THEOREM. Let $G$ be a finite group acting smoothly on $M^n$, and let $\mathcal{F}$ be a compatible codimension $k$ foliation on $M^n$. Suppose that $g : V^k \longrightarrow M^n$ is a smooth equivariant embedding which is transverse to the leaves of $\mathcal{F}$.* Then there is a closed tubular neighborhood $(E, \partial E)$ of $V^k$ s.t.

* we might as well assume $g[V]$ is closed.

(i) $\partial E$ is transverse to the leaves of $\mathcal{F}$ s.t. $\mathcal{F}$ determines a pair of codim $k$ foliations on $(E, \partial E)$,

(ii) the leaves of $(E, \partial E)$ are the fibers of the projections of $E, \partial E$ to $V$.

SKETCH OF PROOF. Let $T\mathcal{L}(M) \subseteq T(M)$ be the subbundle of tangents along the leaves of $M$, and take a suitable (defined on the next page) $G$-compatible spray on $T(M)$. The hypotheses imply that the normal bundle to $g[V]$ is equal to $g^* T\mathcal{L}(M)$. By construction, an invariant tubular neighborhood is $G$-diffeomorphic to an invariant neighborhood of the $0$-section, and the diffeomorphism is given by the spray's exponential map.

SUITABILITY means that near $g[V]$, take a metric which is an orthogonal direct sum of a metric on $T(V)$ and a metric on $T\mathcal{L}(M)$. — This will imply that, near $V$, the fibers of the tubular neighborhood are totally geodesic submanifolds.

The exponential map sends a vector $w \in g^* T\mathcal{L}(M)$, which projects to $x$ in $V$, into a point $w'$ s.t. $w'$ and $g(v)$ are joined by a geodesic whose initial condition is $w \in g^* T\mathcal{L}(M)$. Since the fibers are totally geodesic and $T\mathcal{L}(M)$ is the bundle of tangents along the fibers of $E$ & along the leaves of $\mathcal{F}$, it follows that $w'$ lies in the leaf of $\mathcal{F}$ containing $g(x)$:

"exp" maps a neighborhood of $0$ in the fiber $E_x$ ~~diffeomor~~ 1-1 into a neighborhood of $g(x)$ in the leaf $\mathcal{L}_{g(x)}$ containing it.

By Invariance of Domain or the Inverse Function Thm, it follows that the image of "exp" contains an open neighborhood of $g(x)$ (and in fact the map is open).

Hence locally the points in $E_x$ correspond to points in $L_{g(x)}$ and conversely.

Transversality to the boundary implies that $\partial E$ will be transverse to the leaves if we choose it to be sufficiently close to the zero section.