CODIMENSION ONE FOLIATIONS AND ODD ORDER GROUP ACTIONS

CHRISTOPHER CARLSON AND REINHARD SCHULTZ

Abstract. Given a smooth action of a finite group on a closed manifold, we derive simple sufficient conditions for the existence of a compatible codimension one foliation in terms of Euler characteristics, and we also show the conditions are also necessary if the group has odd order. The proof combines results of W. Thurston on the existence of codimension one foliations for manifolds not equipped with group actions, results of the first author on compatible foliations, and an inductive construction based upon the usual stratification of a smooth group action.

If M is a compact connected smooth manifold, then the Poincaré-Hopf Index Theorem (see [28], p. 35) is a fundamental result relating the global analytic properties of M to the Euler characteristic, which is a purely topological invariant. In particular, this result shows that the Euler characteristic of M is zero if and only if M has a tangent vector field which is nowhere zero [20]. A much deeper theorem along the same lines, due to W. Thurston [37], involves a concept known as a codimension 1 foliation (*e.g.*, see [40]). Roughly speaking, this structure is a partition of M into immersed $(n-1)$ dimensional submanifolds $\{N_\alpha\}$, which are called the *leaves* of the foliation, such that the tangent spaces to the submanifolds are given locally by given by the kernels of nowhere zero differential 1-forms with certain additional properties. It is often more convenient to reformulate this in terms of tangent vector fields rather than 1-forms, in which case the locally defined nowhere zero vector field will be complementary to the tangent spaces of the leaves. From this perspective Thurston's result states that every nowhere zero vector field X can be approximated by a vector field X' such that X' is the normal bundle for some smooth codimension 1 foliation on M.

In [9] S. Costenoble and S. Waner proved a partial analog of the Poincaré-Hopf Index Theorem for manifolds with smooth compact Lie group actions and nowhere zero equivariant vector fields. This paper establishes an analogous result for smooth actions of odd order finite groups on compact manifolds and

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codimension 1 foliations which are compatible with such group actions in an appropriate sense.

A precise statement of the main result requires input from two distinct branches of topology and geometry that are not often seen together; namely, the theories of compact transformation groups and of foliations on manifolds. The necessary background on compact transformation groups can be found in Chapters I, II, IV and VI of Bredon's book [3] on the subject, with additional input from the first two chapters of [14] (see also [33] and [10]), while the books by Candel and Conlon ([5] and [6]) and Tondeur [40] contain the necessary background on foliations. We shall also use results from the first named author's doctoral dissertation [7] about compatible foliations on manifolds with smooth finite group actions (see Section 1 for a discussion of compatibility).

Theorem 0.1. *Let G be a group of odd order, and suppose we are given a smooth action of G on a compact connected manifold M. Then there is a codimension 1 foliation on M which is compatible with the group action if and only if for every subgroup H and every component C of the H-fixed point set M^H, the Euler characteristic $\chi(C)$ is zero.*

Remarks. Although the Euler characteristics of M and the H-fixed point sets M^H satisfy some standard identities (compare [38], Section 1.3), there are many examples such that $\chi(M) = 0$ but $\chi(M^H) \neq 0$ for some subgroup H; one simple example is the $\mathbb{Z}_2$-action on $S^3 \subset \mathbb{R}^4$ which sends (x, y, z, t) to $(x, y, z, -t)$, and in fact for each odd prime p one can also construct examples of $\mathbb{Z}_p$-manifolds M such that $\chi(M) = 0$ but the Euler characteristic of the fixed point set is nonzero. Furthermore, the condition $\chi(M^H) = 0$ for all subgroups H, which plays a key role in equivariant homotopy theory (compare [38]), is much weaker than the condition in the theorem. Elementary examples for the preceding two sentences are given in [35].

The significance of the odd order hypothesis. We shall prove that the Euler characteristic hypotheses are also sufficient for the existence of compatible foliations if G has even order, but the following example shows that these conditions are not necessary if $G = \mathbb{Z}_2$. Let $\mathbb{Z}_2$ act on S^1 by the complex conjugation map sending $(x, y) \in S^1$ to $(x, -y)$, let F be any positive dimensional closed smooth manifold whose Euler characteristic is nonzero, and consider the codimension 1 foliation on $S^1 \times F$ whose leaves are the slices $\{z\} \times F$ for $z \in S^1$. This foliation is clearly compatible with the product involution on $S^1 \times F$ (with trivial action on the second coordinate), but the fixed point set is $S^0 \times F$ and the Euler characteristic of each component in the latter is $\chi(F) \neq 0$. More will be said about this example after the statement of Proposition 1.2.

Here is a brief outline of the paper: Section 1 develops the basic setting for studying codimension 1 foliations which are compatible with smooth actions of finite groups. Much of this material is contained in [7], but the discussion is taken further in a few directions, and this leads to the proof of the "only if" implication in Theorem 0.1 (see Theorem 1.3). A proof of the converse implication for special cases is given in [7], and the next two sections describe the tools needed to generalize this approach so that it applies to a wide range of smooth actions of finite groups. In [7] the orbit structure of the group action was describable in terms of a partially ordered set of smooth submanifolds, and for more general cases we need to analyze the standard description of the orbit structure as a *stratification* in the sense of R. Thom [36] and J. Mather [27] (see also [17]); for the sake of clarity, we shall interpret some aspects of this stratification in terms of the approach in work of M. Davis [10]. The material in Sections 2 and 3 leads to a proof of the "if" implication in Theorem 0.1; in fact, we prove a more general result which also applies to groups of even order (see Theorem 3.3). Section 3 describes the construction of the foliation; this is done inductively on the strata in the orbit structure, and at each step the two main issues are extending a well-behaved nowhere zero vector field to each stratum and applying the results of [37] to find codimension 1 foliations associated to some approximations of these vector fields. Finally, Section 4 discusses the corresponding problem for noncompact smooth G-manifolds, and the analogous existence result in this case is stated.

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1. Leaves and fixed point sets

We shall begin by summarizing some basic facts about codimension 1 foliations which are compatible with smooth actions of finite groups. Details and more general results appear in [7].

Since every smooth action of a compact Lie group on a manifold M has a G-invariant riemannian metric, we shall choose such metrics whenever it is convenient to do so. However, we do not make any assumptions about the extent to which deeper riemannian structures on M (*e.g.*, the geodesics) are compatible with a given foliation; a considerable amount of work has been done on foliations which are compatible with such structures (riemannian foliations), but our emphasis lies in an entirely different direction; one reference for riemannian foliations is [40].

1.1. **Some basic definitions.** If M is a smooth manifold, we shall let $\tau(M)$ denote the tangent bundle of M, and we shall let $T(M)$ denote its total space (*i.e.*, the tangent space of M). Given a foliation $\mathcal{F}$ on a smooth manifold M^n with arbitrary codimension q (where $0 < q < n$), the tangent bundle $\tau(M)$ splits into a direct sum of two vector bundles $\tau(\mathcal{F}) \oplus \nu(\mathcal{F})$, where $\tau(\mathcal{F})$ is the bundle of tangents along the leaves and the restriction of $\nu(\mathcal{F})$ to each leaf L^{n-q} is the normal bundle of the immersion $L \to M$. Note that if $q = 1$ then the restriction of $\mathcal{F}$ to some open subset U is defined by a 1-form with special properties if and only if the restriction of $\nu(\mathcal{F})$ to U is the trivial real line bundle (compare [5], Exercise I.2.15, p. 28).

We shall say that a diffeomorphism $f : M \to M$ preserves a smooth codimension q foliation $\mathcal{F}$ if the induced diffeomorphism on the tangent space of M, which we shall call $T(f) : T(M) \to T(M)$, sends the subbundle of tangents along the leaves to itself. If a diffeomorphism preserves a foliation $\mathcal{F}$, then f permutes the leaves of the foliation, and conversely if this condition holds then f preserves the foliation in the sense of our definition. When $q = 1$ and the foliation is defined locally by a nowhere zero 1-form ω on some f-invariant open subset U, then on this subset then f preserves the foliation if and only if the pullback form $f^*\omega$ is equal to $h \times \omega$ for some nowhere zero smooth function $h : U \to \mathbb{R}$.

