STABLE HOMOTOPY THEORY

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CHAPTER V - DUALITY AND THOM SPECTRA

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In this chapter we consider the formal properties of Thom spectra, and how they arise in Spanier-Whitehead duality. There are many peculiar homomorphisms in algebraic topology, defined in widely differing ways; one of our objects is to unify several of these under the name 'transfer homomorphism'.

We also introduce the bordism homology and cobordism cohomology theories (see [C5]). We show how to define generally the cobordism characteristic classes of a vector bundle over a CW-complex; these take values in the cobordism cohomology ring of the base space.

This chapter comprises the sections:

§1. Thom spectra.

In [A6], Atiyah considered the Thom complex of a vector bundle over a finite CW-complex from the stable point of view, and observed that its stable homotopy type depended only on the stable class of the bundle. Now that we have the correct stable homotopy theory to work in, we can carry this through for vector bundles over arbitrary CW-complexes, and indeed for virtual vector bundles.

We shall assume that all our vector bundles have been given an orthogonal structural group. Those with fibre dimension n are classified by means of a universal bundle γ_n over a classifying space $B\underset{\sim}{O}(n)$. We shall assume that for the various n these fit together nicely:

(a) We have a CW-complex $B\underset{\sim}{O}$ filtered by subcomplexes

$$\ldots B\underset{\sim}{O}(n) \subset B\underset{\sim}{O}(n+1) \ldots,$$

(b) We have a universal vector bundle γ_n over $B\underset{\sim}{O}(n)$, with fibre dimension n,

(c) We have for each n a bundle isomorphism $\gamma_{n+1}|B\underset{\sim}{O}(n) \cong \gamma_n \oplus 1$, where 1 stands for the trivial line bundle,

(d) We have bundle isomorphisms $\mu^*\gamma_{m+n} \cong \gamma_m \times \gamma_n$ (cross product of vector bundles, over $B\underset{\sim}{O}(m) \times B\underset{\sim}{O}(n)$), where $\mu: B\underset{\sim}{O}(m) \times B\underset{\sim}{O}(n) \to B\underset{\sim}{O}(m+n)$ is induced by $\underset{\sim}{O}(m) \times \underset{\sim}{O}(n) \subset \underset{\sim}{O}(m+n)$.

(e) The bundle isomorphisms in (c) and (d) are compatible. This can conveniently be done by using the universal bundles constructed by Milnor [M4].

Bundles are determined by their classifying maps. We shall work with spaces over $BQ(n)$ rather than with vector bundles themselves. Let Z be the additive group of integers, with 0 as base point.

1.1 Definition The category A of finite CW-complexes over $BQ \times Z$ has as objects pairs (X, f), where X is a finite CW-complex, without base point, and $f:X \to BQ \times Z$ is a map. A morphism from (X, f) to (Y, g) is a map $h:X \to Y$ such that $g \circ h = f$. Composition is evident. We have also the subcategory $I(A)$, with the same objects, which contains the morphism h if and only if h is an inclusion of CW-complexes.

1.2 Definition The category A_W of CW-complexes over $BQ \times Z$, with the subcategory $I(A_W)$, is the W-extension of the pair of categories $I(A) \subset A$ (see Chapter I).

We observe that A, and hence A_W, is a topological category. By means of $\mu:BQ \times BQ \to BQ$ and group addition in Z, we have a multiplication on $BQ \times Z$. The definition of μ is not obvious (see Chapter II). Given a vector bundle ξ over X, with fibre dimension n, we take $f:X \to BQ \times Z$ to have a classifying map as first component, and n as second. If the fibre dimension varies, we treat each component of X separately.

1.3 Definition We define

$$\underset{\sim}{KO}(X) = [X^0, B\underset{\sim}{O} \times \underset{\sim}{Z}];$$

the set of unbased homotopy classes of maps from X to $B\underset{\sim}{O} \times \underset{\sim}{Z}$. It is an abelian group. We call the elements virtual vector bundles over X. The projection $X \to \underset{\sim}{Z}$ is called the rank of the virtual vector bundle.

When X is finite, this is the usual Grothendieck group of vector bundles over X. When X is infinite, $\underset{\sim}{KO}(X)$ is much bigger than the Grothendieck group - a virtual bundle is not in general the difference between two honest bundles, else the universal Stiefel-Whitney classes would not be algebraically independent.

Our object is to construct the Thom spectrum of a virtual vector bundle. Given $f: X \to B\underset{\sim}{O}(n)$, the Thom complex of $\xi = f^*\gamma_n$ is obtained from the unit disk bundle in ξ by identifying the boundary sphere bundle to a base point o. It has a natural cell structure. We follow [A6], and write X^ξ for this space. Also, adding a trivial line bundle to ξ simply suspends X^ξ. In particular X^0 is the disjoint union of X and o, as before! If we write n for the trivial bundle of fibre dimension n, and Σ for a point, we see that Σ^n is an n-sphere!

Write $\underset{\sim}{A}_\infty$ for the category of finite CW-complexes over $B\underset{\sim}{O}$, and $\underset{\sim}{A}_n$ for the subcategory of complexes over $B\underset{\sim}{O}(n)$. By compactness, $\underset{\sim}{A}_\infty$ is the union of the subcategories $\underset{\sim}{A}_n$. We can multiply $(X, f) \in \underset{\sim}{A}_m$ with $(Y, g) \in \underset{\sim}{A}_n$ by means of $\mu: B\underset{\sim}{O} \times B\underset{\sim}{O} \to B\underset{\sim}{O}$ to form $(X \times Y, \mu \circ (f \times g))$; this induces a functor $\underset{\sim}{A}_\infty \times \underset{\sim}{A}_\infty \to \underset{\sim}{A}_\infty$. The elementary information about Thom complexes is summarized in:

<u>1.4 Lemma</u> For each $n \geq 0$, we have the Thom complex functor

$$T_n : \underset{\sim}{A}_n \to \underset{\sim}{F}, \quad \underset{\sim}{I}(\underset{\sim}{A}_n) \to \underset{\sim}{I}(\underset{\sim}{F}),$$

where $\underset{\sim}{F}$ is the category of finite CW-complexes with base point. We have natural isomorphisms

a) $ST_n \approx T_{n+1} : \underset{\sim}{A}_n \to \underset{\sim}{F}$,

b) $T_m\alpha \wedge T_n\beta \approx T_{m+n}(\alpha \times \beta)$, $(\alpha \in \underset{\sim}{A}_m, \ \beta \in \underset{\sim}{A}_n)$

which yield the commutative diagram

$$\begin{array}{ccccc}
ST_m\alpha \wedge T_n\beta & \approx & S(T_m\alpha \wedge T_n\beta) & \approx & T_m\alpha \wedge ST_n\beta \\
\downarrow & & \downarrow & & \downarrow \\
 & & ST_{m+n}(\alpha\times\beta) & & \\
 & & \downarrow & & \\
T_{m+1}\alpha \wedge T_n\beta & \approx & T_{m+n+1}(\alpha\times\beta) & \approx & T_m\alpha \wedge T_{n+1}\beta \ .
\end{array}$$]]]

We now feed all this material into the categorical machinery developed in Chapters I and II. We recall that the suspension category $\underset{\sim}{F}_S$ was defined as the 'limit' of the sequence

$$\cdots\cdot \underset{\sim}{F}_{-2} \xrightarrow{S} \underset{\sim}{F}_{-1} \xrightarrow{S} \underset{\sim}{F}_0 \xrightarrow{S} \underset{\sim}{F}_1 \xrightarrow{S} \underset{\sim}{F}_2 \cdots\cdot,$$

in which each $\underset{\sim}{F}_n$ is a copy of $\underset{\sim}{F}$.

<u>1.5 Lemma</u> We have the Thom spectrum functor

$$T: \underset{\sim}{A} \rightarrow \underset{\sim}{F}_S, \quad \underset{\sim}{I}(\underset{\sim}{A}) \rightarrow \underset{\sim}{I}(\underset{\sim}{F}_S),$$

and a natural isomorphism

$$T(\alpha \times \beta) \approx T\alpha \wedge T\beta \qquad (\alpha, \beta \in \underset{\sim}{A}).$$

<u>Proof</u> Take a map $f: X \rightarrow B\underset{\sim}{O} \times \underset{\sim}{Z}$, where X is a finite CW-complex, which is an object of $\underset{\sim}{A}$. By compactness of X, there exists a least n such that f factors through $f': X \rightarrow B\underset{\sim}{O}(n) \times \underset{\sim}{Z}$. We assume X is connected, for the moment. Then f' has first component $f_1: X \rightarrow B\underset{\sim}{O}(n)$ and second component $r \in \underset{\sim}{Z}$, say. We define the functor T on this object of $\underset{\sim}{A}$ to be $T_n(X, f_1) \in \underset{\sim}{F}_{n-r}$, an object of $\underset{\sim}{F}_S$. Now suppose $g: Y \rightarrow X$, where Y is also finite, connected. Then $f \circ g: Y \rightarrow B\underset{\sim}{O} \times \underset{\sim}{Z}$ factors through $B\underset{\sim}{O}(m) \times \underset{\sim}{Z}$, say, where $m \leq n$. We require a map $Tg: T_m(Y, f_1 \circ g) \rightarrow T_n(X, f_1)$ in $\underset{\sim}{F}_S$. Now $T_m(Y, f_1 \circ g) \in \underset{\sim}{F}_{m-r}$ is isomorphic in $\underset{\sim}{F}_S$, canonically, to $S^{n-m}T_m(Y, f_1 \circ g) \in \underset{\sim}{F}_{n-r}$. Since T_n is a functor, we have a map $T_n(Y, f_1 \circ g) \rightarrow T_n(X, f_1)$ in $\underset{\sim}{F}_{n-r}$. Naturality in 1.4 yields a map $S^{n-m}T_m(Y, f_1 \circ g) \rightarrow T_n(Y, f_1 \quad)$. The required map Tg is the composite of these three. One can verify that T is a functor, defined so far for connected X.

If X is not connected, we treat each component of X separately, and take the wedge in $\underset{\sim}{F}_S$, so that T respects sums.

If $\alpha, \beta \in \underset{\sim}{A}$, the natural isomorphisms of 1.4 yield a natural isomorphism of $T(\alpha \times \beta)$ with the smash product $T\alpha \wedge T\beta$ (see II). Care is needed at this stage, but the machinery of II is equal to the task.]]]

<u>1.6 Lemma</u> Suppose the maps $f, g: X \to BO \times Z$ are homotopic. Then the Thom spectra of (X, f) and (X, g) are isomorphic, in $I(E_S)$.

<u>Proof</u> This derives from the covering homotopy property for bundles.]]]

We can now take W-extensions of everything (see I). Also, our functors are all continuous, and we may take homotopy classes.

<u>1.7 Theorem</u> We have the Thom spectrum functor

$$T: A_W \to E_{SW} = S, \quad I(A_W) \to I(S), \quad A_{Wh} \to S_h.$$

We write the Thom spectrum of the virtual vector bundle α over the CW-complex X as X^α. There are canonical natural isomorphisms

$$(X \times Y)^{\alpha\times\beta} \approx X^\alpha \wedge Y^\beta, \qquad S^n X^\alpha \approx X^{\alpha+n}$$

for each of the above three functors. The first is coherently commutative and associative.]]]

In particular, when $Y = X$, the diagonal map $\Delta: X \to X \times X$ induces from $\alpha \times \beta$ over $X \times X$ the <u>Whitney sum</u> $\alpha + \beta$, which makes $KO(X)$ an abelian group.

<u>1.8 Corollary</u> There is a canonical natural <u>diagonal</u> map

$$\Delta: X^{\alpha+\beta} \to X^\alpha \wedge X^\beta,$$

which is commutative and associative.]]]

Given a topological group G and a continuous orthogonal representation $G \to O(n)$, or $G \to O$, we have the Borel map $BG \to BO(n)$ or $BG \to BO$ (see [B2] or [M4]). Hence a map $BG \to BO \times Z$, with second component n in the first case, 0 in the second.

<u>1.9 Definition</u> The <u>Thom spectrum</u> MG is the Thom spectrum of the virtual vector bundle $BG \to BO \times Z$. In particular MO is the Thom spectrum of the identity representation of O.

<u>1.10 Definition</u> Denote by γ the universal virtual vector bundle of rank 0, so that $BO^{\gamma} = MO$. Then any virtual vector bundle over X, α say, of rank 0, is induced from γ by a <u>classifying map</u> $X \to BO$, unique up to homotopy. The Thom spectrum functor applied to the classifying map of α yields the <u>classifying map</u> $X^{\alpha} \to MO$.

More generally, a virtual vector bundle α over X of constant rank n has a classifying map of Thom spectra $X^{\alpha} \to MO$ of degree $-n$.

We have observed that a genuine vector bundle ξ over X gives rise to a (homotopy class of) virtual vector bundle over X whose rank is the fibre dimension of ξ. By convention, we write X^{ξ} for its Thom spectrum; this is consistent with what we already have.

<u>1.11 Theorem</u> Given a genuine vector bundle ξ over X, let N

be its unit disk bundle, ∂N its unit sphere bundle, and $\pi : N \to X$ the bundle projection. Then for any virtual vector bundle α over X, we have an isomorphism of Thom spectra

$$X^{\xi+\alpha} \approx N^{\pi^*\alpha}/\partial N^{\pi^*\alpha}$$

in $\underset{\sim}{S}$, $\underset{\sim}{I}(\underset{\sim}{S})$, or $\underset{\sim}{S}_h$.

Proof. Our categorical machinery requires simply a natural isomorphism of CW-complexes defined when X is a finite CW-complex and α is a genuine vector bundle, and this isomorphism must commute with the suspension operations on α, $N^{\pi^*\alpha}$, and $X^{\xi+\alpha}$. We do this canonically in each fibre. This amounts to finding for each p, q, a $\underset{\sim}{O}(p) \times \underset{\sim}{O}(q)$-equivariant homeomorphism $D^p \times D^q/\partial(D^p \times D^q) \cong D^{p+q}/\partial D^{p+q}$, which has to be associative. This becomes a trivial matter if we first choose for each p an equivariant homeomorphism of $\underset{\sim}{R}^p$ with the interior of D^p.]]]

As an application, suppose given genuine vector bundles ξ, η over X, Y respectively, having unit disk bundles M and N. Then $X^\xi = M/\partial M$, and $Y^\eta = N/\partial N$. Suppose we are given an embedding of N in M as a tubular neighbourhood of $Y \subset M$, not meeting ∂M. Then identification induces a map of Thom spaces $\varphi : X^\xi \to Y^\eta$.

