPLACE WITH DISCUSSION IN THE APPENDIX

We might as well choose g and f to be smooth (take an approximation to f, form g as above and note that the degree of the map $S^2 \longrightarrow S^2$ won't change. Now look at a regular value near the equator, and consider the degree mod 2.

Over the tropical zone there is a single preimage of this regular value. Also,

SEE THE APPENDIX

<u>CLAIM</u> f and g make equal contributions* to the mod 2 degree (prove this!). Hence the local degree is odd — contradiction!

So the line bundle embedding over K does not extend to $S^1 \times_{\mathbb{Z}_2} D^2$.

Application to foliations

Suppose we have a foliation as described in Claim 2 on pp 2-3.

Display the action of $\mathbb{Z}_{2k}$ on S^3 via some rep $\mathbb{R} \oplus L \oplus W$, where $\mathbb{Z}_{2k}$ acts trivially on $\mathbb{R}$, nontrivially on L with $\dim L = 1$, and by rotation on the 2-dim vector space W. Take the usual splitting

$$S(\mathbb{R} \oplus L \oplus W) = \left(S(\mathbb{R} \oplus L) \times D(W)\right) \cup \left(D(\mathbb{R} \oplus L) \times S(W)\right)$$

Note that the orbit space of the second piece is a solid Klein bottle.

<u>CLAIM</u> We can isotop the foliation so that it is a product foliation given by $\{p.t.\} \times D(W)$ on the first piece.

<u>Two cases, same strategy</u>

<u>$k>1$</u> Look at $\mathbb{Z}_k$ action, whose fixed set is $S(\mathbb{R}\oplus L)\times\{0\}$. If ω is a 1-form of suitable type which defines the foliation, then $\mathbb{Z}_k$ preserves ω. Show the foliation is transverse to $S(\mathbb{R}\oplus L)$.

<u>$k=1$</u> Find a smooth curve $\mathbb{Z}_2$ isotopic to $S(\mathbb{R}\oplus L)$ s.t. the foliation is transverse to this curve.

<u>Step 1</u> Do this near the two fixed points. One has a choice of directions, and for almost all of them the tangent planes to the foliation are transverse. Rotate to

make these tangent lines to the curve equal to the tangents for $S(\mathbb{R}\oplus L)$.

<u>Step 2</u> Make the curves transverse everywhere else by ordinary transversality and equivariance.

Using the claim, can say that the product foliation on $S(\mathbb{R}\oplus L) \times S(W)$ extends to $D(\mathbb{R}\oplus L) \times S(W)$. Take orbit spaces, and use the first part to get a contradiction.

APPENDIX ON (LOCAL) DEGREES (MOD 2)

Background:

Dold, Lectures on Algebraic Topology (2nd Ed.), §IV.5

The objective is to derive a contradiction, showing that $\varphi \mid \partial([0,\pi)\times D^2)$ has odd degree. Actually, we shall use the deformation of this map described on pp. 7–8. Call it f.

NH = open northern hemisphere of S^2 $\quad x_1 > 0$
TZ = open "tropical zone" $\quad$ –"– $\quad -\delta < x_1 < \delta$
SH = open southern hemisphere of S^2 $\quad x_1 < 0$

We view S^2 as the unit sphere in $\mathbb{R}\oplus\mathbb{C}$, with equator S^1 = unit circle in $0\oplus\mathbb{C}$; so the north and south poles are $\pm(1;0,0)$.

CLAIM The local degrees of the restrictions $f|NH$, $f|TZ$ and $f|SH$ at $y \in \{0\}\times S^1$ are all definable. Furthermore, $Y = f^{-1}[\{y\}]$

is a union of pairwise disjoint open-closed subsets $Y \cap NH$, $Y \cap TZ$, and $Y \cap SH$.

<u>Consequence</u>: Using the additivity property for local degrees (see Dold), we have

$$\deg(f) = \mathrm{locdeg}_y(f) = \mathrm{locdeg}_y(f|NH) + \mathrm{locdeg}_y(f|TZ) + \mathrm{locdeg}_y(f|SH).$$

$\boxed{\text{LIKEWISE MOD } 2}$

Since $f|TZ$ is a diffeomorphism onto its image, it follows that the second term is ± 1 ($\Rightarrow$ the reduction mod 2 equals 1).

<u>Verification of claim</u>. Since $Y \cap TZ = \{pt.\}$ and this point lies on the equator (map is a diffeo on TZ), we know that

$$(Y \cap NH) \cap (Y \cap TZ) = \emptyset = (Y \cap SH) \cap (Y \cap TZ)$$

Also $(Y \cap NH) \cap (Y \cap SH) \subseteq NH \cap SH = \emptyset$.
So the three sets are pairwise disjoint, open in Y, and their union is Y. Hence they are also closed; but

$Y \subseteq S^2$ is compact, so each of these sets is compact.

It will now suffice to check that

$$\left.\begin{array}{l} \text{loc deg}_y\,(f|NH) \bmod 2 = \\ \text{log deg}_y\,(f|SH) \bmod 2. \end{array}\right\} \begin{array}{l}\text{for}\\ \text{some } y \text{ on}\\ \text{the equator.}\end{array}$$

The restrictions of f to NH and SH are related by the identity $\varphi_0(\pi, w) = -A\varphi_0(0, \overline{w})$. This implies that

$$\text{loc deg}_y\,(f|SH) \bmod 2 =$$
$$\text{loc deg}_{-Ay}\,(f|NH) \bmod 2.$$

Therefore the local degrees are equal if $y = -Ay$; by construction $-Ay = -\overline{y}$, so we want y s.t. $y = -\overline{y}$. This happens when $y = \pm i$, so for this choice the local degrees will be the same, so that

$$\deg(f) \bmod 2 = \mathrm{locdeg}_y (f \,|\, TZ) \bmod 2$$
$$= 1 \bmod 2.$$

This yields the contradiction at the bottom of page 8.