Definition. Suppose that the group G acts on a smooth manifold M by diffeomorphisms, and let $\mathcal{F}$ be a smooth codimension q foliation on M. Then the group action is compatible with $\mathcal{F}$ if for each $g \in G$ the associated diffeomorphism $\Phi_g : M \to M$ preserves the foliation $\mathcal{F}$. In this paper we are almost exclusively interested in actions for which the group G is finite.

Example 1. If G acts smoothly on M_1 and M_2 is an arbitrary smooth manifold with a trivial action of G, then the slices $M_1 \times \{z\} \subset M_1 \times M_2$ are the leaves of a codimension q foliation for $q = \dim M_2$, and the product action of G on $M_1 \times M_2$ is compatible with this foliation.

Example 2. View S^3 as the unit sphere in $\mathbb{R}^4 \cong \mathbb{C}^2$, and let $T \subset S^3$ be the torus in $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ defined by $|z_1| = |z_2|$. It follows that the subsets of S^3 defined by $|z_1| \leq |z_2|$ and $|z_1| \geq |z_2|$ are both diffeomorphic to $S^1 \times D^2$, and if $\mathcal{F}$ is the Reeb foliation of S^3 ([5], p. 93) associated to this decomposition, then the action of S^1 on S^3 by scalar multiplication is compatible with $\mathcal{F}$.

Example 3. If $\mathcal{F}$ is a codimension q foliation of a smooth manifold N and $h : M \to N$ is a smooth submersion, then there is a pullback foliation $h * \mathcal{F}$ on M whose leaves are given by the sets $h^{-1}[L]$, where L is a leaf of $\mathcal{F}$ in N (see [25], p. 373, for even more general statements). When M and N have smooth actions of a compact Lie group G such that h is G–equivariant and the action of G on N is compatible with $\mathcal{F}$, then the action of G on M will be compatible with $h^*\mathcal{F}$.

1.2. **The induced map on normal vectors.** If $\mathcal{F}$ is a smooth codimension q foliation on M and $f : M \to M$ preserves $\mathcal{F}$, then by definition f maps the subbundle $\tau(\mathcal{F}) \subset \tau(M)$ to itself, and passing to quotients we obtain an isomorphism from $\nu(\mathcal{F})$ to itself which sends normal vectors over the point $x \in M$ to normal vectors over the point $f(x)$. From this point on, we shall only consider codimension 1 foliations unless there is an explicit statement to the contrary.

For our purposes it is particularly important to understand how this map of normal vectors behaves near a fixed point of f when f comes from a smooth action of a finite group G. One way to recognize the significance of such questions is to note that the proof of the main theorem on compatible codimension 1 foliations for free smooth actions of a finite group G on a compact manifold M quickly reduces to the corresponding existence question for codimension 1 foliations on the orbit space M/G: *The existence of either is equivalent to the Euler characteristic condition* $\chi(M) = 0$, *which in turn is equivalent to the condition* $\chi(M/G) = 0$ *because* $\chi(M) = |G| \cdot \chi(M/G)$. For the sake of completeness, here is a quick proof: If $\chi(M) = \chi(M/G) = 0$, then by [37] there is a codimension 1 foliation on M/G, and the pullback of this foliation to M will be a compatible foliation for the free G-action on M. Conversely, if M has a compatible codimension 1 foliation, then its tangent bundle splits as $\alpha \oplus \beta$, where β is a line bundle. If β is trivial, then $\chi(M) = 0$ by the Poincaré-Hopf Theorem, and hence $0 = \chi(M) = \chi(M/G)$ implies that M/G has a codimension 1 foliation; if β is nontrivial, then there is some 2-sheeted covering $M' \to M$ such that the pullback of β to M' is trivial, and hence by the Poincaré-Hopf Theorem we have $0 = \chi(M') = 2\chi(M)$, so that $\chi(M) = 0$. Therefore the new content of the main result concerns smooth group actions for which the isotropy subgroup at some point is nontrivial.

Suppose now that G acts effectively and smoothly on the manifold M, which is not assumed to be compact at this point, let $\mathcal{F}$ be a codimension 1 foliation such that the group action is compatible with $\mathcal{F}$, and let x lie in the fixed point set M^H for some nontrivial subgroup $H \subset G$. The cyclic subgroup generated by h will be denoted by $C(h)$, and $\varphi(h)$ will denote the diffeomorphism of M, or some $C(h)$-invariant open subset, given by the group action.

We shall need the following refinement of a key result in [7]:

Proposition 1.1. *Let M be a compact smooth manifold without boundary, let $\mathcal{F}$ be a codimension 1 foliation on M, let G act smoothly on M, and assume that the group action is compatible with $\mathcal{F}$. Also, let $x \in M$ be fixed by the nontrivial element $h \in G$ such that x is not isolated in $M^{C(h)}$, and let U be a $C(h)$-invariant open neighborhood of x on which $\mathcal{F}$ is locally defined by a nowhere zero 1-form.*

Then either $\mathcal{F}$ *is transverse to the fixed point set* $M^{C(h)}$ *near* x or else *the component of* $M^{C(h)}$ *containing* x *is contained in a leaf of* $\mathcal{F}$. *In the first case there is a nowhere zero* 1*-form* ω *defined near* x *such that* $\varphi(h)^*\omega(x) = \omega(x)$, *and in the second case there is a nowhere zero* 1*-form* ω *defined near* x *such that* $\varphi(h)^*\omega(x) = -\omega(x)$.

Proof. Details are given in [7], but we shall sketch the argument because part of our main result relies heavily on this proposition. The first step is to choose a G-invariant riemannian metric on M.

The conclusion only concerns the behavior in a $C(h)$-invariant open neighborhood of x, so by the local linearity of smooth actions (*e.g.*, see [3], Corollary VI.2.4, p. 308) we might as well assume that M is an open disk in some orthogonal representation V of $C(h)$ with $x = 0 \in V$ and $C(h)$ acting orthogonally on V. General considerations imply that the original G-compatible foliation on M restricts to a $C(h)$-compatible foliation on V. Note also that the hypotheses imply $\dim V^{C(h)} > 0$.

Since V is contractible, the foliation on V has a trivial normal bundle and hence is defined by some 1-form ω; if f is a nowhere zero smooth function, then $f \cdot \omega$ is also a defining 1-form for the foliation, and therefore we may as well assume that $|\omega| = 1$ (here we take the $C(h)$-invariant inner product on the dual space V^* which is naturally associated to the given invariant inner product on V). Since the metric is invariant under the group action, we must have $|\varphi(g)^*\omega| = 1$ for all $g \in C(h)$, and since $C(h)$ is compatible with the foliation on V it follows that $\varphi(g)^*\omega(x)$ must be a scalar multiple of $\omega(x)$. These observations combine to imply that $\varphi(g)^*\omega(x) = \varepsilon(g) \cdot \omega(x)$ where $\varepsilon(g) = \pm 1$. The map $g \to \varepsilon(g)$ is a homomorphism, and therefore it follows that $W = \mathrm{Kernel}(\omega(x))$ is a $C(h)$-invariant subspace of V and likewise for the complementary 1-dimensional subspace $W^\perp$. The discussion of this paragraph shows that the action of h on $W^\perp$ is multiplication by either $+1$ or -1.

If the action h on $W^\perp$ is through multiplication by $+1$, then it follows that $W^\perp \subset V^{C(h)}$, which means that the linear subspaces W and $V^{C(h)}$ intersect transversely; in other words, the foliation is transverse to the fixed point set near x. On the other hand, if the action is through multiplication by -1, then W contains $V^{C(h)}$, which means that the latter is tangent to the leaf L_x of the foliation such that $x \in L_x$.