1.12 Theorem Under these conditions, we have also a canonical map of Thom spectra

$$X^{\xi+\alpha} \to Y^{\eta+f^*\alpha}$$

for any virtual vector bundle α over X, where $f : Y \to X$ is the composite $Y \subset N \subset M \to X$. This map is compatible with the diagonal maps, in the sense that the diagram

$$\begin{array}{ccccc} X^{\xi+\alpha} & & \xrightarrow{\quad\Delta\quad} & & X^{\xi}\wedge X^{\alpha} \\ \downarrow & & & & \downarrow{\scriptstyle \varphi\wedge 1} \\ Y^{\eta+f^{*}\alpha} & \xrightarrow{\Delta} & Y^{\eta}\wedge Y^{f^{*}\alpha} & \xrightarrow{1\wedge f_{*}} & Y^{\eta}\wedge X^{\alpha} \end{array}$$

commutes.]]]

§2. Combinatorial Poincaré Duality

In this section we translate G.W. Whitehead's duality theorem [W4] into our theory, with the various simplifications possible.

Let X be any finite triangulated simplicial complex. Given any subcomplex K, the supplement $K^{\sim}$ of K is the union of all simplexes of the first derived complex X' that do not meet K. We observe that

$$(K \cup L)^{\sim} = K^{\sim} \cap L^{\sim}, \quad (K \cap L)^{\sim} = K^{\sim} \cup L^{\sim}, \quad K \subset L \text{ implies } L^{\sim} \subset K^{\sim}.$$

There is a unique simplicial map $X' \to K^{\sim} * K$ (the join) extending the inclusions of K^{-} and K. Let $s:K^{-}*K \to [0, 1]$ be the simplicial map taking K^{-} to 1 and K to 0. Then we set, as [W4],

$$N(K) = s^{-1}[0, \tfrac{1}{2}], \quad N(K^{-}) = s^{-1}[\tfrac{1}{2}, 1],$$

which are triangulable subspaces of X. We see that we have homotopy equivalences

<u>2.1</u> $$K^{-} \subset N(K^{-}) \subset X - K, \quad K \subset N(K) \subset X - K^{-},$$
$$K/L \simeq N(K)/N(L), \qquad K^{-}/L^{-} \simeq N(K^{-})/N(L^{-}).$$

If $L \subset K$, we can define a map

$$\Delta:X^0 \to (N(L^{-})/N(K^{-})) \wedge (N(K)/N(L))$$

in the obvious way on $N(L^{-}) \cap N(K)$, and zero elsewhere.

<u>2.2 Definition</u> Given subcomplexes $L \subset K$ of X, the <u>diagonal map</u>

$$\Delta:X^0 \to (L^{-}/K^{-}) \wedge (K/L)$$

is defined from the above map up to homotopy, by using the homotopy equivalences 2.1.

<u>2.3 Remark</u> When $K = X$, $L = \emptyset$, Δ is the usual diagonal $\Delta:X^0 \to X^0 \wedge X^0$ (recall $X/\emptyset = X^0$).

The diagonal has the expected naturality properties. Given subcomplexes $K \supset L \supset M$ of X, we have $i:L/M \subset K/M$, $p:K/M \to K/L$, and in $\underset{\sim}{S}_{h*}$ the boundary map $\delta:K/L \to L \to L/M$ of degree -1, and similarly for K^{-}, L^{-}, M^{-}. We consider the diagrams

(a)

$$\begin{array}{ccc} X^0 & \xrightarrow{\quad\Delta\quad} & (M^\infty/L^\infty) \wedge (L/M) \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle 1\wedge i} \\ (M^\infty/K^\infty) \wedge (K/M) & \xrightarrow[p\wedge 1]{} & (M^\infty/L^\infty) \wedge (K/M) \end{array}$$

(b)

$$\begin{array}{ccc} X^0 & \xrightarrow{\quad\Delta\quad} & (M^\infty/K^\infty) \wedge (K/M) \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle 1\wedge p} \\ (L^\infty/K^\infty) \wedge (K/L) & \xrightarrow[i\wedge 1]{} & (M^\infty/K^\infty) \wedge (K/L) \end{array}$$

(c)

$$\begin{array}{ccc} X^0 & \xrightarrow{\quad\Delta\quad} & (L^\infty/K^\infty) \wedge (K/L) \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle 1\wedge\delta} \\ (M^\infty/L^\infty) \wedge (L/M) & \xrightarrow[\delta\wedge 1]{} & (L^\infty/K^\infty) \wedge (L/M) \end{array}$$

<u>2.4 Lemma</u> The diagrams (a) and (b) commute, and (c) anticommutes, up to homotopy.

<u>Proof</u> (a) and (b) are obvious. Although (c) looks forbidding, it is sufficient, by (a) and (b), to take $K = X$, $M = \emptyset$.]]]

The diagonal induces cap products. Given a map $\mu: A \wedge B \to C$ of spectra, and $z \in H_n(X^0; A)$, we have the cap product

$$z \cap : H^i(L^\infty/K^\infty; B) \to H_{n-i}(K/L; C).$$

The naturality of the diagonal in 2.4 shows that we have the diagram, for any subcomplexes $K \supset L \supset M$ of X, 2.5

$$\begin{array}{ccccccc} H^i(M^-/L^-;\ B) & \to & H^i(M^-/K^-;\ B) & \to & H^i(L^-/K^-;\ B) & \to & H^{i+1}(M^-/L^-;\ B) \ \ldots \\ \downarrow z\cap & & \downarrow z\cap & & \downarrow z\cap & & \downarrow z\cap \\ H_{n-i}(L/M;\ C) & \to & H_{n-i}(K/M;\ C) & \to & H_{n-i}(K/L;\ C) & \to & H_{n-i-1}(L/M;\ C) \ \ldots \end{array}$$

in which the first two squares commute and the third commutes up to the sign $(-)^{n+1}$.

Now suppose that K and L differ by one simplex P; $K = L \cup P$, and the boundary ∂P of P lies in L. Let c be the barycentre of P, and $V = V_c$ the closed star of c in X', the union of all simplexes containing c. Let Q be the union of all simplexes of X' that meet P only in c, and R the subcomplex of Q consisting of those simplexes not meeting P, the 'link' of P in X'. Then $K/L = P/\partial P$, $L^-/K^- = Q/R$, and $V/\partial V \cong (Q/R) \wedge (P/\partial P)$, where ∂V is the frontier of V.

2.6 Lemma Under these conditions, the diagonal

$$\Delta : X^0 \to (L^-/K^-) \wedge (K/L) \cong (Q/R) \wedge (P/\partial P) \cong V/\partial V$$

agrees, up to homotopy, with the identification map

$$p_c : X^0 \to X/Cl(X - V) = V/\partial V. \quad]]]$$

Homology manifolds

2.7 Definition We say X is a combinatorial homology n-manifold if it is a triangulable space having the same local homology groups as an n-sphere.

We shall always choose a fixed triangulation. Various facts are more or less immediate from the definition. Assume for simplicity that X is compact and connected, for the moment. Then $H^n(X; \mathbb{Z}) \cong \mathbb{Z}$ or $\mathbb{Z}_2$. Every simplex is contained in some n-simplex. Let V_c be the closed star of c in X', for any vertex c of X', and $p_c: X^0 \to V_c/\partial V_c$ the identification map; then $V_c/\partial V_c$ is a homotopy n-sphere (being a suspension), and $p_c^*: H^n(V_c/\partial V_c; \mathbb{Z}) \to H^n(X; \mathbb{Z})$ is epi. We write $q_c: X^0 \to \Sigma^0$ for the desuspended composite $X^0 \to V_c/\partial V_c \simeq \Sigma^n$ of degree $-n$, determined up to sign. Then by a theorem of Hopf, $H^n(X; \mathbb{Z}) \cong \{X^0, \Sigma^0\}^n$, and is generated by q_c for any c.

Orientability

Let X be any triangulated compact combinatorial homology manifold. Suppose given a spectrum A, and a 'unit' map $i: \Sigma^0 \to A$, of degree 0.

2.8 Definition We say $z \in H_n(X; A)$ is a fundamental class of X if for every vertex c of X', $\langle z, q_c \rangle = \pm i \in \pi_0(A)$. We then say X is A-oriented.

Duality

We suppose given $i: \Sigma^0 \to A$ as above.

2.9 Definition We say the spectrum B has A-action if we are given a morphism, $\mu: A \wedge B \to B$, such that the composite

$$B \approx \Sigma^0 \wedge B \xrightarrow{i \wedge 1} A \wedge B \to B$$

is the identity morphism of B.

Remark From IV, the only spectra with $K(\mathbb{Z})$-action are the graded Eilenberg-MacLane spectra.

2.10 Theorem Let X be a triangulated compact combinatorial homology n-manifold, A-oriented by $z \in H_n(X; A)$. Then for any subcomplexes $K \supset L$ of X and any spectrum B with A-action,

$$z \cap : H^r(L^-/K^-; B) \cong H_{n-r}(K/L; B), \quad H^r(K/L; B) \cong H_{n-r}(L^-/K^-; B).$$

Proof If K and L differ by one simplex, the theorem holds by 2.6 and the definition 2.8 of orientability. If the result is true for K/L and L/M, it is true for K/M by exactness, commutativity of 2.5, and the five-lemma. The result therefore follows by induction.]]]

We may take $K = X$, $L = \emptyset$.

2.11 Corollary $z \cap : H^r(X^0; B) \cong H_{n-r}(X^0; B)$.]]]

There is no longer any need to work with a fixed triangulation.

2.12 Theorem Let X be a compact combinatorial homology n-manifold, A-oriented by $z \in H_n(X; A)$. Let $K \supset L$ be closed subsets of X which are subcomplexes in some triangulation of X, and $K' \subset L'$ a pair of closed subsets of X homeomorphic to a CW-complex and a subcomplex, such that K' and L' are deformation retracts of $X - K$ and $X - L$ respectively. Then $z \cap$ induces

$$H^r(L'/K'; B) \cong H_{n-r}(K/L; B), \quad H^r(K/L; B) \cong H_{n-r}(L'/K'; B),$$

for any spectrum B with A-action.]]]

Clearly the n-sphere Σ^n is Σ^0-orientable, and any spectrum B has a unique Σ^0-action. Then given subcomplexes $K \supset L$ such that $K \neq \Sigma^n$, $L \neq \emptyset$, the diagonal map $\Delta:\Sigma^n \to (L^-/K^-) \wedge (K/L)$ induces the isomorphisms of 2.10 for any B. After desuspending, our recognition result (see IV) for dual spectra shows that we have duals here. We have

<u>2.13 Theorem</u> Given subcomplexes $K \supset L$ of Σ^n, $K \neq \Sigma^n$, $L \neq \emptyset$, we have, as spectra, $L^-/K^- \simeq S^n D(K/L)$. The hypotheses may be weakened as in 2.12.]]]

In particular, take L to be a point; then $\delta: L^-/K^- \simeq SK^-$.

<u>2.14 Corollary</u> $K^- \simeq S^{n-1}DK$.]]]

Historically, 2.14 was used [S3] as the definition of the dual.

We may also add duality isomorphisms (see IV) $H^r(X^0; B) \cong H_{-r}(DX^0; B)$ and $H_r(X^0; B) \cong H^{-r}(DX^0; B)$ to 2.11 to give a new form to Poincaré duality.

<u>2.15 Theorem</u> Let X be a compact A-oriented combinatorial homology n-manifold. Then we have isomorphisms, for any spectrum B with A-action,

$$H^r(X^0; B) \cong H^{r-n}(DX^0; B), \quad H_r(X^0; B) \cong H_{r-n}(DX^0; B). \text{]]]}$$

<u>Remark</u> The proof of 2.10 did not make essential use of spectra. It could have been expressed entirely in terms of half-exact functors.

§3. The Thom construction

We give the elementary facts about the Pontryagin–Thom construction, expressed in a form suitable for our applications.

In this section, all manifolds shall be smooth C^∞. Triangulation theorems show that we may freely use the results in §2. We again write n for a trivial vector bundle of fibre dimension n.

3.1 Lemma (Milnor, Spanier [M5]). Let M be a smooth compact manifold, and α any virtual vector bundle over M. Then the dual spectrum $DM^\alpha \simeq M^{-\tau-\alpha}$, where τ is the tangent bundle of M. In particular, $DM^0 \simeq M^{-\tau}$. These equivalences are canonical.

Proof Since $DS'X = S'^{-1}DX$ for any spectrum X, and M is compact, by suspending we may assume α is a genuine vector bundle. If n is large enough, we can embed M in Σ^n smoothly, and the bundle α in the normal bundle ν to M in Σ^n, so that $\nu = \alpha \oplus \beta$, say. Then the disk bundle, with total space K, boundary L, of α, is embedded in a tubular neighbourhood of M in Σ^0. We have $M^\alpha = K/L$. We see geometrically, for suitable representatives, that in the notation of 2.12 $L'/K' = M^\beta$. Hence, by 2.13, $DM^\alpha = S'^{-n}M^\beta \simeq M^{\beta-n} = M^{-\tau-\alpha}$ provided $\alpha \not\simeq 0$, since $\tau \oplus \alpha \oplus \beta = n$. If we take $\beta = 0$, we find $DM^{-\tau} = M^0$.]]]

Let $f: X \subset \mathbb{R}^m \times Y$ be a smooth embedding, where X and Y are compact, with normal bundle ν. Then the Thom construction [T1] yields a map $\underset{\sim}{R}^m \times Y \to X^\nu$, which, compactified, gives a map $S^m Y^0 \to X^\nu$, or $Y^m \to X^\nu$.

<u>3.2 Definition</u> The <u>Thom map</u> $T(f)$ of f is this map $Y^m \to X^\nu$, or any map $T(f): Y^\alpha \to X^{\nu - m + f_1^* \alpha}$ constructed from it by 1.12, where α is a virtual vector bundle over Y, and $f_1: X \to Y$ is the composite of f with projection.

In particular take $-\alpha = \tau(Y)$, the tangent bundle of Y. Then over X we have $f_1^* \tau(Y) \oplus m = \tau(X) \oplus \nu$. Hence a Thom map $T(f): Y^{-\tau(Y)} \to X^{-\tau(X)}$.

<u>3.3 Lemma</u> Under the identifications of 3.1, the Thom map $T(f): Y^{-\tau(Y)} \to X^{-\tau(X)}$ agrees with the dual $Df_1: DY^0 \to DX^0$.

