Local results on tangency and transversality

CHANGES TO §1.3 Some reasoning is needed to justify the condition $g^*\omega = \pm\omega$. Here is what we need:

The natural notion of structure preserving diffeomorphism $f: M \to M$ for a contact structure, codimension 1 foliation, or indeed any hyperplane field on a manifold, is that the tangent space map of the diffeomorphism f sends the hyperplane H_x at $x \in M$ to the hyperplane $H_{f(x)}$ at $f(x)$. If the hyperplane field is defined by a smooth 1-form ω (which is true for contact structures and at least locally for foliations near a point fixed by f*), then the structure preserving condition is equivalent to $f^*\omega = h \cdot \omega$ where $h: M \longrightarrow \mathbb{R}$ is smooth and never zero.** Assume M is connected, so that h is either always positive or negative.

* if f is periodic of finite order

** Note ω & $h \cdot \omega$ define the same hyperplane field.

PROPOSITION. Suppose ~~f is periodic~~ G is a finite group acting smoothly on M and $\mathcal{H}$ is a smooth hyperplane field on M which is G-invariant.

Assume further that $\mathcal{H}$ is defined by a 1-form θ (for foliations this is true near fixed pts. of the group action by local linearity). Then there is a smooth positive valued function $k : M \to \mathbb{R}$ such that for each $g \in G$ we have $g^*(k \cdot \theta) = \varepsilon(g) k\theta$ where $\varepsilon(g) = \pm 1$.

<u>Sketch of proof</u> Assume we are given an invariant Riemann metric and if $x \in M^G$ there is an associated G-invariant nbhd of x which is equivalent to the unit disk in an orthogonal representation (Bochner's result in a weak form). The metric induces a metric on the

cotangent bundle τ_M^*

Choose k such that $|k \cdot \theta| = 1$.

Since G acts by isometries, it follows that for each 1-form λ on M we have $|g^*\lambda| = |\lambda|$. In particular, $|g^*(k\theta)| = |k\theta|$. On the other hand,

$$g^*(h\,\theta) = (k\circ g)\cdot g^*\theta = (k\circ g)\cdot h\theta$$

where $k : M \to \mathbb{R} - \{0\}$ is as before. In other words, both $g^*(k\theta)$ and $k\theta$ are nonzero multiples of both $k\theta$ and θ. Since

$g^*(k\theta) =$ nonzero mult. of $k\theta$ and

$|g^*(k\theta)| = |k\theta| = 1$, this means that $g^* k\theta$ must be $\pm\, k\theta$, or if $\omega = k\theta$, then.

$g^*\omega = \pm\omega$. Choose $\varepsilon(g) = \pm 1$ depending upon which sign appears. ∎

This justifies the 1st 3 lines of the T & T draft. Here is an altanate approach to the latter using local linearity.

<u>Altarnate proof of thm. 1</u> Since transvarsality is a local property, it is enough to prove the result near a fixed point and to restrict to an invt. open nbhd on which the action is orthogonal. We shall also use the isomorphism between forms and vectorfields associated to a specified invt. R metric. Without loss of generality, the fixed point is the 0 in the orthog rep V.

Consider the image of the vector field $X_\omega \leftrightarrow \omega$ in the tangent space $T(V) \underset{G}{\cong} V \times V$. the condition on $g^*\omega$ translates into saying that

$$X(v) = (v, \varphi(v)) \quad \text{where}$$

$$\varphi : V \longrightarrow V^G. \quad \text{and } v \in V^G$$

This means that $\varphi(v)$, which is normal to the

$\mathcal{L}_v$

leaf through v, is also tangent to the fixed set $V^G \subseteq V$, so that $\mathcal{L}_v$ and V^G meet transversely.

Alternate approach to thm. 2: Assume G is cyclic & generated by g with $\cancel{x} \in V^G$
$\updownarrow$
0

and $g^*\omega = -\omega$. Let $V_- \subseteq V$ be the eigenspace of -1. Then we have that φ (as in thm. 1) maps V to $V_- \subseteq V$. Since G acts orthogonally on V, we know that $V^G \perp V_-$ ⊛, so for all $v \in V$ we have that the hyperplane $H_v = \text{Kernel}\, \omega(v) = \varphi(v)^\perp$ must contain V^G.

⊛ linear algebra exercise

In particular, if $\gamma(t) = tv$ joins 0 to $v \in V^G$, then this implies $\omega(\gamma'(t))$ is identically 0 so that all of $\gamma(t)$ — in particular $v = \gamma(1)$ — lies in the leaf containing 0. <u>NOTE</u>: Nothing about orient pres. is assumed here.

Important points

Suppose ω is a 1-form and $k : M \to \mathbb{R}-0$.

Then

(1) ω is contact $\Leftrightarrow$ $k\omega$ is contact

(2) ω defines foliation $\Leftrightarrow$ $k\omega$ does

[only need to prove ($\Rightarrow$); for $\omega = \frac{1}{k} \cdot k\omega$]

(1) $\dim M = 2q+1 \Rightarrow$

$(k\omega) \wedge (dk\omega)^q =$

$\lambda^q =$ q-fold wedge with itself

$k\, \omega \wedge (dk \wedge \omega + k\, d\omega)^q$. All terms with factors $dk \wedge \omega$ make no contribution since $\omega \wedge \omega = 0$, so we are left with $k^{q+1} \omega \wedge (d\omega)^q$

(2) ~~Frobenius~~ Frobenius thm. $\Rightarrow$ $d\omega = \omega \wedge \lambda$ some λ.

is cond for foliation

Hence $d(k\omega) = k\, d\omega + dk \wedge \omega =$

$$\omega \wedge (k\, d\lambda - dk).$$

(Should appear before proposition p. 1)