Finally, we claim that a small neighborhood of x in $V^{C(h)}$ is also contained in the leaf L_x. Suppose that y lies in some small open disk centered at x in the vector subspace $V^{C(h)}$, and let y_t be a linear curve joining x to y. Then $\varphi(h)^*\omega(y_t) = -\omega(y_t)$ for all y_t, which means that for each t the tangent vector at y_t is contained in the kernel of $\omega(y_t)$. Since ω is associated to a foliation, it follows that the entire curve lies in a single leaf, and this leaf must be L_x.

In particular, we have $y \in L_x$, and since y was arbitrary it follows that every point in $C(h)$ which is sufficiently close to x must lie in the leaf L_x. □

If we specialize to odd order groups, we obtain a much stronger conclusion.

Proposition 1.2. *Let M be a compact smooth manifold without boundary, let $\mathcal{F}$ be a codimension 1 foliation on M, let G be a group of odd order whichs act smoothly on M, and assume that the group action is compatible with $\mathcal{F}$.*

If H is a subgroup of G and C is a positive dimensional component of M^H, then $\mathcal{F}$ is transverse to M^H.

Proof. This will be a consequence of Proposition 1.1, so we shall adopt the setting in the proof of that result, starting with the choice of a G-invariant riemannian metric.

By local linearity we might as well assume that M is an open disk in some orthogonal representation V of H with $x = 0 \in V$ where $\dim V^H > 0$. As in the proof of Proposition 1.1, the foliation is locally defined by a smooth 1-form ω with $|omega| = 1$, and there is a homomorphism $\varepsilon : H \to \{\pm 1\}$ such that $h^*\omega(x) = \varepsilon(h)\cdot\omega(x)$ for all $h \in H$. Since H has odd order the homomorphism is trivial. Therefore, if W is the H-invariant subspace Kernel$(\omega(x))$ then V^H is transverse to W by the reasoning in the proof of Proposition 1.1, and hence by the argument proving that result we know that C is transverse to the foliation $\mathcal{F}$. □

In the introduction to this paper we gave an example of a smooth $\mathbb{Z}_2$-action on $S^1 \times F$ such that $\chi(F) \neq 0$ and a compatible codimension 1 foliation whose leaves are the slices $\{z\} \times F$ for $z \in S^1$. In this case each component of the fixed point set is homeomorphic to F, and in fact each component is a leaf of the foliation. This example shows that one cannot extend the conclusion of Proposition 1.2 to actions of even order groups and compatible codimension 1 foliations.

The preceding results yield half of the main theorem.

Theorem 1.3. *Let G be a group of odd order, suppose we are given a smooth action of G on a compact connected manifold M, and suppose further that $\mathcal{F}$ is a codimension 1 foliation on M which is compatible with the group action. Then for every subgroup $H \subset G$ and every component C of the H-fixed point set M^H, the Euler characteristic $\chi(C)$ is zero.*

Proof. Suppose first that C is positive dimensional. By Proposition 1.2 we know that the foliation is transverse to C, and as in ([25], p. 373) it follows that the original foliation induces a codimension 1 foliation on C. As noted earlier, if such a foliation exists then $\chi(C) = 0$.

Suppose now that C is zero dimensional and hence consists of a single point, and assume that M has a compatible codimension 1 foliation $\mathcal{F}$. Take the trivial action on S^2, and consider the foliation of $S^2 \times M$ whose leaves have the form $S^2 \times L$, where L is a leaf of $\mathcal{F}$. Then $S^2 \cong S^2 \times C$ is a component of $\left(S^2 \times M\right)^H$, so by the preceding paragraph its Euler characteristic is zero. Since $\chi(S^2) = 2$ this yields a contradiction. The source of the contradiction was our assumption that M had a compatible codimension 1 foliation and M^H had a zero dimensional component. Therefore at least one of these is false.

The first paragraph shows that all positive dimensional components of each set M^H have zero Euler characteristic, while the second shows that if a compatible codimension 1 foliation exists then there are nonzero dimensional components. Therefore all components C of all subsets M^H must be positive dimensional and satisfy $\chi(C) = 0$. □

2. Stratifications and Euler characteristics

The results in Section 1 show that the Euler characteristic conditions in the main theorem are necessary, and most of the material in the remainder of this section is devoted to showing that the Euler characteristic conditions are also sufficient.

If a finite group Gacts smoothly and freely on a compact connected manifold M, then the orbit space projetion $M \to M/G$ is a covering space projection, and we have already noted that the existence of a G-compatible foliation on M is equivalent to the existence of an ordinary foliation on M/G. More generally, by [3], Theorem II.5.8 (p. 88), the same conclusion holds if the action has exactly one orbit type, so that all isotropy subgroups are conjugate to a single subgroup H. In both cases, one crucial piece of input is the the observation that $\chi(M) = 0$ if and only if $\chi(M/G) = 0$. In particular, it follows that the reasoning in Section 1 yields a proof of the main result if the group action on M has a single orbit type.

Results in [7] prove that the Euler characteristic conditions are sufficient for a class of odd order group actions which are not free but have a relatively simple lattice of isotropy subgroups. Specifically, the assumption on isotropy subgroups is that they are all normal and linearly ordered by inclusion; one motivation for such a hypothesis is that if p is a prime then the isotropy subgroups of a $\mathbb{Z}_{p^r}$ action always satisfy the condition, for in this case the lattice of all subgroups is linearly ordered. One of the main tasks facing any attempt at generalization is to find a setting for handling actions whose lattices of isotropy subgroups can be more or less arbitrary.

In fact there are two related bookkeeping schemes for dealing with the combinatorial issues related to the lattice of isotropy subgroups, and we shall use both of them; one is implicit in the statement of the main result, and the other is needed for the constructing a foliation on increasingly large subsets of a smooth G-manifold which satisfies the hypotheses of the main result.

If G acts smoothly on a manifold M, then by local linearity we know that for each subgroup $H \subset G$ the fixed point set M^H is a union finite of connected, smoothly embedded submanifolds, possibly of varying dimensions. We shall let $\Pi(M)$ denote the family of all subsets $C \subset M$ such that C is a component of M^H for some subgroup H of G (as in Section I.10 of [39], this construction can be viewed in several equivalent ways). There is an associated decomposition of M into pairwise disjoint subsets which can be defined as follows: If $C \subset M^H$ and H is an isotropy subgroup of the action for some point of C, define the relative singular set $\mathrm{RelSing}\,(C)$ to be the intersection of C with $\cup_K M^K$, where K runs through all isotropy subgroups which properly contain H, and define the set of nonsingular points $\mathrm{Nonsing}\,(C)$ to be the difference $C - \mathrm{RelSing}\,(C)$. If $M_{(H)}$ denotes the set of all points whose isotropy subgroups are conjugate to H, then $\mathrm{Nonsing}\,(C) = M_{(H)} \cap C$, and $M_{(H)}$ is the union of the sets $\mathrm{Nonsing}\,(g \cdot C)$, where f runs through all the elements of G; standard considerations show that $g_1 \cdot C = g_2 \cdot C$ if and only if $g_1^{-1} g_2$ lies in the normalizer of H in G.

By definition the sets $M_{(H)}$ partition M into pairwise disjoint subsets which are smoothly embedded submanifolds by local linearity (possibly with varying dimensions), and if we fix H then the subsets $\mathrm{Nonsing}\,(g \cdot C)$ form a similar partition of $M_{(H)}$. The submanifolds $\mathrm{Nonsing}\,(g \cdot C)$ define smooth stratifications in the sense of Thom [36] and Mather [27] (see also [17]); one proof of this appears at the end of Chapter 2 in [14], and a more general approach is given in in Chapter 4 of [33].