<u>Proof</u> Let N be the disk bundle over Y having $\underset{\sim}{R}^m \times Y$ as interior. Let M be a tubular neighbourhood of X in N. We embed N smoothly in Σ^k. Then we find tubular neighbourhoods Q of Y, P of X, in Σ^k, $P \subset Q$. By definition the identification map $N/\partial N \to M/\partial M$ is the Thom map $T(f)$, and we see from 1.12 that $Q/\partial Q \to P/\partial P$ is an associated Thom map $T(f)$. Comparison of 2.4(a) with the definition of Df_1 shows that the latter map, after desuspension, gives Df_1.]]]

<u>3.4 Corollary</u> The stable homotopy class of $T(f)$ depends only on the stable homotopy class of f.]]]

This was clear anyway.

<u>3.5 Lemma</u> Under the above hypotheses, the diagram commutes

for any virtual vector bundles α and β over Y:

$$\begin{array}{ccccc}
Y^{\alpha+\beta} & & \xrightarrow{\quad\Delta\quad} & & Y^{\alpha} \wedge Y^{\beta} \\
\downarrow{\scriptstyle T(f)} & & & & \downarrow{\scriptstyle T(f)\wedge 1} \\
X^{\nu+f_1^*\alpha+f_1^*\beta} & \xrightarrow{\ \Delta\ } & X^{\nu+f_1^*\alpha} \wedge X^{f_1^*\beta} & \xrightarrow{1\wedge f_{1*}} & X^{\nu+f_1^*\alpha} \wedge Y^{\beta}\ .
\end{array}$$

<u>Proof</u> This is clear for genuine vector bundles. By compactness of Y the virtual bundles may be suspended to give genuine bundles.]]]

Now suppose we have a second smooth embedding, $g:Y \subset \underset{\approx}{R}^n \times Z$, with normal bundle μ. Then $(1 \times g) \circ f : X \subset \underset{\approx}{R}^{m+n} \times Z$ is a third, with normal bundle $\nu \oplus f_1^*\mu$.

<u>3.6 Lemma</u> Under these hypotheses,

$$T((1 \times g) \circ f) \simeq T(f) \circ T(g) : Z^{\alpha} \to X^{f_1^* g_1^* \alpha + \nu + f_1^* \mu},$$

for any virtual vector bundle α over Z.

<u>Proof</u> Directly, or from 3.3.]]]

§4. Thom isomorphisms

We give in this section the abstract theory behind the Thom isomorphisms, which applies regardless of the honesty or otherwise of the bundles involved. We deduce that a smooth manifold is A-orientable if and only if its stable normal bundle is A-orientable.

Let us take a CW-complex X, and a virtual vector bundle α over X, having constant rank n. Take also a spectrum A with a 'unit' $i:\Sigma^0 \to A$.

4.1 Definition We say $u:X^\alpha \to A$ (of codegree n, i.e. degree $-n$) is a fundamental class of X^α or of α if for every point $x \in X$, the composite $\Sigma^n \cong x^{\alpha|x} \subset X^\alpha \xrightarrow[u]{} A$ is $\pm i$. We then say that X^α and α are A-oriented.

4.2 Definition Given a fundamental class $u:X^\alpha \to A$ of α, and $\mu:A \wedge B \to C$, we have the Thom homomorphisms

$$\Phi^\alpha = u \cup : \{X^\beta, B\}^m \to \{X^{\beta+\alpha}, C\}^{m+n}$$

$$\Phi_\alpha = (-)^{mn}(\cap u) : \{\Sigma^0, X^{\beta+\alpha} \wedge B\}_m \to \{\Sigma^0, X^\beta \wedge C\}_{m-n}$$

induced by the diagonal map (1.8)

$$X^{\alpha+\beta} \xrightarrow[\Delta]{} X^\beta \wedge X^\alpha \xrightarrow[1\wedge u]{} X^\beta \wedge A,$$

where β is any virtual vector bundle over X.

Also, by means of the diagonal $\Delta:K^{\beta+\alpha}/L^{\beta+\alpha} \to (K^\beta/L^\beta) \wedge X^\alpha$, we can define useful Thom homomorphisms

4.3 $$\Phi^{\alpha}:\{K^{\beta}/L^{\beta}, B\}^{m} \to \{K^{\beta+\alpha}/L^{\beta+\alpha}, C\}^{m+n}$$
$$\Phi_{\alpha}:\{\Sigma^{0}, (K^{\beta+\alpha}/L^{\beta+\alpha})\wedge B\}_{m} \to \{\Sigma^{0}, (K^{\beta}/L^{\beta})\wedge C\}_{m-n}$$

for any subcomplexes $L \subset K$ of X, natural in K and L, including boundary maps.

4.4 Theorem [Dold] Suppose given $i:\Sigma^{0} \to A$, and a spectrum B with A-action (see 2.9) $\mu:A\wedge B \to B$. Suppose the virtual vector bundle α over X is A-oriented. Then

$$\Phi^{\alpha}:\{X^{\beta}, B\}^{m} \cong \{X^{\beta+\alpha}, B\}^{m+n}$$
$$\Phi_{\alpha}:\{\Sigma^{0}, X^{\beta+\alpha}\wedge B\}_{m} \cong \{\Sigma^{0}, X^{\beta}\wedge B\}_{m-n}$$

are isomorphisms, for any virtual vector bundle β over X. We also have Thom isomorphisms 4.3 for any subcomplexes $L \subset K$ of X.

Proof By induction on cells. Suppose first that $L \subset K \subset X$, and $K = L \cup e^{k}$, i.e. K is obtained from L by adjoining one k-cell. Let $\chi:D^{k} \to K$ be its characteristic map; then to prove 4.4 for K/L we need only prove 4.4 for $D^{k}/\partial D^{k}$ for the virtual bundles $\chi^{*}\alpha$ and $\chi^{*}\beta$, which are trivial. For either Thom homomorphism, this result follows from 4.1 and the hypothesis on μ, by suspending.

Let X_{r} be the r-skeleton of X. Then by additivity, from the previous case, we have 4.4 for X_{r}/X_{r-1}. Suppose, by induction on r, that 4.4 holds for X_{r-1}. Then by naturality of 4.3 and the five lemma, we deduce the isomorphism for X_{r}. Hence we have the Thom isomorphism for X_{r} for all r, by induction. By Milnor's lemma (IV), we therefore have the isomorphisms for general X.

In particular, we have isomorphisms for the restricted bundles $\alpha|K$ and $\alpha|L$, whenever $L \subset K \subset X$ are subcomplexes. The theorem for K/L now follows from the five lemma.]]]

Remark Again, when the vector bundles are genuine, this theorem does not make essential use of spectra, and could be expressed in terms of half-exact functors. Theorem 4.4 is in some sense 'dual' to 2.10.

Because we have allowed for the possibility $\beta \neq 0$, Thom isomorphisms can evidently be composed.

4.5 Lemma Suppose given spectra $i:\Sigma^0 \to A$, $i:\Sigma^0 \to B$, and a spectrum C with A- and B-actions $\mu:A \wedge C \to C$, $\mu:B \wedge C \to C$ that commute, in the sense that they yield only one $(A \wedge B)$-action $\mu:A \wedge B \wedge C \to C$ on C. Suppose given virtual vector bundles α, β over X, such that α is A-oriented, and β is B-oriented. Then we can $(A \wedge B)$-orient $\alpha + \beta$ canonically. With these orientations, we have $\Phi^{\alpha+\beta} = \Phi^{\alpha}\Phi^{\beta}$, $\Phi_{\alpha+\beta} = \Phi_{\alpha}\Phi_{\beta}$.

Proof Let $u:X^{\alpha} \to A$ and $v:X^{\beta} \to B$ be the given orientations of α and β. We orient $\alpha + \beta$ by

$$X^{\alpha+\beta} \xrightarrow[\Delta]{} X^{\alpha} \wedge X^{\beta} \xrightarrow[u \wedge v]{} A \wedge B.$$

The commutativity and associativity of Δ yield the composition laws.]]]

We always orient the zero bundle by means of the obvious projection $X^0 \longrightarrow \Sigma^0 \xrightarrow{i} A$.

4.6 Lemma Then Φ^0 and Φ_0 are identity homomorphisms.]]]

These two results yield a simpler proof of the Thom isomorphism, in the common cases. Given an A-action on A, $\mu: A \wedge A \to A$, commutative and associative, one deduces from 4.5 and 4.6 that $\Phi^{-\alpha}$ is inverse to Φ^{α}, and that $\Phi_{-\alpha}$ is inverse to Φ_{α}.

Let X be a compact smooth n-manifold with tangent bundle τ, and $i: \Sigma^0 \to A$ a map of spectra. We can consider A-orientability of X, or of τ.

4.7 Theorem With X and A as above, X is an A-orientable manifold if and only if $-\tau$ is an A-orientable bundle. The possible fundamental classes correspond, under the isomorphisms

$DX^0 = X^{-\tau}$ (see 3.1), and $\{DX^0, A\}^* \approx \{\Sigma^0, X^0 \wedge A\}_*$ (see IV).

The Thom isomorphisms $\Phi^{-\tau}: \{X^0, B\}^* \cong \{X^{-\tau}, B\}^*$ and $\Phi_{-\tau}: \{\Sigma^0, X^{-\tau} \wedge B\}_* \cong \{\Sigma^0, X^0 \wedge B\}_*$ agree with the isomorphisms 2.15, for any spectrum B with A-action.

Proof We have the stated isomorphisms. We must check that, if $u: X^{-\tau} \to A$ corresponds to $z: \Sigma^0 \to X^0 \wedge A$, the local conditions on u and z for these to be fundamental classes are equivalent.

We embed X smoothly in Σ^{n+k}, with normal bundle ν. Then we may conveniently take $u: X^{\nu} \to A$, of degree $-k$, instead of the given map $X^{-\tau} \to A$, by suspending, since $\tau + \nu = n + k$. The Thom construction gives a map $\Sigma^{n+k} \to X^{\nu}$. Then z is obtained from u as the composite

$$\Sigma^{n+k} \longrightarrow X^{\nu} \xrightarrow[\Delta]{} X^{0} \wedge X^{\nu} \xrightarrow[1\wedge u]{} X^{0} \wedge A,$$

desuspended as necessary. The assertion about Thom isomorphisms follows.

Take any point x of X. Then we have the composite $v:\Sigma^{k} \to A$, defined by inclusion of a fibre, $\Sigma^{k} \cong x^{\nu|x} \subset X^{\nu} \xrightarrow[u]{} A$. Take a disk neighbourhood D^{n} of x in X, and let $q:X^{0} \to \Sigma^{n}$ be the Thom construction applied to this disk neighbourhood D^{n} of B in X. Then our two local conditions are that for all $x \in X$, the maps $v:\Sigma^{k} \to A$ and $(q \wedge 1) \circ z:\Sigma^{n+k} \to \Sigma^{n} \wedge A$ are each ± 1, apart from suspensions. But it is immediate from the diagram below that these conditions are equivalent. This diagram is made up of Thom constructions, and commutes up to homotopy (compare 1.11).

$$\begin{array}{ccccccc}
\Sigma^{n+k} & \longrightarrow X^{\nu} \xrightarrow[\Delta]{} & X^{0} \wedge X^{\nu} & \xrightarrow[1\wedge u]{} & X^{0} \wedge A \\
\downarrow \simeq & & \downarrow q\wedge 1 & & \downarrow q\wedge 1 \\
\Sigma^{n} \wedge \Sigma^{k} & \longrightarrow & \Sigma^{n} \wedge X^{\nu} & \xrightarrow[1\wedge u]{} & \Sigma^{n} \wedge A.
\end{array}$$

]]]

Multiplicative structure

Take $1:\Sigma^{0} \to A$, and let B and C be spectra with A-action, such that $B \wedge C$ inherits a well-defined A-action $\mu:A \wedge B \wedge C \to B \wedge C$. Let ξ be an A-oriented virtual vector bundle of rank r over X.

Then we have seen that the diagonal $X^{\xi} \xrightarrow{\Delta} X^{0} \wedge X^{\xi} \to X^{0} \wedge A$ induces Thom isomorphisms. Also, the diagonals $\Delta: X^{\xi} \to X^{0} \wedge X^{\xi}$ and $\Delta: X^{0} \to X^{0} \wedge X^{0}$ induce cup and cap products. Then by commutativity and associativity of cup products and diagonals Δ (see 1.8), and the mixed rule for cup and cap products, we deduce the multiplication formulae

4.8 $\quad \Phi^{\xi}(\alpha \cup \beta) = \Phi^{\xi}\alpha \cup \beta = (-)^{mr}\, \alpha \cup \Phi^{\xi}\beta \quad (\alpha \in \{X^{0}, B\}^{m},\ \beta \in \{X^{0}, C\}^{n})$

4.9 $\quad \Phi_{\xi}(x \cap \alpha) = (-)^{mr}\, x \cap \Phi^{\xi}\alpha = \Phi_{\xi}x \cap \alpha$

$$(x \in \{\Sigma^{0}, X^{\xi} \wedge B\}_{m}, \quad \alpha \in \{X^{0}, C\}^{n}).$$

Naturality

Consider the Thom maps $T(f): Y^{\alpha} \to X^{\mu + f^{*}\alpha}$ induced by a smooth embedding $f: X \subset Y \times \mathbb{R}^{k}$ of smooth manifolds, as in 3.2, where α may be any virtual vector bundle over Y.

4.10 Lemma Let β be an A-oriented virtual vector bundle over Y. Then the Thom maps $T(f)$ and Thom isomorphisms Φ^{β} yield commutative diagrams, for any spectrum B with A-action,

$$\begin{array}{ccc}
\{X^{\mu+f_1^{*}\alpha}, B\}^{*} & \xrightarrow{T(f)^{*}} & \{Y^{\alpha}, B\}^{*} \\
\cong \downarrow \Phi^{f_1^{*}\beta} & & \cong \downarrow \Phi^{\beta} \\
\{X^{\mu+f_1^{*}\alpha+f_1^{*}\beta}, B\}^{*} & \xrightarrow{T(f)^{*}} & \{Y^{\alpha+\beta}, B\}^{*}
\end{array}$$

and

$$\begin{array}{ccc}
\{\Sigma^{0}, Y^{\alpha+\beta} \wedge B\}_{*} & \xrightarrow{T(f)_{*}} & \{\Sigma^{0}, X^{\mu+f_1^{*}\alpha+f_1^{*}\beta} \wedge B\}_{*} \\
\cong \downarrow \Phi_{\beta} & & \cong \downarrow \Phi_{f_1^{*}\beta} \\
\{\Sigma^{0}, Y^{\alpha} \wedge B\}_{*} & \xrightarrow{T(f)_{*}} & \{\Sigma^{0}, X^{\mu+f_1^{*}\alpha} \wedge B\}_{*}
\end{array}$$

Proof Both parts are immediate from 3.5.]]]

