Compatible
Foliations ~~for~~

Odd Order Group Actions

Rough Sketch

(One should probably
modify this to include
the noncompact case)

Theorem. G = finite group of odd order, M is a closed smooth G-manifold. Then (G, M) has a ~~strongly~~ compatible codimension 1 foliation $\Longleftrightarrow$ for each subgroup $H \subseteq G$ and each component C of M^H we have $\chi(C) = 0$.

What if $C = pt$? No 0 dim strata

Note| We know that a compatible codimension 1 foliation's leaves are transverse to each C. Hence each C inherits a codim 1 foliation and this implies $\chi(C) = 0$ (even if $\nu_{\mathcal{F}}^{\pm}$ is trivial). Thus the real work involves the converse.

see 1A

PROOF OF THE CONVERSE. Consider the standard stratification of M/G whose closed strata S_α^* are the orbit spaces of fixed point sets of the various isotropy subgroups G_α's ($S_\alpha \subseteq M$ = inverse image of S_α^*). Let $\Sigma_\alpha^* \subseteq S_\alpha^*$ be all orbits with isotropy subgroups properly containing G_α.

CLAIM $\chi(S_\alpha^* - \Sigma_\alpha^*) = 0$.

Note 2 Examples show this is stronger than assuming $\chi(M^H) = 0$ for all subgps H.

<u>VERIFICATION</u> We are given that $\chi(S_\alpha)=0$ all S_α. There are only finitely many strata and they are partially ordered by inclusion. Extend this to a linear ordering of the strata; If we have a minimal stratum S_ω^* then $\Sigma_\omega^* = \phi$ and $S_\omega \to S_\omega^*$ is a finite covering, so that $\chi(S_\omega)=0 \Rightarrow \chi(S_\omega^*)=0$. <u>Proceed by induction on the strata</u>. So we assume the result for all strata preceding S_α^*.

We shall use the triangulation results of A. Verona, which imply that Σ_α^* is a union of lower closed strata S_β^* s.t. any two intersect in a finite union of substrata, with everything in sight triangulable, i.e. $S_{\beta_1}^* \cap S_{\beta_2}^* \cong \bigcup S_{\gamma_i}$.

By induct ion and the additivity of χ, each union of lower strata $\bigcup S_{\gamma_i}^*$ has $\chi(\cup S_{\gamma_i}^*)=0$. Likewise,

$\chi(\cup S_{\gamma_i}) = 0$. These imply $\chi(\Sigma_\alpha^*) = \chi(\Sigma_\alpha) = 0$ where $\Sigma_\alpha \subseteq S_\alpha$ = inverse image of Σ_α^*.

We know (from the theory of stratified sets) that $S_\alpha^* - \Sigma_\alpha^*$ is diffeomorphic to the interior of a manifold with boundary W_α^* such that there is a map of pairs $f: (W_\alpha^*, \partial W_\alpha^*) \longrightarrow (S_\alpha^*, \Sigma_\alpha^*)$ where f maps $W_\alpha^* - \partial W_\alpha^*$ to an open subset of $S_\alpha^* - \Sigma_\alpha^*$ by a homotopy equivalence. In fact, $S_\alpha^* - \Sigma_\alpha^* \cong W_\alpha^* \cup \partial W_\alpha^* \times [0,1)$.

~~This yields~~ $\cancel{H_*(W_\alpha^*, \partial W_\alpha^*) \xrightarrow[\cong]{f_*} H_*(S_\alpha^*, \Sigma_\alpha^*)}$ ~~etc.~~
~~By duality the left side is isomorphic to~~ $\cancel{H^{*SHIFT}(W_\alpha^* - \partial W_\alpha^*)}$
$\cancel{\cong H^{*SHIFT}(S}$

Hence $H_*(S_\alpha^* - \Sigma_\alpha^*) \cong H_*(W_\alpha^*)$; furthermore, by the basic properties of stratified objects we have know that $\overbrace{\partial W_\alpha^* \times [0,1) \cup \Sigma_\alpha'^*}^{N_\alpha^*} \supseteq \Sigma_\alpha^*$ is a homotopy equivalence, and ~~therefore~~ in fact we also have

$$H_*(S_\alpha^*, \Sigma_\alpha^*) \cong H_*(S_\alpha^*, N_\alpha^*) \cong H_*(W_\alpha^*, \partial W_\alpha^*).$$

$$H_*(S_\alpha, \Sigma_\alpha) \cong H_*(S_\alpha, N_\alpha) \cong H_*(W_\alpha, \partial W_\alpha)$$

where Σ_α, N_α = inverse images inside M etc.