The stratification of M in the previous paragraph also passes to a stratification of the orbit space M/G which will be useful for our purposes. In the setting of the preceding paragraph, the strata are given by the components of the subsets $M_{(H)}/G$; by [3], Theorem II.5.8, p. 88, the restricted orbit space projection $M_{(H)} \to M_{(H)}/G$ is a finite covering with $|G/H|$ sheets, and by [3], p. 89, for each isotropy subgroup H the strata in M/G given by orbits of type G/H all have the form $\mathrm{Nonsing}\,(C)/N(H)$, where C runs through the closed substrata of M which are components of M^H.

2.1. **Default hypothesis.** At this point it is convenient to impose the following restriction on the group actions under consideration, and unless explicitly stated otherwise we shall assume it holds for all group actions which arise:

Codimension ≥ 2 Gap Hypothesis. *If C is a component of M^H for some subgroup $H \subset G$ and* $\mathrm{Nonsing}\,(C)$ *is nonempty, then the relatively singular set* $\mathrm{RelSing}\,(C)$ *satisfies* $\dim \mathrm{RelSing}\,(C) \leq \dim C - 2$.

This assumption has two simple but far-reaching consequences. First, standard general position considerations imply that Nonsing (C) is connected. Second, it establishes the following strong connection between the set $\Pi(M)$ defined above and the stratification. Namely, if K is a subgroup of G and C is a component of M^K, then there is a subgroup H such that C is a also a component of M^H and C and $C_{(H)}$ is nonempty. In the terminology of [14] this means that C is a *closed substratum* of the group action. In the discussion which follows, when we discuss an element C of $\Pi(M)$ we shall view it as a component of M^H for such a subgroup H, and as in [14] we shall say that H is the *subprincipal isotropy subgroup* for C. Basic results in group theory imply that if C is a closed substratum with subprincipal isotropy subgroup H and $g \in G$, then the subprinicipal isotropy subgroup for the closed substratum $g \cdot C$ is the conjugate subgroup $g^{-1}Hg$.

Before proceeding, we shall verify that the Codimension ≥ 2 Gap Hypothesis always holds for smooth actions of odd order groups.

Proposition 2.1. *Let G be a group of odd order acting smoothly on the manifold M, and assume that C is a closed substratum of the group action with subprincipal isotropy subgroup H. Then* $\dim \mathrm{RelSing}\,(C) \leq \dim C - 2$.

Proof. We need to show that if L is a subgroup of G which properly contains H and $x \in M^L \cap C$ then the connected component D of M^L containing x is a proper subset of C and $\dim D \leq \dim C - 2$. Note that the containment $D \subset C$ is trivial because D is a connected subset of M^H (which contains M^L) and C is a component of M^H such that $x \in C \cap D$.

Let V denote the tangent space to x in M, and consider the local representation of L on V associated to the action of L on the equivariant tangent bundle of M. By local linearity the tangent spaces at C and D correspond to V^H and V^L respectively, and therefore we have $V^L \subset V^H$. Now D is a compact smooth submanifold of C, so if $V^H = V^L$ then $\dim D = \dim C$ and hence $C = D$ by Invariance of Domain. Therefore we must have $\dim V^H \neq \dim V^L$, which means that $\dim D < \dim C$ and hence by local linearity D is a proper subset of C.

In fact, by local linearity it will now suffice to verify that $\dim V^H - \dim V^L \geq 2$; local linearity implies that the L-representation V splits as a direct sum $V^L \oplus W_L$, where W_L has no nontrivial irreducible summands, and similarly the restriction of the representation to H splits as a direct sum $V^H \oplus W_H$, where W_H has no nontrivial irreducible summands.

We claim that both W_H and W_L are even dimensional. More generally, if Ω is an arbitrary nontrivial representation of an odd order group, then a fundamental result of representation theory states that Ω is given by restricting the scalars for some irreducible unitary representation, so that its real dimension is automatically even (compare [15], Exercise 3.38, p. 41); since W_H and W_L are both direct sums of nontrivial irreducible representations, it follows that their dimensions are also even. Therefore the difference

$$\dim V^H \; - \; \dim V^L \; = \; \dim W_L \; - \; \dim W_H$$

is even, and since this difference is positive it must be at least 2. □

2.2. **Euler characteristic identities.** The proof of the main result requires information about the Euler characteristics of the strata in M/G, where G acts smoothly on M and M is compact. Everything we need is contained in the following result:

Theorem 2.2. *Let G be a finite group acting smoothly on the compact connected manifold M, and assume this action satisfies the Codimension ≥ 2 Gap Hypothesis. Let $\Pi(M) = \{M_\alpha\}$ denote the set of closed substrata in M, and define the relative singular and nonsingular subsets of each M_α as above. If $\chi(M_\alpha) = 0$ for each closed substratum M_α, then we also have $\chi(\mathrm{RelSing}\,(M_\alpha)\,) = 0$ and $\chi(\mathrm{Nonsing}\,(M_\alpha)\,) = 0$ for all α. Furthermore, if V_α denotes the image of* Nonsing (M_α) *in M/G, then $\chi(V_\alpha) = 0$ for all α.*

Proof. For each substratum M_α let H_α denote its subprincipal isotropy subgroup. Order the substrata linearly so that $M_\alpha \leq M_\beta$ if the subprincipal isotropy subgroup H_β is contained in H_α. If M_α is the first substratum, then its relative singular set is empty and hence the theorem is true in this case for trivial reasons. We shall prove the general conclusion inductively, so assume the theorem is known to be true for all closed substrata $M_\gamma < M_\alpha$.

If we can prove that $\chi(\mathrm{RelSing}\,(M_\alpha)\,) = 0$ then by Lefschetz Duality (with mod 2 coefficients in general) we have

$$\chi(\mathrm{Nonsing}\,(M_\alpha)\,) \;=\; \pm \left(\, M_\alpha, \mathrm{RelSing}\,(M_\alpha)\,\right) \;=\;$$
$$\pm\; (\chi(M_\alpha) \;-\; \chi(\,\mathrm{Nonsing}\,(M_\alpha)\,)\,)$$

and since the two summands in the right hand expression are zero it follows that the left hand side is zero. Therefore proving $\chi(\mathrm{Nonsing}\,(M_\alpha)\,) = 0$ reduces to verifying that the Euler characteristic of RelSing (M_α) is zero.

By definition RelSing$(M_\alpha) = \cup_\gamma\, M_\gamma$, where the union runs over all closed substrata M_γ such that the subprincipal isotropy subgroup H_γ strictly contains M_α and $M_\gamma \subset M_\alpha$. If we list these closed substrata in order as $M_{\gamma(1)}, \; \cdots \;, M_{\gamma(k)}$, then we shall prove by induction that

$$\chi\left(M_{\gamma(1)} \cup \; \cdots \; \cup M_{\gamma(i)}\right) = 0$$

for $1 \leq i \leq k$. Since each summand has Euler characteristic zero, we can use the standard Mayer-Vietoris identity $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$, the distributive laws for unions and intersections, and induction to prove the Euler characteristic identity for finite unions if we can prove the Euler characteristic vanishes for each finite intersection

$$A \;=\; M_{\gamma(j_1)} \cap \;\cdots\; \cap M_{\gamma(j_q)}$$

of closed substrata.

Let K be the subgroup generated by the subprincipal isotropy subgroups $H_{j_1},\;\cdots\; H_{j_q}$, then K fixes every point of A, and in fact A is the disjoint union of all components in M^K which are contained in $M_{\gamma(j_1)},\;\cdots\;, M_{\gamma(j_q)}$. By Proposition I.2.3 in [14] each such component of M^K is a closed substratum M_δ for some δ with $H_\delta \supset K$. Since we know that $\chi(M_\delta) = 0$ and A is a disjoint union of some subsets of the form M_δ, it follows that $\chi(A) = 0$, and as indicated before this proves that the Euler characteristic of Nonsing (M_α) is zero.