§5. Bordism and cobordism theories

Thom's fundamental lemma [T1] relating cobordism classes to homotopy classes shows that the use of Thom spectra as coefficient spectra gives rise to geometrically interesting homology and cohomology theories.

Let ξ be any vector bundle, rank k, over a CW-complex B. Let M be any smooth $(n + k)$-manifold, and M_c its one-point compactification.

5.1 Definition   A ξ-submanifold, or submanifold with ξ-structure, is a smooth compact submanifold V^n of M, with a bundle map from its normal bundle ν to ξ. Two ξ-submanifolds V_1 and V_2 are cobordant if there exists a ξ-submanifold W of $M \times I$, with boundary $V_i = W \cap (M \times i)$ $(i = 0, 1)$, where W meets $M \times 0$ and $M \times 1$ transversely, and the ξ-structure on W extends that on V_0 and V_1, under the natural identifications $\nu | V_i = \nu_i$ $(i = 0, 1)$, where ν, ν_0, ν_1, are the normal bundles of W in $M \times I$, V_0 in M, V_1 in M.

In particular, if the same submanifold V is given two homotopic structure maps $\nu \to \xi$, the two resulting ξ-submanifolds are cobordant. Cobordism is an equivalence relation.

The Thom construction applied to the ξ-submanifold V of M yields a map $M \to B^\xi$ with compact support, and therefore a map $M_c \to B^\xi$. Uniqueness of tubular neighbourhoods and the definition of cobordism show that this construction yields a well defined map from the set of cobordism classes of ξ-submanifolds of M, to $[M_c, B^\xi]$.

5.2 Lemma [after Thom] The Thom construction induces an isomorphism

$$L(M; \xi) \cong [M_c, B^\xi],$$

where $L(M; \xi)$ denotes the set of cobordism classes of ξ-submanifolds of M.

Proof The method of proof in [T1] is valid for the case when ξ is a smooth vector bundle over a smooth manifold B. We reduce the general case to this case.

Since any ξ-submanifold V of M is compact, its structure map $\nu \to \xi$ factors through $\xi|C$, for some finite subcomplex C of B. Similarly for the structure map of a cobordism manifold between two ξ-submanifolds. Hence

$$L(M; \xi) = \varinjlim L(M; \xi|C); \text{ and } [M_c, B^\xi] = \varinjlim [M_c, C^{\xi|C}];$$

as C runs through finite subcomplexes of B. Thus we need consider only the case when B is a finite CW-complex.

We may clearly replace B by any space B' of the same homotopy type, and ξ by the induced bundle ξ' over B'. We can choose B' to be a smooth manifold (e.g. an open neighbourhood of B in $\mathbb{R}^m$ for some suitable m), and give ξ' a smooth structure.]]

Remark The condition on B can be removed.

We now stabilize. We shall be concerned only with the case $M = \mathbb{R}^{n+k}$; then $M_c \cong \Sigma^{n+k}$, a sphere. We may replace B by its $(n+1)$-skeleton without loss of generality; then if $k > n$ any virtual vector bundle ξ over B of rank k is isomorphic to a genuine vector bundle. Also, if $k > n+1$ the particular embedding of V in $\mathbb{R}^{n+k}$ becomes irrelevant, and we ignore it.

We next give the stable version of 5.1. Let ξ be a virtual vector bundle with base B (i.e. a map $\xi : B \to B\mathbb{Q}$), of rank 0.

5.3 Definition The smooth n-manifold V^n is a ξ-manifold if we are given a map $-\tau \to \xi - n$ of virtual vector bundles, where τ is the tangent bundle of V. Two compact ξ-manifolds without boundary V_0^n and V_1^n are said to be cobordant if there is a compact ξ-manifold W^{n+1} with boundary $V_0 \cup V_1$, whose ξ-structure extends those of V_0 and V_1. We define $L_n(\xi)$ as the set of cobordism classes of compact smooth ξ-manifolds without boundary.

The word 'extends' needs amplification. In comparing the structures of W and V_i over V_i $(i = 0, 1)$, we need to make use of bundle isomorphisms $\tau | V_i \cong \tau_i \oplus 1$, where τ, τ_0, τ_1, are the tangent bundles of W, V_0, V_1, respectively, and the extra trivial bundle 1 represents the inward normal bundle of V_0 in W or the outward normal bundle of V_1 in W.

With the help of the remarks preceding 5.3, we apply 5.2 with $M = \underset{\sim}{R}^{n+k}$, $k \geq n + 2$, to $\xi + k$, which bundle may be assumed honest.

5.4 Theorem Let ξ be a virtual vector bundle over B, and $L_n(\xi)$ the set of cobordism classes of ξ-manifolds of dimension n. Then the Thom construction induces the isomorphism

$$L_n(\xi) \cong \{\Sigma^0, B^{\xi}\}_n. \quad]]]$$

This theorem leads to the computation of $L_n(\xi)$ in various well known cases [T1, M1, A6]. As examples, we have:

ξ	ξ-manifolds	$L_*(\xi)$
zero bundle over point	stably framed manifolds	framed cobordism groups
identity virtual bundle over $\underset{\sim}{BO}$	unoriented manifolds (i.e. no extra structure)	$\underset{=}{N}_*$
universal virtual bundle over $\underset{\sim}{BSO}$	oriented manifolds	Ω_*
universal virtual bundle over $\underset{\sim}{BU}$	'unitary' manifolds	$\underset{=}{U}_*$
universal virtual bundle over B Spin	spin manifolds	spin cobordism groups

The 'unitary' manifolds are commonly called 'weakly almost complex' manifolds. Spin cobordism appears not to have been properly defined until [M6].

Track addition in $L_n(\xi)$ in 5.4 is clearly expressed geometrically by disjoint union of ξ-manifolds. If we are given a bundle map $\xi \times \xi \to \xi$, we can introduce multiplication into $L_*(\xi)$ in the obvious way, which corresponds under 5.4 to that induced by the map of Thom spectra $B^\xi \wedge B^\xi \cong (B \times B)^{\xi \times \xi} \to B^\xi$ (from 1.7).

Singular manifolds

Suppose we are given a space X and a virtual vector bundle ξ over B of rank 0.

5.5 Definition A singular ξ-manifold of X is a pair (V, f), where V is a ξ-manifold and $f: V \to X$ is an (unbased) map. Two singular manifolds (V, f) and (V', f') are bordant if there is a cobordism ξ-manifold W between V and V', and a map $g: W \to X$ extending f and f'. Denote by $B_n(X; \xi)$ the set of bordism classes of singular ξ-manifolds of dimension n. (Compare [C5].)

Thus the structure of a singular ξ-manifold (V, f) of X consists of a bundle map $-\tau \to \xi - n$ and a map $V \to X$. We may combine these into a single virtual bundle map $-\tau \to \eta - n$, where η is the virtual bundle over $X \times B$ induced from ξ by projection. So singular ξ-manifolds of X correspond to η-manifolds, and cobordism classes correspond. We have, therefore, $B_n(X; \xi) \cong L_n(\eta)$. By 1.7, $(X \times B)^\eta \approx X^0 \wedge B^\xi$.

5.6 Theorem The Thom construction induces an isomorphism, natural in X and ξ,

$$B_*(X;\ \xi) \cong \{\Sigma^0,\ X^0 \wedge B^\xi\}_*,$$

between the graded group $B_*(X;\ \xi)$ of bordism classes of singular ξ-manifolds of X and the stable homotopy groups of $X^0 \wedge B^\xi$.]]]

Thus $B_*(\ ;\xi)$, apart from reduction, is the standard homology theory (see IV) with B^ξ as coefficient spectrum. It is therefore prudent to introduce the associated cohomology theory. In accordance with general policy, we define all homology and cohomology theories in the reduced form.

5.7 Definition
$$\underline{\underline{N}}_n(X) = \{\Sigma^0,\ X \wedge \underset{\sim}{MO}\}_n, \quad \underline{\underline{N}}^n(X) = \{X,\ \underset{\sim}{MO}\}^n$$
$$\underline{\underline{U}}_n(X) = \{\Sigma^0,\ X \wedge \underset{\sim}{MU}\}_n, \quad \underline{\underline{U}}^n(X) = \{X,\ \underset{\sim}{MU}\}^n$$
$$\Omega_n(X) = \{\Sigma^0,\ X \wedge \underset{\sim}{MSO}\}_n, \quad \Omega^n(X) = \{X,\ \underset{\sim}{MSO}\}^n$$

(see 1.9 for $\underset{\sim}{MU}$ and $\underset{\sim}{MSO}$).

Then $\underline{\underline{N}}_*(X^0)$ are the bordism groups of X; but $\underline{\underline{N}}_*$ and $\underline{\underline{N}}^*$ are now defined on all spectra. The products $\underset{\sim}{MO} \wedge \underset{\sim}{MO} \to \underset{\sim}{MO}$, etc. from 1.8. induce commutative and associative products in all the above pairs of theories. In particular, these are modules over the coefficient rings $\underline{\underline{N}} = \{\Sigma^0,\ \underset{\sim}{MO}\}_*$, etc.

Conner and Floyd show in [C5] that when A is a subspace of X, the relative bordism group $\underline{\underline{N}}_n(X,\ A) = \{\Sigma^0,\ (X/A) \wedge \underset{\sim}{MO}\}_n$, etc., can also be given a geometric interpretation. Elements are represented as equivalence classes of singular manifolds with boundary, $f:(V,\ \partial V) \to (X,\ A)$, under a rather artificial equivalence relation. This has some uses. For purely homological considerations, relative groups are superfluous: instead a Mayer-Vietoris boundary homomorphism is all that is required. Berstein has given an elegant method of doing this. Let $X = A \cup B$, where A and B are open subsets, and let $f:M \to X$ be a singular manifold of X. Then $f^{-1}(X - A)$ and $f^{-1}(X - B)$ are disjoint closed subsets of M. Take a smooth Urysohn function $\varphi:M \to \underset{\sim}{R}$, 0 on $f^{-1}(X - A)$, 1 on $f^{-1}(X - B)$, and transverse to $\frac{1}{2}$. Then $N = \varphi^{-1}(\frac{1}{2})$ is a smooth $(n - 1)$-manifold, and the class of the singular manifold $g = f|N:N \to A \cap B$ is the required boundary.

Trivially, any virtual bundle over any CW-complex X is $\underset{\sim}{MO}$-oriented (4.1) by means of its classifying map $X \to \underset{\sim}{MO}$ (assuming it has constant rank), where $i:\Sigma^0 \to \underset{\sim}{MO}$ is the classifying map of the zero bundle over a point. Hence we always have Thom isomorphisms for the $\underset{\sim}{MO}$ theories. Let us give the geometric interpretation in terms of singular manifolds.

<u>5.8 Lemma</u> Let ξ be a smooth vector bundle over the manifold X. Let $f:(M, \partial M) \to (X^\xi, o)$ be a singular manifold of (X^ξ, o), smooth near and transverse to $X \subset X^\xi$. Put $N = f^{-1}(X)$, and $g = f|N$, so that $g:N \to X$ is a singular manifold of X. Then the Thom isomorphism $\Phi_\xi:\underline{\underline{N}}_*(X^\xi) \to \underline{\underline{N}}_*(X^0)$ is given by $\Phi_\xi[M, f] = [N, g]$.

<u>Proof</u> In effect, Φ_ξ is induced by the map of Thom spectra

$$X^{\xi} \xrightarrow{\Delta} X^0 \wedge X^{\xi} \to X^0 \wedge MO$$

over the map of base spaces $X \to X \times BO$. This, composed with f, is homotopic to the map $M \to X^0 \wedge MO$ obtained by applying the Thom construction to N in M.]]]

In particular, let X be a smooth submanifold of the smooth manifold Y with normal bundle ν, $f:M \to Y$ a singular manifold of Y transverse to X, $N = f^{-1}(X)$, and $g = f|N$, so that $g:N \to X \subset Y$ is a singular manifold of Y. We recall that the Thom isomorphism is a cap product. Let $\alpha:X \to MO$ be the classifying map of ν. Then naturality of Thom spectra yields the geometric interpretation of cap products:

5.9 Lemma We have $[M, f] \cap \alpha = [N, g]$.]]]

Again, any smooth manifold is canonically MO-oriented by means of the identity singular manifold. One can deduce that in this case Poincaré duality is given by the Thom map (see 3.2) : given $f:M \subset X \times R^n$, we use the map $X^n \xrightarrow{T(f)} M^{\nu} \to MO$.

Evidently, everything we have done for the MO theories carries over to the other theories, provided the bundles and manifolds have suitable structures.

It is well known when bundles are orientable for ordinary cohomology.

5.10 Definition

We have the fundamental classes

$\sigma_O:MO \to K(Z_2)$, $\sigma_{SO}:MSO \to K(Z)$, hence $\sigma_U:MU \to K(Z)$, defined in the usual way (e.g. [T1]).

§6. Transfer homomorphisms

There are certain well-known important homomorphisms in algebraic topology which do not quite fit into the usual functorial framework. We propose to call them transfer homomorphisms, by analogy with the representation theory of groups. There is one for each homology and cohomology theory.

The transfer homomorphism is defined like a function on a manifold - we define it locally, on various domains of definition, and show that it is well defined on the intersection of any two of these domains. We shall content ourselves with eight types of transfer homomorphism; there are many more in common use.

Since our applications are to smooth manifolds, we shall often restrict attention to this easy case, even though it is well known, and sometimes obvious, that the definitions work much more generally. We also say nothing about manifolds with boundary. We give bordism transfer homomorphisms only for the theory $\underline{\underline{N}}_*$; but again the definitions hold for other bordism theories, under the obvious orientation conditions.

We shall use the same symbol $\natural$ for all transfer homomorphisms, however defined, to distinguish them from ordinary induced homomorphisms (this differs from some current practice).