Now $\chi(S_\alpha) = 0 = \chi(\Sigma_\alpha) \Longrightarrow \chi(S_\alpha, \Sigma_\alpha) = 0$
$\Longrightarrow \chi(W_\alpha, \partial W_\alpha) = 0$. By duality, $\chi(W_\alpha - \partial W_\alpha) = 0$

and since $W_\alpha - \partial W_\alpha \longrightarrow W_\alpha^* - \partial W_\alpha^*$ is a covering space projection (only one orbit type) we must have $\chi(W_\alpha^* - \partial W_\alpha^*) = 0$. Finally, $H_*(S_\alpha^* - \Sigma_\alpha^*) \cong H_*(W_\alpha^* - \partial W_\alpha^*) \Longrightarrow$ $\chi(S_\alpha^* - \Sigma_\alpha^*) = 0$. <u>This proves the claim.</u>

Note Also, $\chi(\partial W_\alpha^*) = 0$ WHY?

<u>NEXT CLAIM</u> There is a controlled nowhere zero vector field on the stratified object M/G.

VERIFICATION — Again induction on strata.

So the controlled vector field is already given on Σ_α^*

Assume this is known for all strata up to S_α^*. Let $\varphi_\alpha : \partial W_\alpha^* \to \Sigma_\alpha^*$ be a smooth carpeting function (TAPIS) in the sense of Thom. Then by the definition of carpeting function there is a ~~well defined path~~ canonical pullback of the controlled vector field on Σ_α to ∂W_α^* and in fact to N_α^*. We want to extend this to a smooth vector field on W_α^*.

A result of M. Morse (Gottlieb's Law of Vector Fields) states that if we are given a nowhere zero vector field $\mathcal{V}$ on the boundary ∂W of a smooth manifold, then we can extend $\mathcal{V}$ to W provided $\chi(W)=0$. Since $W_\alpha^* \underset{\text{homotopy}}{\cong} W_\alpha^* - \partial W_\alpha^*$ and $\chi(W_\alpha^* - \partial W_\alpha^*) = 0$, we know this holds in our setting and hence $\mathcal{V}$ extends to a vector field on W_α^* and a controlled vector field on S_α^*.

CONCLUSION. The controlled vector field on M/G lifts to a similar vector field on M.

We now use Thurston's existence results for codim 1 foliations in its relative form.

The absolute version of Thurston's result implies that there are compatible foliations on the minimal strata, and they come from the controlled vector field. Now assume there are compatible foliations on all strata preceding $\left\{ \begin{matrix} \mathcal{S}_\alpha^* \\ S_\alpha \end{matrix} \right\}$ and that they are transverse to substrata whenever this makes sense. Then one can use the carpeting function to ~~pull~~ extend these foliations compatibly to $\left\{ \begin{matrix} N_\alpha^* \\ N_\alpha \end{matrix} \right\}$ (compatibly). Finally, use the Law of Vector Fields and Thurston's results to extend the foliation to all of $\left\{ \begin{matrix} W_\alpha^* \\ W_\alpha \end{matrix} \right\}$ hence compatibly to all of $\left\{ \begin{matrix} \mathcal{S}_\alpha^* \\ S_\alpha \end{matrix} \right\}$.

<u>Cor.</u> $|G|$ odd, dim M odd $\Longrightarrow$
compatible foliations always exist.

<u>Proof</u> It suffices to ~~show~~ know that if $|G|$ is odd, then every nontrivial rep of G (& its subgps) comes from an irred unitary rep.* For then each C is odd dim and hence we have $\chi(C) = 0$.

* This fact is also needed in proving the necessity of the $\chi = 0$ condition. It follows directly from the Artin Induction Theorem (and the elementary fact that it's true for odd order cyclic groups).

<u>Reference</u>

I. M. Isaacs, Character Theory of Finite Groups | Thm (5.21) p. 72
| Cor (5.23) p. 73.

↑
MORE TO THE POINT.

$\boxed{\begin{array}{l} \mathrm{ind}_C^G \alpha \overset{\text{DEF}}{=} \mathbb{R}[G] \otimes_{\mathbb{R}[C]} \alpha \\ \text{induction} \end{array}}$

<u>Artin</u> in $RO(G)$ we have

$$|G| = \sum a_C \, \mathrm{ind}_C^G(1)$$

where C runs through all cyclic subgroups of G.

<u>Cor.</u> For every rep we have

$$|G|\rho = \sum b_C \, \mathrm{ind}_C^G(\rho_C) \quad \text{for suitable reps } \rho_C.$$

<u>Derivation</u>

$$|G|\rho = \sum a_C \, \mathrm{ind}_C^G 1 \cdot \rho =$$

$$\sum a_C \, \mathrm{ind}_C^G\left(\mathrm{res}_C^G \rho\right)$$

where res_C^G is restriction from G to C.

More generally, if $H \subseteq G$ then

$$(\mathrm{ind}_H^G \alpha) \otimes \beta = \mathrm{ind}_H^G(\alpha \otimes \mathrm{res}_H^G \beta).$$

References

Verona, Springer Lecture Notes 1102

Dovermann-Schultz, —"— 1443

Pflaum , —"— 1763

Jubin, ArXiv 0903.0697v2

Gottlieb, See above for references.
et al 9,10

Thom, Bulletin AMS 1969

Thurston, Existence of cod 1 Annals 1976

—"— , Local constr AMS Sympos
Proc., Stanford 1973.

Morse, ref 7 in Jubin's paper