To prove the final assertion in the theorem, recall that the restricted orbit space projection from Nonsing (M_α) to V_α is a finite covering, so that one vanishes if the other does. Since the first Euler characteristic is zero, it follows that $\chi(V_\alpha) = 0$. □

2.3. **Properties of the stratification.** Thus far we have used a limited amount of data from the stratification of a smooth G-manifold, but in Section 3 we shall need more information about the ways in which the various strata fit together in a smooth G-manifold, so in the remainder of this section we shall summarize the basic setting.

By local linearity, for each closed substratum M_α the subset Nonsing (M_α) and its image V_α in M/G are smooth manifolds. Furthermore, since smooth G-manifolds have equivariant triangulations (compare [22]), some standard piecewise linear topology and the Cairns-Hirsch Theorem (see [21], Section I.7) show that both V_α and its finite covering Nonsing (M_α) are canonically diffeomorphic to the interiors of compact bounded smooth manifolds B_α and C_α. One can view the data associated to a stratification as the raw material for constructing attaching maps

$$\partial B_\alpha \;\longrightarrow\; \mathrm{RelSing}\,(M_\alpha)/G\;, \qquad \partial C_\alpha \;\longrightarrow\; \mathrm{RelSing}\,(M_\alpha)\;.$$

The stratification data imply that these attaching maps have some special properties, and we shall need some of the latter in Section 3. For our purposes the description of stratification data related to [10] is particularly useful. One central concept in [10] is the notion of a *normal system* associated to the relative singular set; an equivalent construction in [14] is called a vector bundle system. These are compatible families of closed tubular neighborhoods for the closed substrata M_β which are contained in RelSing(M_α), and the union of T_α of these closed tubular neighborhoods is a closed neighborhood of $\mathrm{RelSing}(M_\alpha)$ such that the inclusions $\mathrm{RelSing}(M_\alpha)$ in T_α and its interior are homotopy equivalences.

As noted in Chapter IV of [10], the closed complement $P_\alpha = \overline{M_\alpha - T_\alpha}$ is a compact smooth manifold with corners in the sense of [8] and the appendix to [2] (see [11], [12] and [13] for additional details).

Given a smooth submanifold B of a smooth manifold A and an open neighborhood U of B in A, it is always possible to find a tubular neighborhood of B which is contained in U, and if M is a smooth G-manifold (where G is finite) then for each closed substratum M_α one has a similar result for tubular neighborhoods of $\mathrm{RelSing}(M_\alpha)$ in M_α.

Proposition 2.3. *In the setting described above, assume that M is compact and let U be an open neighborhood of* $\mathrm{RelSing}(M_\alpha)$ *in M_α. Then one can construct the splitting $M_\alpha = T_\alpha \cup P_\alpha$ so that $T_\alpha \subset U$.*

This follows because T_α is a union of closed tubular neighborhoods given by disks of radius ε for some small, fixed $\varepsilon > 0$; if ε is sufficiently small, then by compactness all of these closed tubular neighborhoods will be contained in U.

The functional stratification data of Section 8 in [27] (see (A6)–(A9) in particular) have simple interpretations in terms of the normal vector bundle system over $\mathrm{RelSing}(M_\alpha)$; namely, if the closed substratum M_β is properly contained in M_α then the retraction $\pi_{\beta\alpha}$ is given by the bundle projection in an equivariant tubular neighborhood of M_β in M_α, and the tubular function $\rho_{\beta\alpha}$ corresponds to the length of a normal vector with respect to a suitable G-invariant riemannian metric on M (since T_α is a union of disk bundles, this concept of normal vector length is defined in a natural way). These functions have a few properties which are stronger than the general conditions for stratification data; for example, the function $\pi_{\beta\alpha}$ is a smooth submersion on an entire tubular neighborhood W of $\mathrm{Nonsing}(M_\beta)$ in M_α and not just on $W - \mathrm{Nonsing}(M_\beta)$, and the function $\rho^2_{\beta\alpha}$ is also smooth on all of W and not just on $W - \mathrm{Nonsing}(M_\beta)$.

Since P_α is a manifold with corners, its boundary has a decomposition into faces $\partial_s P_\alpha$ as defined in Section IV.2 of [10]. In the setting of [10], each face of ∂P_α corresponds to an orthogonal sphere bundle over a piece of some submanifold $\mathrm{Nonsing}(M_\beta)$. The corners arise as follows: Over the closed substrata in M_α the normal bundles are presented as orthogonal direct sums $\alpha_1 \oplus \cdots \oplus \alpha_k$, and the intersection of P_α with the fiber over a point corresponds to all k-tuples of vectors $w = (v_1, \cdots, v_k)$ such that $|v_j| \geq \varepsilon$ for all j and for some suitably small value of ε; if we are given a k-tuple of vectors w such that $|v_j| \geq \varepsilon$ for all j,it follows that w has a neighborhood in P_α which is smoothly equivalent to an open neighborhood of the origin in $[0,\infty)^k \times \mathbb{R}^{m-k}$, where $m = \dim M_\alpha$.

Finally, if we factor out the action of the group in the preceding description of the stratification, then we obtain corresponding interpretations for the stratification data associated to the orbit space M/G.

3. Controlled vector fields and extensions of foliations

In Thurston's result on the existence of codimension 1 foliations [37], the underlying idea is to begin with a nowhere zero vector field and to construct an approximation which generates the bundle of normals to the leaves for some codimension 1 foliation. In order to apply Thurston's methods to manifolds with group actions, we need to construct a nowhere zero vector field X on a smooth G-manifold M which is G-invariant (*i.e.*, $g_*X = X$ for all $g \in G$) and is well-behaved with respect to the stratification on M. More precisely, we need to find a G-invariant smooth vector field with satisfies slight strengthenings of the control conditions in [36], Section 1.F, p. 256, or [27], Section 9. The main conditions on such a vector field X are (i) if a point p lies in the closed substratum M_α $X(p)$ is tangent to M_α, (ii) if the closed substratum M_B is properly contained in M_α and $\varphi : E \to M_\beta$ is an open G-invariant tubular neighborhood of M_β in M_α, then the restriction $X|E$ can be viewed as a pullback of $X|M_\beta$ with respect to the smooth submersion φ (in fact, if we dualize to 1-forms via some invariant riemannian metric, then the associated 1-form on E is the pullback of the 1-form on M_β). The main difference between our control conditions and those of [36] and [27] involves the fact that the submersions into the proper substrata M_β are defined on the entire neighborhood E and not just on the complement of the zero section.

3.1. **Controlled foliations.**

We shall construct foliations which satisfy analogs of the defining conditions for a controlled foliation.

Definition. Suppose that we are given a smooth G-manifold with associated stratification data, and let (F) be a compatible codimension 1 foliation on M which is transversely oriented (*i.e.*,, defined by a G-invariant 1-form ω) and transverse to every closed substratum. Assume further that we are given some fixed but unspecified smooth G-invariant riemannian metric on M, and let X be the the G-invariant vector field which is dual to ω. Then (F) is said to be *controlled* with respect to the stratification data if X is controlled with respect to the stratification data for M. More generally, if $\Sigma \subset M$ is a G-invariant union of closed substrata (*i.e.*, $M_\alpha \subset \Sigma$ implies $g \cdot M_\alpha \subset \Sigma$ for all $g \in G$), then we can define a compatible controlled codimension 1 foliation on Σ to be a family of foliations $\mathcal{F}_\alpha$ which are G-compatible on Σ and controlled on the closed substrata in Σ.

3.2. **Extending controlled codimension 1 foliations.** As in [7] we shall construct compatible foliations inductively, assuming that we have what we want on the relative singular set of a closed substratum and extending everything to the entire closed substratum. More accurately, we shall work instead with the orbits of closed substrata under the action of G which was described in the preceding section, with an inductive hypothesis that the foliation is already given on $G \cdot \mathrm{RelSing}(M_\alpha)$ for some closed substratum M_α, and the goal is to find an extension to all of $G \cdot M_\alpha$.