Axiomatic description

We recall that $X^0 = X/\emptyset$. For certain maps $f:X \to Y$ of spaces with additional structure, in addition to the ordinary induced homomorphisms

$$f^*:\{Y^0, B\}^* \to \{X^0, B\}^* \qquad f_*:\{\Sigma^0, X^0 \wedge B\}_* \to \{\Sigma^0, Y^0 \wedge B\}_*$$

we have transfer homomorphisms, both of the same generally non-zero degree, m say,

$$f_\natural:\{X^0, B\}^* \to \{Y^0, B\}^* \qquad f^\natural:\{\Sigma^0, Y^0 \wedge B\}_* \to \{\Sigma^0, X^0 \wedge B\}_*.$$

These are functorial to the extent that $1_\natural$ and $1^\natural$ are identities, and that if we have $g_\natural$ and $g^\natural$, where $g:Y \to Z$, then

6.1 $\qquad (g \circ f)_\natural = g_\natural f_\natural \qquad (g \circ f)^\natural = (-)^{mn} f^\natural g^\natural,$

where $f_\natural$, $g_\natural$, and $(g \circ f)_\natural$ have degrees m, n, m + n. Under the conditions favourable to cup and cap products, we have

6.2 (a) $\quad f_\natural(\alpha \cup f^*\beta) = f_\natural\alpha \cup \beta \qquad (\alpha:X^0 \to B,\ \beta:Y^0 \to C)$

(b) $\quad f_\natural(f^*\alpha \cup \beta) = (-)^{m|\alpha|}\, \alpha \cup f_\natural\beta \qquad (\alpha:Y^0 \to B,\ \beta:X^0 \to C)$

(c) $\quad f^\natural(x \cap \alpha) = f^\natural x \cap f^*\alpha \qquad (x:\Sigma^0 \to Y^0 \wedge B,\ \alpha:X^0 \to C)$

(d) $\quad f_*(f^\natural x \cap \alpha) = (-)^{m|x|}\, x \cap f_\natural\alpha \qquad (x:\Sigma^0 \to Y^0 \wedge B,\ \alpha:X^0 \to C);$

in particular, for Kronecker products,

(e) $\quad \langle f^\natural x, \alpha \rangle = (-)^{m|x|} \langle x, f_\natural\alpha \rangle \qquad (x:\Sigma^0 \to Y^0 \wedge B,\ \alpha:X^0 \to C).$

If we are also given C = B = A and a multiplication $\mu:A \wedge A \to A$, then $f_\natural$ and $f^\natural$ are homomorphisms of $\{\Sigma^0, A\}_*$-modules. We shall give the products in the simplest forms available; they may all be embellished with suitable multiplication maps $B \wedge C \to D$, etc.

Various transfer homomorphisms

We shall always need a spectrum A with a map $i:\Sigma^0 \to A$, and a spectrum B with A-action $\mu:A \wedge B \to B$ (see 2.9).

(n) Poincaré duality transfer

Let X and Y be A-oriented combinatorial homology manifolds, of dimensions m and n, with fundamental classes $z_X:\Sigma^0 \to X^0 \wedge A$, and $z_Y:\Sigma^0 \to Y^0 \wedge A$. Then $|z_X| = m$, $|z_Y| = n$. Take $f:X \to Y$.

6.3 Definition We define the Poincaré duality transfers

$$f_\natural:\{X^0, B\}^* \to \{Y^0, B\}^* \qquad f^\natural:\{\Sigma^0, Y^0 \wedge B\}_* \to \{\Sigma^0, X^0 \wedge B\}_*$$

by the formulae

$$z_Y \cap f_\natural \alpha = f_*(z_X \cap \alpha) \qquad (\alpha:X^0 \to B)$$

$$f^\natural(z_Y \cap \beta) = (-)^{(m-n)n}\, z_X \cap f^*\beta \quad (\beta:Y^0 \to B).$$

Then $f_\natural$ and $f^\natural$ have degree $m - n$, and the diagrams

$$\begin{array}{ccc} \{X^0, B\}^* & \xrightarrow{f_\natural} & \{Y^0, B\}^* \\ \cong\downarrow z_X\cap & & \cong\downarrow z_Y\cap \\ \{\Sigma^0, X^0 \wedge B\}_* & \xrightarrow{f_*} & \{\Sigma^0, Y^0 \wedge B\}_* \end{array} \qquad \begin{array}{ccc} \{Y^0, B\}^* & \xrightarrow{f^*} & \{X^0, B\}^* \\ \cong\downarrow z_Y\cap & & \cong\downarrow z_X\cap \\ \{\Sigma^0, Y^0 \wedge B\}_* & \xrightarrow{f^\natural} & \{\Sigma^0, X^0 \wedge B\}_* \end{array}$$

commute up to sign. The definition works in virtue of the duality isomorphisms 2.11.

For these transfers, 6.1 is trivial.

Suppose we have also a spectrum C with A-action $A \wedge C \to C$, such that the two resulting A-actions on $B \wedge C$ coincide. Then the multiplicative formulae 6.2 follow from the standard formulae (IV) for associativity, commutativity, and induced homomorphisms of cup and cap products, by algebraic manipulation or commutative diagrams, according to taste. (To prove (a) and (b), apply $z_Y\cap$ to each side. To prove (c) and (d), express x in the form $z_Y \cap \beta$.)

Next, assume that $B = C = A$, and put $\beta = 1 \in \{Y^0, B\}$, the 'unit element', in 6.3. Then

6.4 $$f^\natural z_Y = (-)^{(m-n)n} z_X.$$

If we put $x = z_Y$ in 6.2 (d) and (c) and substitute from 6.4, we recover the formulae of 6.3. Thus

6.5 Lemma In this case, 6.4 and the multiplicative formulae 6.2 characterize $f^\natural$ and $f_\natural$.]]]

Remark The Principle of Signs breaks down in 6.3, because z_X and z_Y have non-zero degree. In any case, we cannot regard $f^\natural$ or $f_\natural$ as being obtained from f by a unary operation. We have chosen the signs to make 6.1 hold.

(b) Spanier-Whitehead duality transfer

Let X and Y be A-oriented combinatorial homology manifolds, of dimensions m and n, with fundamental classes $z_X : \Sigma^0 \to X^0 \wedge A$, $z_Y : \Sigma^0 \to Y^0 \wedge A$. Let $f : X \to Y$ be a map.

6.6 Definition We define the Spanier-Whitehead duality transfers

$$f_\natural : \{X^0, B\}^* \to \{Y^0, B\}^*, \quad f^\natural : \{\Sigma^0, Y^0 \wedge B\}_* \to \{\Sigma^0, X^0 \wedge B\}_*$$

by requiring the diagrams

$$\begin{array}{ccc} \{X^0, B\}^p & \xrightarrow{f_\natural} & \{Y^0, B\}^{p-m+n} \\ \cong\downarrow & & \cong\downarrow \\ \{DX^0, B\}^{p-m} & \xrightarrow{(Df)^*} & \{DY^0, B\}^{p-m} \end{array} \qquad \begin{array}{ccc} \{\Sigma^0, Y^0 \wedge B\}_p & \xrightarrow{f^\natural} & \{\Sigma^0, X^0 \wedge B\}_{p+m-n} \\ \cong\downarrow & & \cong\downarrow \\ \{\Sigma^0, DY^0 \wedge B\}_{p-n} & \xrightarrow{(Df)_*} & \{\Sigma^0, DX^0 \wedge B\}_{p-} \end{array}$$

to commute up to the signs $+1$ and $(-)^{(m-n)n}$ respectively, in which D denotes dual as in IV, and the vertical isomorphisms are provided by 2.15.

<u>6.7 Lemma</u> The Spanier-Whitehead and Poincaré duality transfers agree.

<u>Proof</u> We recall from 2.15 the definition of the vertical isomorphisms. This shows that the first diagram of 6.6 can be expanded to give the commutative diagram

$$\begin{array}{ccc}
\{X^0, B\}^p & \xrightarrow{f_\natural} & \{Y^0, B\}^{p-m+n} \\
\cong \downarrow z_X \cap & & \cong \downarrow z_Y \cap \\
\{\Sigma^0, X^0 \wedge B\}_{m-p} & \xrightarrow{f_*} & \{\Sigma^0, Y^0 \wedge B\}_{m-p} \\
\cong \downarrow & & \cong \downarrow \\
\{DX^0, B\}^{p-m} & \xrightarrow{(Df)^*} & \{DY^0, B\}^{p-m},
\end{array}$$

which shows the result for $f_\natural$. Similarly for $f^\natural$.]]]

<u>(c) The transfer of an oriented map</u>

Let $f: X \to Y$ be a map of CW-complexes, which need not now be finite.

<u>6.8 Definition</u> We say f is an <u>A-oriented map</u> if we are given a map of spectra $\hat{f}: Y^0 \to X^0 \wedge A$, of degree n, say, such that

$$\begin{array}{ccccc}
Y^0 & & \xrightarrow{\Delta} & & Y^0 \wedge Y^0 \\
\downarrow \hat{f} & & & & \downarrow 1 \wedge \hat{f} \\
X^0 \wedge A & \xrightarrow{\Delta \wedge 1} & X^0 \wedge X^0 \wedge A & \xrightarrow{f \wedge 1 \wedge 1} & Y^0 \wedge X^0 \wedge A
\end{array}$$

commutes (up to homotopy). It induces transfers

$$f_\natural: \{X^0, B\}^* \to \{Y^0, B\}^*, \quad f^\natural: \{\Sigma^0, Y^0 \wedge B\}_* \to \{\Sigma^0, X^0 \wedge B\}_*$$

as follows: Given $\alpha : X^0 \to B$, we have $Y^0 \xrightarrow[\hat{f}]{} X^0 \wedge A \xrightarrow[\alpha \wedge 1]{} B \wedge A \xrightarrow[\mu]{} B$, and we define $f_\natural \alpha = (-)^{n|\alpha|} \mu \circ (\alpha \wedge 1) \circ \hat{f}$. Given $x : \Sigma^0 \to Y^0 \wedge B$, we have $\Sigma^0 \xrightarrow[x]{} Y^0 \wedge B \xrightarrow[\hat{f} \wedge 1]{} X^0 \wedge A \wedge B \xrightarrow[1 \wedge \mu]{} X^0 \wedge B$, and we define $f^\natural x = (1 \wedge \mu) \circ (\hat{f} \wedge 1) \circ x$. (One could always take $\hat{f} = 0$, which would make $f_\natural$ and $f^\natural$ zero.)

Suppose that we are also given a spectrum C with A-action, such that the two resulting A-actions on $B \wedge C$ agree. Then the deduction of the multiplicative formulae 6.2 from the commutative diagram of 6.8 is another exercise in manipulating commutative diagrams or algebraic formulae, according to taste.

The multiplicative properties of the Thom isomorphisms (see §4) are included as a special case.

Now assume that we are in the simplified multiplicative situation with $A = B = C$, and we have a commutative and associative map $\mu : A \wedge A \to A$. Then given maps $f : X \to Y$, $g : Y \to Z$, A-oriented by $\hat{f} : Y^0 \to X^0 \wedge A$, $\hat{g} : Z^0 \to Y^0 \wedge A$, we can A-orient $g \circ f : X \to Z$ by putting

<u>6.9</u> $$\widehat{g \circ f} = (-)^{mn} (1 \wedge \mu) \circ (\hat{f} \wedge 1) \circ \hat{g} : Z^0 \to Y^0 \wedge A \to X^0 \wedge A \wedge A \to X^0 \wedge A,$$

where m and n are the degrees of $\hat{f}$ and $\hat{g}$. The commutative diagram of 6.8 for $\widehat{g \circ f}$ follows immediately.

We next compare this transfer with the Poincaré duality transfer. Let X and Y be A-oriented manifolds, of dimensions m and n.

6.10 Definition We say the A-orientation of $f:X \to Y$ is compatible with the A-orientations of X and Y if f has degree $m-n$, and the transfer induced by $\hat{f}$ gives $f^{!}z_Y = (-)^{(m-n)n} z_X$, where z_X and z_Y are the fundamental classes of X and Y.

6.11 Lemma Suppose the map $f:X \to Y$ of A-oriented manifolds is A-oriented compatibly, by $\hat{f}:Y^0 \to X^0 \wedge A$, where $\mu:A \wedge A \to A$ is commutative and associative. Then the transfer induced by $\hat{f}$ agrees with the Poincaré duality transfer. If also Z is an A-oriented manifold, and $g:Y \to Z$ is oriented compatibly, then the A-orientation of $g \circ f$ is compatible with the A-orientations of X and Z.

Proof It is immediate from 6.5 that the two transfers agree, for they agree on z_Y by 6.4 and 6.10, and are both multiplicative.]]

(d) The Grothendieck transfer

Let $f:X \to Y$ be a map of compact smooth manifolds, whose tangent bundles are $\tau(X)$ and $\tau(Y)$. We suppose that the virtual vector bundle $f^*\tau(Y) - \tau(X)$ over X is A-oriented, and deduce a transfer.

We lift f, up to homotopy, to a smooth embedding $f_1:X \subset Y \times \underline{R}^k$. Let ν be the normal bundle. Then $\nu = k + f^*\tau(Y) - \tau(X)$ is A-oriented. The Thom construction (3.2) yields a map $T(f):Y^0 \to X^\nu$, of degree k, and hence, by using diagonals (1.8), a map of spectra $\hat{f}:Y^0 \to X^\nu \xrightarrow[\Delta]{} X^0 \wedge X^\nu \to X^0 \wedge A$.

This A-orients the map f (the commutative diagram of 6.8 is immediate), and hence induces transfer homomorphisms $f_\natural$ and $f^\natural$. However, we can express these transfers slightly differently, as being induced by the composites of the ordinary homomorphisms induced by T(f) with Thom isomorphisms.

<u>6.12 Definition</u> In this situation, the <u>Grothendieck transfers</u>

$$f_\natural : \{X^0, B\}^* \to \{Y^0, B\}^*, \quad f^\natural : \{\Sigma^0, Y^0 \wedge B\}_* \to \{\Sigma^0, X^0 \wedge B\}_*$$

are the composite homomorphisms

$$\{X^0, B\}^* \xrightarrow[\Phi^\nu]{} \{X^\nu, B\}^* \xrightarrow[T(f)^*]{} \{Y^0, B\}^*$$

and

$$\{\Sigma^0, Y^0 \wedge B\}_* \xrightarrow[T(f)_*]{} \{\Sigma^0, X^\nu \wedge B\}_* \xrightarrow[\Phi_\nu]{} \{\Sigma^0, X^0 \wedge B\}_*.$$

We see from 3.4, or by direct geometric construction, that these transfers are well defined. We deduce the multiplicative formulae 6.2 from those for Thom isomorphisms (4.8 and 4.9), and the composition law 6.1 from 3.6.

We have already observed that the Grothendieck transfers are special cases of transfers induced by an oriented map.

<u>6.13 Lemma</u> Suppose that X and Y are also A-oriented manifolds, and that we are given $\mu : A \wedge A \to A$, commutative and associative. Then there is a canonical A-orientation for $f^*\tau(Y) - \tau(X)$, and with this orientation the Grothendieck and Spanier-Whitehead duality transfers agree.