The first step is to extend the foliation to a small neighborhood of $\mathrm{RelSing}(M_\alpha)$ in M_α. This extension should be equivariant with respect to the $N(H_\alpha)/H_\alpha$ action on M_α induced by the action of G on M, and if this is the case then one can extend it to the strata $g \cdot M_\alpha$ — where g runs through a sequence of representatives for the cosets in $G/N(H_\alpha)$ — by equivariance.

Proposition 3.1. *Let G be a finite group, let M be a compact connected smooth G-manifold which satisfies the Codimension ≥ 2 Gap Hypothesis, let let M_α be a closed substratum of the group action. Suppose that $\mathcal{F}$ is a compatible codimension 1 foliation which is defined on* $\mathrm{RelSing}(M_\alpha)$*, transverse to each closed substratum $M_\beta \subset \mathrm{RelSing}(M_\alpha)$ and controlled with respect to the stratification data. If T_α is the tubular neighborhood of* $\mathrm{RelSing}(M_\alpha)$ *given by* [10]*, then $\mathcal{F}$ extends to a compatible, controlled codimension 1 foliation $\mathcal{F}'$ on a neighborhood of T_α in M_α such that $\mathcal{F}'$ is transverse to ∂T_α and $\mathcal{F}'$ is also transverse to each closed substratum $M_\beta \subset \mathrm{RelSing}(M_\alpha)$.*

In this proposition, compatibility involves the associated action of the subquotient $H(H_\alpha)/H_\alpha$ on M_α, where H_α is the subprincipal isotropy subgroup of the closed substratum M_α.

Proof. For each closed substratum $M_\beta \subset \mathrm{RelSing}(M_\alpha)$ we can extend $\mathcal{F}$ to an open tubular neighborhood N_β of M_β in M_α using the bundle projection $N \to M_\beta$, which is a smooth submersion. Furthermore, the control hypothesis on $\mathcal{F}$ and the compatibility properties for the bundle projections (compare [10], property (iv), p. 352, and [27], axiom (A9), p. 492) imply that these pulled back foliations agree on the intersections of two suitably chosen tubular neighborhoods, and therefore one obtains a well-defined compatible codimension 1 foliation on the union N of these neighborhoods. By construction, this extended foliation is also transverse to each closed substratum $M_\beta \subset \mathrm{RelSing}(M_\alpha)$. Furthermore, we claim that the normal vector field X to the leaves satisfies the conditions for a controlled vector field ([27], (9.1) and (9.2), p. 493) on N. Condition (9.1) will follow from the local triviality of the orthogonal disk bundles, for by construction the integral curves of the pulled back vector field are contained in spheres of fixed radii in the disk bundles, and therefore the Lie derivatives of the tubular functions with respect to X must vanish. Condition (9.2) is satisfied because the normal vectpor field X is formed by pulling back the normal bundle to the leaves on the closed substrata $M_\beta \subset N$.

It remains to verify that if T_α is a closed tubular neighborhood of $\mathrm{RelSing}(M_\alpha)$ in N, then the extended foliation $\mathcal{F}'$ is also transverse to ∂T_α; recall that the latter is also the boundary of the manifold with corners P_α. Since the corner and collar structures on a neighborhood of ∂P_α are determined by the tubular functions, the foliation will be transverse to all the faces in ∂P_α if X is orthogonal to $\nabla\rho$ for each of the tubular functions defined on open subsets of $\mathrm{Nonsing}(M_\alpha)$. The orthogonality conditions are a consequence of the following sequence of equations

$$\langle X, \nabla\rho\rangle \;=\; d\rho(X) \;=\; X\,\rho \;=\; 0$$

in which the first equation is the definition of a gradient vector field in a riemannian manifold, the second is the definition of the 1-form $d\rho$ via its values on vector fields, and the third equation is merely the previously cited (9.1) from [27]. These observations show that the extended foliation is transverse to each face of $\partial P_\alpha = \partial T_\alpha$. □

We shall need a slightly stronger version of the preceding result:

Corollary 3.2. *Let G be a finite group, let M be a compact connected smooth G-manifold which satisfies the Codimension ≥ 2 Gap Hypothesis, let let M_α be a closed substratum of the group action. Suppose that $\mathcal{F}$ is a G-compatible codimension 1 foliation which is defined on $G \cdot \mathrm{RelSing}(M_\alpha)$, transverse to each closed substratum $g \cdot M_\beta \subset g \cdot \mathrm{RelSing}(M_\alpha)$ and controlled with respect to the stratification data. If T_α is the G-invariant tubular neighborhood of $G \cdot \mathrm{RelSing}(M_\alpha)$ given by* [10]*, then $\mathcal{F}$ extends to a compatible, controlled codimension 1 foliation $\mathcal{F}^*$ on a neighborhood of $G \cdot T_\alpha$ in $G \cdot M_\alpha$ such that $\mathcal{F}^*$ is transverse to $G \cdot \partial T_\alpha$ and $\mathcal{F}^*$ is also transverse to each closed substratum $g \cdot M_\beta \subset G \cdot \mathrm{RelSing}(M_\alpha)$.*

Proof. We shall work in the setting for the proof of 3.1, in which V is a neighborhood of $\mathrm{RelSing}(M_\alpha)$ in M_α on which an extension of (F) has been defined. The action of G on $G \cdot M_\alpha$ is equivalent to the action on the balanced product $G \times_{N(H_\alpha)} M_\alpha$, and since the extended foliation $\mathcal{F}'$ from the proposition is $N(H_\alpha)$-compatible, the balanced product extension $\mathcal{F}^* = G \times_{N(H_\alpha)} \mathcal{F}'$ defines a further extension to a neighborhood $G \times_{N(H_\alpha)} N$ of $G \times_{N(H_\alpha)} \mathrm{RelSing}(M_\alpha)$ in $G \times_{N(H_\alpha)} M_\alpha$. □

3.3. **Rounding corners.** The next major step in constructing a compatible foliation on the G-manifold M is to extend everything from the invariant neighborhood $G \cdot N$ of $G \cdot \mathrm{RelSing}(M_\alpha)$ to all of $G \cdot M_\alpha$. In terms of the splitting

$M_\alpha = T_\alpha \cup P_\alpha$, this step is equivalent to constructing a suitable extension of the foliation from an invariant neighborhood of $G \cdot \partial P_\alpha$ to all of P_α. Since the induced action of G on $G \cdot P_\alpha$ has a single orbit type (namely, $G?H_\alpha$), the foliation on a neighborhood of $G \cdot \partial P_\alpha$, the compatible foliation on the latter is equivalent to an ordinary foliation on the corresponding neighborhood of $\partial P_\alpha / N(H_\alpha) \cong G \cdot \partial P_\alpha / G$ in the orbit space $Q_\alpha = P_\alpha / N(H_\alpha) \cong G \cdot P_\alpha / G$, and therefore the extension problem reduces to finding an extension of an ordinary foliation on a neighborhood of ∂Q_α to a foliation on all of Q_α. The results of [37] provide criteria for finding such extensions of foliations if we are working with a smooth manifold W whose boundary is also smooth. It seems clear that such results should also be valid for smooth manifolds with (boundary) corners such as Q_α, but since the details do not seem to have been verified we shall explain how one can derive a version of this principle which suffices for our purposes. The underlying idea is to approximate the original manifold with (boundary) corners by a smooth manifold with a smooth boundary, and such an approximation process is generally described as *rounding* (or *straightening*) *the corners* in the boundary.