<u>Proof</u> Write $\nu = f^*\tau(Y) - \tau(X)$. By 4.7, we have A-orientations $v : Y^{-\tau(Y)} \to A$ and $w : X^{-\tau(X)} \to A$. We choose $u : X^\nu \to A$ corresponding

to w under the Thom isomorphism

$$\Phi^{-f^*\tau(Y)}:\{X^\nu,\ A\}^* \cong \{X^{-\tau(X)},\ A\}^* \ ;$$

naturality of Thom isomorphisms for the inclusion in X of any point shows that $\bar{u}$ is an A-orientation of the virtual bundle ν. Further, by means of 4.5 and μ, the orientations u and v give back the orientation w, and $\Phi^{-\tau(X)} = \Phi^{-f^*\tau(Y)}\ \Phi^\nu$. We consider the diagram

$$\begin{array}{ccccc} \{X^0,\ B\}^* & \xrightarrow[\Phi^\nu]{} & \{X^\nu,\ B\}^* & \xrightarrow{\Phi^{-f^*\tau(Y)}} & \{X^{-\tau(X)},\ B\}^* \\ & & \downarrow T(f)^* & & \downarrow T(f)^* \\ & & \{Y^0,\ B\}^* & \xrightarrow[\Phi^{-\tau(Y)}]{} & \{Y^{-\tau(Y)},\ B\}^*, \end{array}$$

which commutes by 4.10. From 4.7, the Thom isomorphisms of $-\tau(X)$ and $-\tau(Y)$ are the Spanier-Whitehead duality isomorphisms 2.15 for X and Y, apart from putting $DX^0 = X^{-\tau(X)}$ and $DY^0 = Y^{-\tau(Y)}$, and from 3.3 $T(f)^* = (Df)^*$; we are back to 6.6.

Similarly for homology.]]]

(e) Integration over the fibre

We now consider a fibre bundle $\pi: E \to B$ whose fibre F is a compact n-manifold, with fundamental classes $z_F \in H_n(F;\ G)$ and $u_F \in H^n(F;\ G)$, where $G = \mathbb{Z}$ or $\mathbb{Z}_2$. In the case $G = \mathbb{Z}$ we also require the fundamental group of B to act trivially on $H_n(F;\ G)$. Then we have transfer homomorphisms (see e.g. [B3]) known as 'integration over the fibre'. The picturesque name arises from the case when B, E, and F are smooth manifolds and π is a smooth bundle, in which the cohomology transfer can be expressed in terms of integrating differential forms.

We shall also call these transfers the spectral sequence transfers, and use the definition [B3] in terms of the Leray-Serre spectral sequences of π.

6.14 Definition We define the spectral sequence transfers, or integration over the fibre, $\pi_{\natural}: H^i(E) \to H^{i-n}(B)$, $\pi^{\natural}: H_i(B) \to H_{i+n}(E)$, in terms of the spectral sequences of π as follows:

$\pi_{\natural}$ is the composite $H^i(E) \to E_{\infty}^{i-n,n} \subset E_2^{i-n,n} \cong H^{i-n}(B)$,

$\pi^{\natural}$ is the composite $H_i(B) \cong E^2_{i,n} \to E^{\infty}_{i,n} \subset H_{i+n}(E)$,

where the isomorphisms are $\otimes u$ and $\otimes z$.

We know (from IV) that we can put cup and cap products into these spectral sequences. It is easy to deduce from this fact the multiplicative formulae 6.2.

It is clear that $\pi_{\natural}$ and $\pi^{\natural}$ are natural for maps of bundles with fibre F, because the spectral sequences are natural.

6.15 Lemma (Chern [C2]) Suppose B is a manifold. Then E is also a manifold, and the spectral sequence transfers agree with the Poincaré duality transfers.

Proof Both pairs of transfers are multiplicative, and hence, by 6.5, we need only check $\pi^{\natural} z_B = \pm z_E$. This is evident from the definition 6.14, assuming we choose the correct orientation z_E for E.]]

(f) The pullback transfer

One would expect that for a geometrically defined homology theory such as bordism theory various transfers could be defined geometrically. This is indeed the case, although we shall restrict attention to the theory $\underline{\underline{N}}_*$ for simplicity.

Suppose we are given a smooth map $f:X \to Y$ of compact smooth manifolds, of dimensions m and n. Given a singular manifold $h:N \to Y$ of Y, we can construct the pullback space M and a map $g:M \to X$. Under a suitable transversality condition (viz. $f \times h:X \times N \to Y \times Y$ transverse to the diagonal of $Y \times Y$) $g:M \to X$ is a singular manifold of X.

6.16 Definition The pullback transfer $f^\natural:\underline{\underline{N}}_i(Y^0) \to \underline{\underline{N}}_{i+m-n}(X^0)$ is defined by taking the class of $h:N \to Y$ to the class of $g:M \to X$.

One can show directly that $f^\natural$ is well defined.

6.17 Lemma The pullback transfer agrees with the Grothendieck transfer.

Proof We lift f to a smooth embedding $f':X \subset Y \times \underline{R}^k$. Then our assertion is evident from two applications of 5.8.]]]

(g) The bundle transfer

There is another case, very similar to the previous, in which a geometric definition can be given. Suppose $\pi:E \to B$ is a fibre bundle whose fibre F is a smooth compact n-manifold, and whose structure group is a Lie group G acting smoothly on F.

Then given a singular manifold $f:M \to B$ of B, we construct the induced bundle $\tilde{M} \to M$ over M and a map $\tilde{f}:\tilde{M} \to E$. We may give this induced bundle a smooth structure, which makes $\tilde{f}:\tilde{M} \to E$ a singular manifold of E.

<u>6.18 Definition</u> The <u>bundle transfer</u> $\pi^{\natural}:\underline{\underline{N}}_i(B^0) \to \underline{\underline{N}}_{i+n}(E^0)$ is defined by taking the class of $f:M \to B$ to the class of $\tilde{f}:\tilde{M} \to E$.

Again one can show that $\pi^{\natural}$ is well defined. It obviously agrees with the pullback transfer when B is a smooth manifold. It is also trivial that $\pi^{\natural}$ is natural for maps of bundles with fibre F.

<u>(h) The Grothendieck bundle transfer</u>

A serious disadvantage of the two previous transfers is that there is no obvious way to define the corresponding cohomology transfer, because $\underline{\underline{N}}^*$ is not a geometric theory. We should like to have multiplicative transfers. Again, integration over the fibre has only been defined for ordinary homology and cohomology. We fill this gap by constructing another transfer, available for general cohomology and homology theories.

Let $\pi:E \to B$ be a fibre bundle whose fibre F is a smooth compact n-manifold, and whose structure group is a compact Lie group acting smoothly on F. We shall need the bundle τ of tangents along the fibre (see e.g. [B3]); this is a vector bundle over E whose restriction to a typical fibre F is the tangent bundle of F.

6.19 Lemma Let F be a smooth compact manifold, and G a compact Lie group acting smoothly on F. Then there exists a finite-dimensional representation space V for G, and a smooth G-equivariant embedding $F \subset V$.

Proof. We use a useful lemma of Mostow [M7], based on the Peter-Weyl theorem. Let S be the algebra of smooth real functions on F; then G acts on S. Take a finite set of elements $\{h_i\}$ of S which separate points of F. By [M7] we can approximate these by $\{h_i'\}$, still separating, such that for all i Gh_i' is contained in a finite-dimensional subspace of S. Let W be the subspace of S spanned by all the sets Gh_i'; it is finite-dimensional. Put $V = \mathrm{Hom}(W, \mathbb{R})$. Then evaluation of W at each point of F yields the required equivariant embedding $F \subset V$.]]]

Given a representation space V as in 6.19, let η be the vector bundle over B with fibre V associated to π [S5]. Then 6.19 yields an embedding of E in the total space of η. Choose an equivariant metric on V, and let U be a metric tubular neighbourhood of F in V. Then U gives rise to an associated subbundle of η having fibre U, total space N, say. We have a tubular neighbourhood disk bundle N of E in η, with normal bundle ν, say. Without loss of generality N is contained in the unit disk bundle of η. The

Thom construction now gives a map $B^{\eta} \to E^{\nu}$, and hence a map of spectra $B^{\eta+\xi} \to E^{\eta+\pi^*\xi}$ by 1.12 for any virtual vector bundle ξ over B. In particular, we have a map of spectra

6.20 $$T(\pi): B^0 \to E^{-\tau},$$

since $\pi^*\eta \cong \tau \oplus \nu$. The Thom map 6.20 is well defined (for if ζ is another bundle containing π, adding ζ to η does not affect $T(\pi)$, and we then have to compare two isotopic embeddings of E in $\eta \oplus \zeta$).

Now we suppose that $-\tau$ is A-oriented, and that C is a spectrum with A-action.

6.21 Definition We define the Grothendieck bundle transfers

$$\pi_{\natural}: \{E^0, C\}^* \to \{B^0, C\}^*, \quad \pi^{\natural}: \{\Sigma^0, B^0 \wedge C\}_* \to \{\Sigma^0, E^0 \wedge C\}_*$$

as the composite homomorphisms

$$\{E^0, C\}^* \xrightarrow[\Phi^{-\tau}]{} \{E^{-\tau}, C\}^* \xrightarrow[T(\pi)^*]{} \{B^0, C\}^*$$

$$\{\Sigma^0, B^0 \wedge C\}_* \xrightarrow[T(\pi)_*]{} \{\Sigma^0, E^{-\tau} \wedge C\}_* \xrightarrow[\Phi_{-\tau}]{} \{\Sigma^0, E^0 \wedge C\}_*$$

Formally we have exactly the same situation as for the Grothendieck transfers, and we shall not trouble to repeat the details. These transfers are multiplicative, i.e. satisfy 6.2. It is clear that they are natural for maps of fibre bundles with fibre F. As before, we have here a particular case of an oriented map.

If B is in fact a smooth manifold, we can choose η to be a trivial vector bundle (it does not have to come from a representation space). Then these transfers agree with the Grothendieck transfers 6.12.

Again, naturality and two applications of 5.8 show that our transfer includes the transfer 6.18 in bordism groups.

6.22 Lemma If the spectral sequence transfers 6.14 are also defined, they agree with the Grothendieck bundle transfers.

Proof We observe that we can relativize the transfers 6.21 as we did Thom isomorphisms, by constructing a map of spectra

6.23 $$T(\pi): B_1/B_2 \to (E_1/E_2) \wedge E^{-\tau}$$

for subcomplexes $B_2 \subset B_1 \subset B$, where $E_i = \pi^{-1}(B_i)$. The spectral sequences of π can also be relativized. By using naturality of both pairs of transfers, we quickly reduce to the case of a trivial bundle over $(D^r, \partial D^r)$, which is clear.]]]

Summary

Let us gather together what we have. For the map $f: X \to Y$ of CW-complexes, under the respective orientation conditions, we have various transfer homomorphisms $f_!$ and $f^!$:

(a) Poincaré duality - X and Y A-oriented manifolds.

(b) Spanier-Whitehead duality - X and Y A-oriented manifolds.

(c) Oriented map - A-orientation $\hat{f}: Y^0 \to X^0 \wedge A$.

(d) Grothendieck - X, Y smooth manifolds, $f^*\tau(Y)-\tau(X)$ A-oriented.

(e) Integration over the fibre - f a bundle, fibre a manifold.

(f) Pullback - X and Y smooth manifolds. (Only in bordism.)

(g) Bundle - f a bundle, fibre a smooth manifold, group a Lie group acting smoothly. (Only in bordism.)

(h) Grothendieck bundle - f a bundle, fibre a smooth manifold, group a compact Lie group acting smoothly, - τ A-oriented.

6.24 Theorem Under suitable conditions, these transfers are multiplicative. If two transfers are defined for f, and the appropriate compatibility conditions on the orientations hold, the two transfers agree.]]]

§7. Riemann-Roch theorems

We give here the formal theory (compare [D2]) of Riemann-Roch theorems for smooth manifolds (e.g. [A8]) and other situations in which transfer homomorphisms are available.

We shall suppose throughout this section that we are given two coefficient spectra A and C, with maps $i:\Sigma^0 \to A$, $i:\Sigma^0 \to C$, and commutative and associative multiplication maps $\mu:A \wedge A \to A$, $\mu:C \wedge C \to C$, such that $A \approx \Sigma^0 \wedge A \xrightarrow[i\wedge 1]{} A \wedge A \xrightarrow[\mu]{} A$ and similarly $C \to C$ are identity maps of spectra. We suppose also that we have a 'homomorphism' $\theta:A \to C$, such that $\theta \circ \mu = \mu \circ (\theta \wedge \theta):A \wedge A \to C$, and $\theta \circ i = i:\Sigma^0 \to C$.

If ξ is a virtual vector bundle over X, with A- and C-orientations, there is no reason for expecting the diagram

$$\begin{array}{ccc} \{X^0, A\}^* & \xrightarrow[\theta_*]{} & \{X^0, C\}^* \\ \cong\downarrow\Phi^\xi & & \cong\downarrow\Phi^\xi \\ \{X^\xi, A\}^* & \xrightarrow[\theta_*]{} & \{X^\xi, C\}^* \end{array}$$

to commute. Indeed, we use this diagram to define a new homomorphism.

<u>7.1 Definition</u> We define a homomorphism

$$\theta_\xi:\{X^0, A\}^* \to \{X^0, C\}^*$$

by putting $\theta_\xi \alpha = (\Phi^\xi)^{-1}\theta_*\Phi^\xi \alpha$.

Associativity of cup products yields the formula

<u>7.2</u> $$\theta_\xi (\alpha \cup \beta) = \theta_\xi \alpha \cup \theta_* \beta. \qquad (\alpha, \beta \in \{X^0, A\}^*).$$

Next, we suppose that X and Y are smooth manifolds, each A- and B- oriented, and $f:X \to Y$ a map. Then the transfer $f_!^A:\{X, A\}^* \to \{Y, A\}^*$ may be defined (6.6 and 4.7) by Thom isomorphisms and the Thom map $Y^{-\tau(Y)} \to X^{-\tau(X)}$ (essentially the dual of f by 3.3), where $\tau(X)$ and $\tau(Y)$ are the tangent bundles of X and Y. Then naturality and 7.1 yields the formula

<u>7.3</u> $$f_!^C \theta_{-\tau(X)} = \theta_{-\tau(Y)} f_!^A:\{X^0, A\}^* \to \{Y^0, C\}^*.$$

Take $\alpha:Y^0 \to A$, $\beta:X^0 \to A$. Then 6.2, 7.2, and 7.3 give

$$f_!^C(\theta_{-\tau(X)} f^*\alpha \cup \theta_*\beta) = f_!^C\theta_{-\tau(X)}(f^*\alpha \cup \beta) = \theta_{-\tau(Y)} f_!^A(f^*\alpha \cup \beta)$$

$$= \theta_{-\tau(Y)}(\alpha \cup f_!^A\beta) = \theta_{-\tau(Y)}\alpha \cup \theta_* f_!^A\beta,$$

i.e.

<u>7.4</u> $$f_!^C(\theta_{-\tau(X)} f^*\alpha \cup \theta_*\beta) = \theta_{-\tau(Y)}\alpha \cup \theta_* f_!^A\beta \quad (\alpha:Y^0 \to A,\ \beta:X^0 \to A).$$

In the cohomology ring $\{X^0, A\}^*$ of X we have the identity element 1, given by $X^0 \to S^0 \xrightarrow{1} A$, where the first map is induced by projecting X to a point.

<u>7.5 Definition</u> We put $\hat{a}(X) = \theta_{-\tau(X)}1 \in \{X^0, C\}$.

<u>7.6 Theorem</u> $f_!^C(\hat{a}(X) \cup \theta_*\beta) = \hat{a}(Y) \cup \theta_* f_!^A\beta \quad (\beta:X^0 \to A)$.

<u>Proof</u> We put $\alpha = 1$ in 7.4, and use 7.5.]]]

The proof of this 'Riemann-Roch' theorem is trivial. It is the verification of orientability and computation of $\hat{a}(X)$ and $\hat{a}(Y)$ that are liable to cause difficulties (e.g. [A8]).

More generally, we need only the difference bundle $f^*\tau(Y) - \tau(X)$ to be oriented, if we use the Grothendieck transfer 6.12.

<u>7.7 Definition</u> If $\nu = f^*\tau(Y) - \tau(X)$ is A- and C- oriented, we put

$$\hat{a}(f) = \theta_\nu 1 \in \{X^0, C\}.$$

If also X and Y are oriented as before, and the difference bundle ν is oriented according to 6.13, it is not difficult to prove

<u>7.8</u> $$\hat{a}(X) = f^*\hat{a}(Y) \cup \hat{a}(f),$$

and that $\hat{a}(Y)$ is invertible in $\{Y^0, C\}$, so that in this case $\hat{a}(f)$ can be determined from this formula.

In the same way as for 7.6, we obtain

<u>7.9 Theorem</u> $f_!^C(\hat{a}(f) \cup \theta_*\beta) = \theta_* f_!^A \beta \quad (\beta : X^0 \to A).$]]]

Again, suppose we have a fibre bundle $\pi : E \to B$ as in the context of the Grothendieck bundle transfer 6.21, and let τ be the bundle of tangents along the fibres. Formally, the situation is exactly that of 7.9. Suppose $-\tau$ is A- and C- oriented.

<u>7.10 Definition</u> We put $\hat{a}(\pi) = \theta_{-\tau} 1 \in \{E^0, C\}$.

<u>7.11 Theorem</u> $\pi_!^C(\hat{a}(\pi) \cup \theta_*\alpha) = \theta_* \pi_!^A \alpha \quad (\alpha : E^0 \to A).$]]]

Thus if θ_* is mono, and we know $\pi_!^C$ and $\hat{a}(\pi)$, we can compute $\pi_!^A$. It is this case that will concern us.

One could, of course, derive Riemann-Roch-type theorems for homology and cap products, along the same lines. There are obvious advantages, however, in arranging the c[illegible]tations so that they only involve cohomology and cup produc[illegible].

§8. Characteristic cobordism classes

Given a complex vector bundle ξ over a CW-complex B, we shall define natural characteristic classes of ξ, called the Chern cobordism classes of ξ, taking values in $\underline{U}^*(B^0)$. (Recall that $B^0 = B/\emptyset$, and that all cohomology is taken reduced.) Similarly we obtain Whitney classes in the real case, and an Euler class. For the Chern cobordism classes, this has been done by Conner and Floyd, under the restriction that B is finite. Now that we have $\underline{U}^*(\ )$ defined satisfactorily for arbitrary CW-complexes, this restriction is irrelevant; also we may parallel Borel's approach [B2] and work only with the universal bundle over $B\underline{U}(n)$, thus guaranteeing naturality. Finally, we show that our definition agrees with that of Conner and Floyd.

The importance of these classes in $\underline{U}^*(B\underline{U}(n)^0)$ and $\underline{N}^*(B\underline{O}(n)^0)$ is that they pick out canonical elements, which makes more precise investigations possible, as we shall see in VI. Further, we shall find in §9 some geometric properties of these classes.

For any honest vector bundle ξ over B, there is a canonical inclusion of B^0 in the Thom complex B^ξ, as the zero section (apart from base point). In particular, we have $B\underline{U}(1)^0 \subset M\underline{U}(1)$, etc

8.1 Definition The first universal Chern cobordism class $C_1 \in \underline{U}^2(B\underline{U}(1)^0)$ is the composite $B\underline{U}(1)^0 \subset M\underline{U}(1) \to M\underline{U}$.
The first universal Stiefel-Whitney cobordism class $W_1 \in \underline{N}^1(B\underline{O}(1)^0)$

is the composite $BO(1)^0 \subset MO(1) \to MO$.

The n th universal Euler cobordism class $X_n \in \{BSO(n)^0, MSO\}^n$ is the composite $BSO(n)^0 \subset MSO(n) \to MSO$.

In each case, the second map of spectra is the classifying map of a Thom spectrum.

These are our initial characteristic classes, from which we shall construct the others. We do this by using the fundamental classes (see 5.10) $\sigma_O : MO \to K(Z_2)$ and $\sigma_U : MU \to K(Z)$, observing that they induce ring homomorphisms from cobordism to ordinary cohomology, and using the results of Borel [B2].

Denote by $T(n)$ the usual maximal torus of diagonal matrices in $U(n)$, and $Q(n)$ the diagonal subgroup of $O(n)$. Then $T(n) \cong T(1) \times T(1) \times \dots \times T(1)$, and we may therefore take $BT(n) = BT(1) \times BT(1) \times \dots \times BT(1)$, and similarly for $Q(n)$. Define the cobordism classes $S_i \in U^2(BT(n)^0)$ $(1 \leq i \leq n)$ induced from $C_1 \in U^2(BT(1)^0) \cong U^2(BU(1)^0)$ by projection $BT(n) \to BT(1)$ to the i th factor; similarly we obtain $T_i \in N^1(BQ(n)^0)$.

In cohomology we have the corresponding cohomology classes s_i and t_i, and by Borel we have

8.2
$$H^*(BT(n)^0; Z) = Z[s_1, s_2, \dots, s_n],$$
$$H^*(BQ(n)^0; Z_2) = Z_2[t_1, t_2, \dots, t_n],$$

graded polynomial rings.

8.3 Lemma (a) $U^*(BT(n)^0) = U[S_1, S_2, \dots, S_n]^\wedge$, and $\sigma_U \circ S_i = s_i$.

(b) $\underline{N}^*(BQ(n)^0) = \underline{N}[T_1, T_2, \ldots, T_n]^\wedge$, and $\sigma_Q \circ T_1 = t_1$.

In each case, completion ^ is with respect to the skeleton topology (see IV), which here is the augmentation ideal generated by the S_i or the T_i, and its powers.

Proof We see from 8.1 that $\sigma_U \circ C_1 = c_1$, the first cohomology Chern class, from [B2] or [H1], and hence $\sigma_U \circ S_1 = s_1$. (We may take the inclusion $BU(1) \subset MU(1)$ as inclusion of a hyperplane in $P_\infty(C)$.) By IV we have a spectral sequence with E_2 term $E_2^{p,q} = H^p(BT(n)^0; \underline{U}^q)$, where we write $\underline{U}^q = \underline{U}_{-q}$. Milnor has shown that $\underline{U}$ is a graded polynomial ring over Z, with one generator in each even negative codegree, and in particular is free abelian [M1]; hence by 8.2 all the differentials vanish. Therefore the derived term RE_ω vanishes, and the spectral sequence converges (see IV; this is a fourth quadrant spectral sequence) to E_∞, associated to a complete Hausdorff filtration of $\underline{U}^*(BT(n)^0)$. The homomorphism induced by σ_U appears here as an edge homomorphism $\underline{U}^p(BT(n)^0) \to E_2^{p,0}$. Since $\sigma_U \circ S_1 = s_1$, and the spectral sequence has products and $\underline{U}$-module structure, $\underline{U}^*(BT(n)^0)$ must be as stated.

Similarly for $BQ(n)$, except that we have to invoke the fact (compare [C5]) that again the differentials vanish, because as we shall see in VI, MO is a graded Eilenberg-MacLane spectrum.]]

Still following Borel, we consider the map $\rho : B\underset{\sim}{T}(n) \to B\underset{\sim}{U}(n)$ induced by inclusion $\underset{\sim}{T}(n) \subset \underset{\sim}{U}(n)$.

8.4 Lemma (a) Inclusion induces the monomorphism $\rho^* : \underline{\underline{U}}^*(B\underset{\sim}{U}(n)^0) \to \underline{\underline{U}}^*(B\underset{\sim}{T}(n)^0)$, whose image is the symmetric subalgebra of $\underline{\underline{U}}^*(B\underset{\sim}{T}(n)^0) = \underline{\underline{U}}[S_1, S_2, \ldots, S_n]^\wedge$.

(b) Inclusion $\underset{\sim}{Q}(n) \subset \underset{\sim}{O}(n)$ induces the monomorphism $\rho^* : \underline{\underline{N}}^*(B\underset{\sim}{O}(n)^0) \to \underline{\underline{N}}^*(B\underset{\sim}{Q}(n)^0)$, whose image is the symmetric subalgebra of $\underline{\underline{N}}^*(B\underset{\sim}{Q}(n)^0) = \underline{\underline{N}}[T_1, T_2, \ldots, T_n]^\wedge$.

Proof The symmetric group G of permutations of n objects acts on $\underset{\sim}{T}(n)$ by permuting the factors, and hence also on $B\underset{\sim}{T}(n)$. However, each permutation can be expressed as conjugation by an element of $\underset{\sim}{U}(n)$, which is path connected. It follows that G acts on $\underline{\underline{U}}^*(B\underset{\sim}{T}(n)^0) = \underline{\underline{U}}[S_1, S_2, \ldots, S_n]^\wedge$ by permuting the S_i, and that the image of ρ^* is contained in the symmetric subalgebra of $\underline{\underline{U}}[S_1, \ldots, S_n]^\wedge$. Consideration of the spectral sequences for $\underline{\underline{U}}^*(B\underset{\sim}{T}(n)^0)$ and $\underline{\underline{U}}^*(B\underset{\sim}{U}(n)^0)$ and of the map between them induced by ρ shows that ρ^* must be mono and its image the whole of the symmetric subalgebra, since $H^*(B\underset{\sim}{U}(n)^0) = \underset{\sim}{Z}[c_1, c_2, \ldots, c_n]$ and $\rho^* H^*(B\underset{\sim}{U}(n)^0)$ is the symmetric subalgebra of $\underset{\sim}{Z}[s_1, s_2, \ldots, s_n]$.

Similarly for $B\underset{\sim}{O}(n)$.]]]

The situation is therefore exactly as we would expect from that in cohomology, apart from the need for completion. We can therefore p[illegible]ed.

<u>8.5 Definition</u> We define the <u>universal Chern cobordism classes</u> $C_i \in \underline{\underline{U}}^{2i}(B\underset{\sim}{U}(n)^0)$ $(1 \leq i \leq n)$ so that $\rho^* C_i$ is the i th elementary symmetric function of the S_j.

We define the <u>universal Stiefel-Whitney cobordism classes</u> $W_i \in \underline{\underline{N}}^i(B\underset{\sim}{O}(n)^0)$ so that $\rho^* W_i$ is the i th elementary symmetric function of the T_i.

This definition is permitted by 8.4.

<u>Remark</u> We can now complement 8.1.

<u>8.6</u> $C_n \in \underline{\underline{U}}^{2n}(B\underset{\sim}{U}(n)^0)$ is the composite $B\underset{\sim}{U}(n)^0 \subset M\underset{\sim}{U}(n) \to M\underset{\sim}{U}$.

$W_n \in \underline{\underline{N}}^n(B\underset{\sim}{O}(n)^0)$ is the composite $B\underset{\sim}{O}(n)^0 \subset M\underset{\sim}{O}(n) \to M\underset{\sim}{O}$.

We are ready for the main theorem.

<u>8.7 Theorem</u>

(a) $\underline{\underline{U}}^*(B\underset{\sim}{U}(n)^0) = \underline{\underline{U}}[C_1, C_2, \ldots, C_n]^\wedge$; $\underline{\underline{N}}^*(B\underset{\sim}{O}(n)^0) = \underline{\underline{N}}[W_1, W_2, \ldots, W_n]^\wedge$.

(b) $\sigma_U \circ C_i = c_i$; $\sigma_O \circ W_i = w_i$.

(c) Inclusion $\underset{\sim}{U}(n) \subset \underset{\sim}{U}(n+1)$ induces the homomorphism

$$\underline{\underline{U}}^*(B\underset{\sim}{U}(n+1)^0) = \underline{\underline{U}}[C_1, \ldots, C_{n+1}]^\wedge \to \underline{\underline{U}}^*(B\underset{\sim}{U}(n)^0) = \underline{\underline{U}}[C_1, \ldots, C_n]^\wedge,$$

which takes C_i to C_i for $1 \leq i \leq n$, and C_{n+1} to 0. Similarly for $\underset{\sim}{O}(n) \subset \underset{\sim}{O}(n+1)$.

(d) By (c), we may define $C_i \in \underline{\underline{U}}^{2i}(B\underset{\sim}{U}^0)$ as the inverse limit of the elements $C_i \in \underline{\underline{U}}^{2i}(B\underset{\sim}{U}(r)^0)$ $(r \geq i)$. Similarly $W_i \in \underline{\underline{N}}^i(B\underset{\sim}{O}^0)$.

(e) $\underline{\underline{U}}^*(B\underset{\sim}{U}^0) = \underline{\underline{U}}[C_1, C_2, \ldots]^\wedge$; $\underline{\underline{N}}^*(B\underset{\sim}{O}^0) = \underline{\underline{N}}[W_1, W_2, \ldots]^\wedge$.

(f) Inclusion $\underset{\sim}{U}(m) \times \underset{\sim}{U}(n) \subset \underset{\sim}{U}(m+n)$ induces the homomorphism

$$\begin{array}{ccc} \underline{\underline{U}}^*(B\underset{\sim}{U}(m+n)^0) & \to & \underline{\underline{U}}^*(\{B\underset{\sim}{U}(m) \times B\underset{\sim}{U}(n)\}^0) \\ \| & & \| \\ \underline{\underline{U}}[C_1, C_2, \ldots, C_{m+n}]^\wedge & \to & \underline{\underline{U}}[C_1 \otimes 1, C_2 \otimes 1, \ldots, C_m \otimes 1, 1 \otimes C_1, \ldots, 1 \otimes C_n] \end{array}$$

in which

$$C_i \rightsquigarrow C_i \otimes 1 + C_{i-1} \otimes C_1 + C_{i-2} \otimes C_2 + \ldots + 1 \otimes C_i,$$

with the convention that $C_j = 0$ in $U^*(BU(r)^0)$ if $j > r$. Similarly for $O(m) \times O(n) \subset O(m+n)$.