There are several approaches to rounding corners in the published and unpublished literature; we shall use the setting in the Appendix to [2] and [11]–[13]. In this approach, the crucial idea is to show that if W is a smooth manifold with boundary corners, then one can find an open collar neighborhood U of ∂W in W such that $U - \partial W$ is diffeomorphic to $L \times (0,1)$ for some smooth manifold L and L is an approximation to ∂Q in an appropriate sense (*e.g.*, L is homeomorphic to ∂Q and the homeomorphism is a diffeomorphism on an open dense subset). We shall sometimes refer to L as a *smoothed out boundary*. The diffeomorphism and smoothed out boundary are constructed by means of a real valued function on a collar neighborhood known as a *carpeting function* (called a *fonction tapissante* in the Appendix to [2]), which is constructed from real valued functions on U that carry the smooth collar and corner neighborhood data for ∂W.

For the example Q_α arising in question about extending foliations, we also want the smoothed out boundary L_α to be transverse to the foliation which was constructed on a neighborhood of ∂Q_α. If X is the vector field defined on a neighborhood of ∂P_α which is transverse to the leaves of the compatible foliation and Y is the quotient vector field on a neighborhood of the orbit space ∂Q_α, then Y will be transverse to L_α if Y and the gradient of the carpeting function are perpendicular at every point of L_α.

We have already noted that the carpeting functions in the Appendix to [2] are given by constructions which start with the real valued functions carrying the collar and corner neighborhood data (*e.g.*, see Section 5in the Appendix to [2], especially page 488). For our example Q_α, we have noted that these functions are given by the tubular functions in the stratification data, and therefore one can view the carpeting function τ as as an expression involving the tubular functions as the coordinate variables. Since the original normal vector field X is perpendicular to the gradients of the tubular functions, it follows from the Chain Rule that the corresponding vector field Y on the orbit space is perpendicular to the gradient $\nabla\tau$, which in turn implies that Y is perpendicular to the smoothed out boundary L_α. In fact, the normal vector field Y is tangent to L_α, and it follows that the original foliation defines a transverse intersection foliation on L_α which extends to a neighborhood of L_α.

To summarize, we have split Q_α into a union of two pieces $K_\alpha \cup C$ such that $L_\alpha = K_\alpha \cap C$, $\partial K_\alpha = L_\alpha$, C is a closed (topological) collar neighborhood of ∂Q_α and K_α is a smooth manifold with a smooth boundary. Furthermore, we can choose everything so that the foliation is defined on a neighborhood of ∂Q_α containing C and it is transverse to L_α.

3.4. **Extension to a closed substratum.** Our objective has been to take the partial foliation defined on $G \cdot \mathrm{RelSing}(M_\alpha)$ and extend it to $G \cdot M_\alpha$. If we can find an extension of an ordinary foliation on L_α to an ordinary foliation on K_α, then we can lift this extension to the covering space $G \cdot P_\alpha$ over K_α, and this lifted extension and the previously defined extension to $G \cdot T_\alpha$ will yield an extension of the original controlled foliation on $G \cdot \mathrm{RelSing}(M_\alpha)$ to all of $G \cdot M_\alpha$. We shall assume that $\chi(M_\beta) = 0$ for all closed substrata which are contained in M_α (this hypothesis includes M_α itself).

By [37] the codimension 1 foliation on a neighborhood of L_α extends to a codimension 1 foliation on K_α if Y extends to a nowhere zero vector field on K_α, and this happens if there is a vector field Y' on K_α which extends Y and has index equal to zero. Since Y is tangent to $L_\alpha = \partial K_\alpha$ and nowhere zero on the boundary, the results of [29] imply that there is a nowhere zero vector field Y' extending Y if the Euler characteristic $\chi(K_\alpha)$ is zero, or equivalently if the Euler characteristic of the homeomorphic manifold Q_α is zero. Since P_α is a finite covering of Q_α, it suffices to check that $\chi(P_\alpha) = 0$. We can prove this using the methods of the preceding section as follows: By duality we know that $\chi(P_\alpha) = \pm\,\chi(M_\alpha, T_\alpha)$, and since $\mathrm{RelSing}(M_\alpha)$ is a deformation retract of T_α we know that $\chi(M_\alpha, T_\alpha) = \chi(M_\alpha, \mathrm{RelSing}(M_\alpha))$. Since we are assuming $\chi(M_\alpha) = 0$ and we have $\chi(\mathrm{RelSing}(M_\alpha)) = 0$ by Theorem 2.2, it follows that $0 = \chi(M_\alpha, \mathrm{RelSing}(M_\alpha)) = \chi(K_\alpha)$, and therefore we have an extension of Y to a nowhere zero vector field Y' on K_α. Using [37] we can use Y' to construct an extension of the foliation on ∂L_α; as noted above we can now lift this to $G \cdot \mathrm{Nonsing}(M_\alpha)$ and combine this lifting with the foliation on T_α to obtain a controlled compatible foliation on all of $G \cdot M_\alpha$.

3.5. **Sufficiency of the Euler characteristic conditions.** With everything we have done thus far, the proof of the following result becomes straightforward.

Theorem 3.3. *Let G be a finite group, let M be a compact connected smooth G-manifold which satisfies the Codimension ≥ 2 Gap Hypothesis, and suppose that for each closed substratum M_α the Euler characteristic $\chi(M_\alpha)$ is equal to zero. Then there is a controlled compatible codimension 1 foliation $\mathcal{F}$ on M such that $\mathcal{F}$ is transverse to each closed substratum and the bundle $\nu(\mathcal{F})$ of normals to the leaves is trivial.*

Proof. Let $\mathcal{S}$ be the set of all G-orbits $\mathbf{O}_\alpha = \{g \cdot M_\alpha \mid g \in G\}$, where M_α is a closed substratum of the group action. Order $\mathcal{S}$ linearly such that if $M_\beta \subset M_\alpha$ then the orbit $(O)_\beta$ precedes $\mathbf{O}_\alpha$. We shall inductively construct a foliation over the substrata in each of the orbits $\mathbf{O}_\alpha$.

We need to begin with the first orbit $\mathbf{O}_1$. On these substrata the group must act with exactly one orbit type, and in this case a compatible codimension 1 foliation with the desired properties is merely a codimension 1 foliation on the orbit manifold. By previous remarks, such a foliation exists if and only if the Euler characteristic of a closed substratum in $\mathbf{O}_1$ is zero; since this is true by hypothesis, we have a compatible foliation on the union of all closed substrata in $\mathbf{O}_1$, and this foliation has all the required properties.

Assume by induction that we have constructed the foliation over all the strata in all the orbits $\mathbf{O}_\beta$ which precede $\mathbf{O}_\alpha$. The discussion preceding the statement of the theorem implies that one can extend the foliation to all the closed substrata in $\mathbf{O}_\alpha$ provided $\chi(M_\alpha) = 0$, and since this is a key assumption in the theorem, it follows that one can always construct such an extension.

We may continue this inductive process of extending the foliation over each union of closed substrata until we finally obtain an extension over the last orbit $\mathbf{O}_{final}$, which consists of M alone. Of course, the foliation constructed on M in this fashion will have all the required properties. □

FINAL REMARK. The results of [24] provide additional information on the index of a controlled vector field on a smoothly stratified set; in particular, if the Euler characteristics of all the strata vanish then one can construct a nowhere zero controlled vector field. However, this conclusion is not quite strong enough for our purposes because the construction of the foliation uses the fact that the stratification data for a finite group action satisfy strengthened versions of the defining conditions for a Thom-Mather stratification.

4. The noncompact case

For noncompact manifolds, it is much easier to construct codimension 1 foliations because of results due to A.Phillips (see [31] and [32]), A. Haefliger [19], and M. Gromov [18] (in fact, the ideas in the latter apply to a wide range of other flexible geometrical structures). In particular, if we combine our results with an equivariant version of Gromov's work due to E. Bierstone [1], we can prove the following result:

Theorem 4.1. *Let G be a group of odd order, and suppose we are given a smooth action of G on a* noncompact *connected manifold M. Then there is a codimension 1 foliation on M which is compatible with the group action if and only if for every subgroup H and every* compact *component C of the H-fixed point set M^H, the Euler characteristic $\chi(C)$ is zero.*

The idea of the proof is straightforward. We can use the methods of this paper to construct the foliation on all of the compact closed substrata, and then we can construct an extension to the rest of the G-manifold by combining a relative version of Bierstone's equivariant Gromov theory with the existence of nowhere zero vector fields on noncompact manifolds; recall that the latter follows because a connected noncompact n-manifold has the homotopy type of an $(n-1)$-dimensional complex (*e.g.*, see [30], Lemma 1.1, p. 176).

References

[1] E. Bierstone, Equivariant Gromov theory Topology **13** (1974), 327–345.

[2] A. Borel and J.-P. Serre. Corners and arithmetic groups. With an appendix: Arrondissement des variétés à coins, by A. Douady and L. Hérault. Comment. Math. Helv. **48** (1973), 436–491.

[3] G. E. Bredon, Introduction to compact transformation groups. Pure and Applied Mathematics, Vol. 46. Academic Press, New York-London, 1972.

[4] Browder, William; Quinn, Frank. A surgery theory for G-manifolds and stratified sets. ManifoldsTokyo 1973 (Proc. Internat. Conf., Tokyo, 1973), pp. 27–36. Univ. Tokyo Press, Tokyo, 1975.

[5] A. Candel and L. Conlon, Foliations. I. Graduate Studies in Mathematics, 23. American Mathematical Society, Providence, RI, 2000.

[6] A. Candel and L. Conlon, Foliations. II. Graduate Studies in Mathematics Vol. 60. American Mathematical Society, Providence, RI, 2003.

[7] C. A. Carlson, Foliations, Contact Structures and Finite Group Actions. Ph.D. Thesis, University of California, Riverside, 2012. Available from ProQuest LLC, Ann Arbor, MI, 2012, Order number 3518640. Also available online at http://escholarship.org/uc/item/98758436.

[8] J. Cerf, Topologie de certains espaces de plongements. Bull. Soc. Math. France **89** (1961), 227–380.

[9] S. R. Costenoble and S. Waner, Equivariant vector fields and self-maps of spheres. J. Pure Appl. Algebra **187** (2004), 87–97.

[10] M. Davis, Smooth G-manifolds as collections of fiber bundles. Pacific J. Math. **77** (1978), 315–363.

[11] A. Douady, Variétés à bord anguleux et voisinages tubulaires. 1961/62 *Séminaire Henri Cartan 1961/62, Exp.* 1, 11 *pp. Secrétariat mathématique, Paris*, 1962. Available online:

http://www.numdam.org/numdam-bin/feuilleter?id=SHC_1961-1962__14_

[12] A. Douady, Théorèmes d'isotopie et de recollement. 1961/62 *Séminaire Henri Cartan 1961/62, Exp.* 2 *, 16 pp. Secrétariat mathématique, Paris*, 1962. Available online: http://www.numdam.org/numdam-bin/feuilleter?id=SHC_1961-1962__14_

[13] A. Douady, Arrondissement des arêtes. 1961/62 *Séminaire Henri Cartan 1961/62, Exp.* 3 *, 25 pp. Secrétariat mathématique, Paris*, 1962. Available online: http://www.numdam.org/numdam-bin/feuilleter?id=SHC_1961-1962__14_

[14] K. H. Dovermann and R. Schultz, Equivariant surgery theories and their periodicity properties. Lecture Notes in Mathematics Vol. 1443. Springer-Verlag, Berlin, 1990.

[15] W. Fulton and J. Harris, Representation theory. A first course. Graduate Texts in Mathematics Vol. 129. Readings in Mathematics. Springer-Verlag, New York, 1991.

[16] D. H. Gottlieb, A de Moivre like formula for fixed point theory. Fixed point theory and its applications (Fixed Point Theory Sub-Conference Proceedings, International Congress of Mathematicians, Berkeley, CA, 1986), 99–105, Contemp. Math. **72**, Amer. Math. Soc., Providence, RI, 1988.

[17] M. Goresky, Introduction to the papers of R. Thom and J. Mather. Bull. Amer. Math. Soc. (2) **49** (2012), 469–474.

[18] M. L. Gromov, Stable mappings of foliations into manifolds, Izv. Akad. Nauk. SSSR Ser. Mat. **33** (1969), 707–734 = Math. USSR Izv. **3** (1969), 671–694.

[19] A Haefliger, Feuilletages sur les variétés ouvertes, Topology **9** (1970), 183-194.

[20] M. W. Hirsch, Differential topology. Corrected reprint of the 1976 original. Graduate Texts in Mathematics Vol. 33. Springer-Verlag, New York, 1994.

[21] M. W. Hirsch and B. Mazur, Smoothings of piecewise linear manifolds. Annals of Mathematics Studies No. 80. Princeton University Press, Princeton, NJ, 1974.

[22] S. Illman, Smooth equivariant triangulations of G-manifolds for G a finite group. Math. Ann. **233** (1978), 199–220.

[23] B. Jubin. A generalized Poincaré-Hopf index theorem. ArXiv e-print, available electronically at http://arxiv.org/pdf/0903.0697.pdf.

[24] H. King and D. Trotman. Poincaré-Hopf theorems on singular spaces. Proc. Lond. Math. Soc. (3) **108** (2014), 682–703.

[25] H. B. Lawson, Foliations. Bull. Amer. Math. Soc. **80** (1974), 369–418.

[26] W. Lück and J. Rosenberg, Equivariant Euler characteristics and $K-$homology Euler classes for proper cocompact $G-$manifolds. Geometry and Topology **7** (2003), 569-613.

[27] J. N. Mather, Notes on topological stability. Bull. Amer. Math. Soc. (2) **49** (2012), 475–506.

[28] J. W. Milnor, Topology from the differentiable viewpoint. Based on notes by David W. Weaver. Revised reprint of the 1965 original. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997.

[29] M. Morse, Singular points of vector fields under general boundary conditions, American Journal of Mathematics **51** (1929), 165–178.

[30] A. Phillips, Submersions of open manifolds. Topology **6** (1967), 171–206.

[31] A. Phillips, Foliations on open manifolds. I. Comment. Math. Helv. **43** (1968), 204–211.

[32] A. Phillips, Foliations on open manifolds. II. Comment. Math. Helv. **44** (1969), 367–370.

[33] M. J. Pflaum, Analytic and geometric study of stratified spaces. Lecture Notes in Mathematics Vol. 1768. Springer-Verlag, Berlin, 2001.

[34] R. Schultz, Exotic spheres admitting circle actions with codimension four stationary sets. Proceedings of the Northwestern Homotopy Theory Conference (Evanston, IL, 1982), 339–368, Contemp. Math. **19**, Amer. Math. Soc., Providence, RI, 1983.

[35] R. Schultz, Connected sums and Euler characteristic invariants for G-manifolds, e-print, University of California Riverside, 2014 — available at http://math.ucr.edu/∼res/miscpapers/csum+echar.pdf.
[36] R. Thom, Ensembles et morphismes stratifiés, Bull. Amer. Math. Soc. **75** (1969), 240-284.
[37] W. P. Thurston, Existence of codimension-one foliations. Ann. of Math. **104** (1976), 249–268.
[38] T. tom Dieck, Transformation groups and representation theory. Lecture Notes in Mathematics, 766. Springer, Berlin, 1979.
[39] T. tom Dieck, Transformation Groups. de Gruyter Studies in Mathematics Vol. 8. W. de Gruyter, Berlin, 1987
[40] P. Tondeur, Geometry of foliations. Monographs in Mathematics, Vol. 90. Birkhäuser Verlag, Basel, 1997.

Christopher Carlson, Department of Mathematics, Centralia College, Centralia, WA 98531-4099

E-mail address: ccarlson@centralia.edu

Reinhard Schultz, Department of Mathematics, University of California Riverside, 900 Big Springs Drive, Riverside, CA 92521

E-mail address: schultz@math.ucr.edu