Proof   We give only the unitary proofs, as the orthogonal proofs are similar. By 8.4, $U^*(BU(n)^0)$ is isomorphic by ρ^* to the symmetric subalgebra of $U[S_1, S_2, \ldots, S_n]^\wedge$, which is known (Newton?) to be a completed graded polynomial algebra on the elements $\rho^* C_i$. We have (a). Since C_i and c_i are both defined in terms of elementary symmetric functions, (b) follows from 8.3. The inclusion $T(n) \subset T(n+1)$ induces a homomorphism taking S_i to S_i $(1 \leqslant i \leqslant n)$ and S_{n+1} to 0, clearly. Hence (c), by 8.4 and 8.5. By Milnor's lemma (see IV) $U^*(BU^0) = \varprojlim U^*(BU(n)^0)$, since we have here a sequence of epimorphisms; hence (d) and (e). In (f), we need only work with maximal tori, $T(m) \times T(n) = T(m+n)$, by 8.4. For these, we have $S_i \rightsquigarrow S_i \otimes 1$ (if $1 \leqslant i \leqslant m$) or $S_i \rightsquigarrow 1 \otimes S_{i-m}$ (if $m < i \leqslant m+n$). The result follows from 8.5.]]]

8.8 Corollary   The comultiplications in $U^*(BU^0)$ and $N^*(BO^0)$ are given by

$$C_i \rightsquigarrow C_i \otimes 1 + C_{i-1} \otimes C_1 + C_{i-2} \otimes C_2 + \ldots + 1 \otimes C_i;$$

$$W_i \rightsquigarrow W_i \otimes 1 + W_{i-1} \otimes W_1 + W_{i-2} \otimes W_2 + \ldots + 1 \otimes W_i. \quad]]]$$

8.9 Definition   Given a virtual vector bundle ξ over the CW-complex X, its Stiefel-Whitney cobordism characteristic classes

$W_i(\xi) \in \underline{\underline{N}}^i(X^0)$ are defined by $W_i(\xi) = \xi^* W_i$ (recall that by definition 1.3 we have $\xi: X \to B\underset{\sim}{O}$). Similarly, $C_i(\xi) \in \underline{\underline{U}}^{2i}(X^0)$ is defined, if ξ is factored through $B\underset{\sim}{U}$.

8.10 Corollary   Let ξ and η be virtual vector bundles over X. Then $W_k(\xi + \eta) = \Sigma_{i+j=k} W_i(\xi).W_j(\eta)$.]]]

Finally we give an alternative description of the Chern (and equally of the Stiefel-Whitney) cobordism classes, which is the version adopted by Conner and Floyd.

Suppose ξ is an honest complex vector bundle over the finite-dimensional CW-complex X, with complex fibre dimension n. Let Y be the total space of the associated projective bundle with fibre $P_{n-1}(\underset{\sim}{C})$, projection $\pi: Y \to X$, and let Z be the unit sphere bundle in ξ. Then the map $Z \to Y$ is a principal $\underset{\sim}{U}(1)$-bundle, with Chern class $C \in \underline{\underline{U}}^2(Y^0)$, say.

8.11 Theorem   By means of $\pi^*: \underline{\underline{U}}^*(X^0) \to \underline{\underline{U}}^*(Y^0)$, $\underline{\underline{U}}^*(Y^0)$ is a free $\{\underline{\underline{U}}^*(X^0)\}$-module with base $\{1, C, C^2, \ldots, C^{n-1}\}$. Multiplicatively, there is one relation

8.12 $C^n - C^{n-1}.\pi^* C_1(\xi) + C^{n-2}.\pi^* C_2(\xi) - \ldots + (-)^n \pi^* C_n(\xi) = 0.$

Proof   We first consider the universal example $\pi: E \to B$. When we have unravelled the various definitions, we find we are to investigate the Borel bundle [B2]

8.13 $\underset{\sim}{U}(n)//\{\underset{\sim}{U}(n-1) \times \underset{\sim}{U}(1)\} \to B\underset{\sim}{U}(n-1) \times B\underset{\sim}{U}(1) = E \underset{\pi}{\to} B\underset{\sim}{U}(n) = B$

induced by $\underset{\sim}{U}(n-1) \times \underset{\sim}{U}(1) \subset \underset{\sim}{U}(n)$. The class C is induced by projection from $C_1 \in \underline{\underline{U}}^2(B\underline{U}(1)^0)$. Write $\underline{\underline{U}}^*(B\underline{U}(n-1)^0) = \underline{\underline{U}}[C_1', C_2', \ldots, C_{n-1}']^\wedge$, by 8.7. Then we know (from the usual spectral sequence) that $\underline{\underline{U}}^*(E^0) = \underline{\underline{U}}[C, C_1', C_2', \ldots, C_{n-1}']^\wedge$. From 8.7, the homomorphism π^* is given by

$$\pi^* C_1 = C_1' + C, \quad \pi^* C_i = C_i' + CC_{i-1}' \ (1 < i < n), \ \pi^* C_n = CC_{n-1}'.$$

Eliminating the C_i' yields the relation 8.12. We also note that for the fibre $F = P_{n-1}(\underline{C}) \cong \underline{U}(n)//\{\underline{U}(n-1) \times \underline{U}(1)\}$, $\underline{\underline{U}}^*(F^0)$ is $\underline{\underline{U}}$-free with base $\{1, C, C^2, \ldots, C^{n-1}\}$, and therefore a free abelian group.

Let us now return to $\pi: Y \to X$. Certainly 8.12 holds, by naturality. We must show there are no new relations.

Consider the Leray-Serre spectral sequences of $\pi: Y \to X$ and $1: X = X$, with $M\underline{U}$ as coefficient spectrum. Let us write them as $(E_r(Y))$ and $(E_r(X))$ respectively, and $\pi^*: E_r(X) \to E_r(Y)$ for the map induced by π. Now $\pi_1(X)$ acts trivially on $\underline{\underline{U}}^*(F^0)$, and

$$E_2(X) = H^*(X; \underline{Z}) \otimes \underline{\underline{U}}^*; \qquad E_2(Y) = H^*(X; \underline{Z}) \otimes \underline{\underline{U}}^*(F^0),$$

since these facts are true of 8.13. Now these are graded rings, and by means of π^*, $E_2(Y)$ is a free $E_2(X)$-module, with base $\{1, C, C^2, \ldots, C^{n-1}\}$. Moreover, the differentials are all derivations, and vanish on C since they do for 8.13. It follows, by induction on r, that $E_r(Y)$ is a free $E_r(X)$-module with base $\{1, C, C^2, \ldots, C^{n-1}\}$. The spectral sequences converge without difficulty, to show that $\underline{U}^*(Y^0)$ is a free $\underline{U}^*(X^0)$-module with base $\{1, C, C^2, \ldots, C^{n-1}\}$.]]]

The dimensional restriction on X can be removed.

§9. Some geometric homomorphisms

We consider here two geometrically defined homomorphisms in cobordism theory discussed by Conner and Floyd [C5], and called by them the Smith homomorphism and J. The second is of crucial importance in the study of fixed points sets of involutions on manifolds, as we shall see in VI. We show here that both homomorphisms are special cases of homomorphisms already considered.

We know [C5] that $\underline{N}_*(BG^0)$ classifies equivariant cobordism classes of manifolds with free smooth G-action, where G is a Lie group (by considering the orbit spaces). Take a manifold

$\tilde{M}$ with free involution, representing $x \in \underset{\approx}{N}_i(B\underset{\sim}{O}(1)^0)$. Its orbit space is a singular manifold $f:M \to B\underset{\sim}{O}(1)$. We take $B\underset{\sim}{O}(1) = P_\infty(\underset{\sim}{R})$. Since M is compact, we may assume $fM \subset P_q(\underset{\sim}{R}) \subset P_\infty(\underset{\sim}{R})$, for some large q, and that f is smooth and transverse to P_{q-1} in P_q. Put $N = f^{-1}(P_{q-1})$, a submanifold of M, and $g = f|N$. Let $\tilde{N} \subset \tilde{M}$ be the double covering of N.

<u>9.1 Definition</u> The <u>Smith homomorphism</u>

$$\Delta:\underset{\approx}{N}_i(B\underset{\sim}{O}(1)^0) \to \underset{\approx}{N}_{i-1}(B\underset{\sim}{O}(1)^0)$$

is defined by taking the class of $f:M \to B\underset{\sim}{O}(1)$ to the class of $g:N \to B\underset{\sim}{O}(1)$.

The importance of N is that the involution on $\tilde{M}$ is trivial on $\tilde{M} - \tilde{N}$.

By 5.9, we are here simply taking the cap product with the class of the Thom map of the normal bundle of P_{q-1} in P_q. Write ξ for the canonical line bundle over P_r, for any r. Then clearly this normal bundle is ξ, and its classifying map $P_{q-1} \to B\underset{\sim}{O}(1)$ is simply inclusion $P_{q-1} \subset P_\infty$.

Let us be more general. The normal bundle η of P_{m-1} in P_{m+n-1} is the Whitney sum of n copies of ξ. Clearly η extends over P_{m+n-1} in the obvious way. We therefore obtain <u>two</u> maps from P_{m+n-1} to P^η_{m+n-1}; the first is inclusion of the base of the Thom complex, and the second is the composite

$$P_{m+n-1} \to P^\eta_{m-1} \subset P^\eta_{m+n-1}.$$

<u>9.2 Lemma</u> These two maps are homotopic. Moreover, they are <u>equivariantly</u> homotopic with respect to the obvious action of $O(m) \times O(n)$ on the pair (P_{m+n-1}, P_{m-1}), if we also make $O(n)$ transform the n copies of ξ in η.

<u>Proof</u> This becomes clear if we note that $P^{\eta}_{m+n-1} = P_{m+n+n-1}/P_{n-1}$, and work with the two obvious inclusion maps $R^m \times R^n \subset R^m \times R^n \times R^n$ omitting either factor R^n; these maps are plainly equivariantly homotopic with respect to the actions of $O(m) \times O(n)$.]]]

In our case, the required map $P_q \to MO(1)$ is homotopic to the composite $P_q \subset P_\infty = BO(1) \subset MO(1)$. The inclusion $BO(1) \subset MO(1)$ gives W_1, by definition 8.1.

<u>9.3 Lemma</u> The Smith homomorphism Δ is given by $\Delta x = x \cap W_1$.]]]

<u>The bordism J-homomorphism</u>

Take a smooth vector bundle ξ over a manifold X^i with fibre dimension n; such are classified up to bordism by $N_i(BO(n)^0)$. Its unit sphere bundle Y^{i+n-1} when equipped with the antipodal involution represents an element $x \in N_{i+n-1}(BO(1)^0)$.

<u>9.4 Definition</u> The <u>bordism J-homomorphism</u>

$$J_n : N_i(BO(n)^0) \to N_{i+n-1}(BO(1)^0)$$

is defined by taking the class of the bundle ξ to x.

Now consider X as a singular manifold of $BO(n)$. We see that Y is the covering singular manifold obtained by the

construction 6.18 of the bundle transfer for the universal ca
over $B\mathbf{O}(n)$. Let E be a universal $\mathbf{O}(n)$-space, and put
$B\mathbf{O}(n) = E//\mathbf{O}(n)$. Then the universal case is the Borel fibre
bundle [B2]

$$P_{n-1}(\mathbf{R}) = \mathbf{O}(n)//\{\mathbf{O}(n-1) \times \mathbf{O}(1)\} \to E//\{\mathbf{O}(n-1) \times \mathbf{O}(1)\} \to E/$$

induced by the inclusion $\mathbf{O}(n-1) \times \mathbf{O}(1) \subset \mathbf{O}(n)$. Now
$E//\{\mathbf{O}(n-1) \times \mathbf{O}(1)\} \simeq B\mathbf{O}(n-1) \times B\mathbf{O}(1)$, and the antipodal invol
on the sphere bundle $E//\mathbf{O}(n-1)$ is classified by the projectio
$B\mathbf{O}(n-1) \times B\mathbf{O}(1) \to B\mathbf{O}(1)$.

Let us write this Borel bundle as

9.5 $\quad P_{n-1}(\mathbf{R}) = \mathbf{O}(n)//\{\mathbf{O}(n-1) \times \mathbf{O}(1)\} \to B\mathbf{O}(n-1) \times B\mathbf{O}(1) \underset{\pi}{\to} B\mathbf{O}($

Then we have proved

9.6 Theorem $\quad J_n : \mathbf{N}_*(B\mathbf{O}(n)^0) \to \mathbf{N}_*(B\mathbf{O}(1)^0)$ is the composite o
the transfer homomorphism $\pi^\natural$ of 9.5 with the homomorphism
$\mathbf{N}_*(\{B\mathbf{O}(n-1) \times B\mathbf{O}(1)\}^0) \to \mathbf{N}_*(B\mathbf{O}(1)^0)$ induced by projection.

For the usual reasons, we would prefer to have multipli
structure available, by means of a similar homomorphism in $\mathbf{N}^*$
For transfer homomorphisms, this can be defined, if we use th
Grothendieck form 6.21 of the bundle transfer.

9.7 Definition $\quad$ The cobordism J-homomorphism

$$J^n : \mathbf{N}^i(B\mathbf{O}(1)^0) \to \mathbf{N}^{i-n+1}(B\mathbf{O}(n)^0)$$

is defined as the composite of the transfer homomorphism $\pi_\natural$ w
the homomorphism induced by the projection $B\mathbf{O}(n-1) \times B\mathbf{O}(1) \to$

Then we have the multiplicative properties 6.2.

We shall need to compare the homomorphisms J_n for different values of n.

9.8 Lemma   Write $i:BO(n) \subset BO(n+1)$ for the map induced by inclusion. Then $J_n x = \Delta J_{n+1} i_* x$ for $x \in \underline{\underline{N}}_*(BO(n)^0)$.

Proof   This can be seen directly from the geometric definitions. It appears as Theorem 26.4 of [C5].]]]

9.9 Corollary $J^n \alpha = i^* J^{n+1}(\alpha \cup W_1)$ for $\alpha \in \underline{\underline{N}}^*(BO(1)^0)$.

Proof   9.8, with 9.3, 9.7, and 6.2, shows that $\langle x, J^n\alpha\rangle = \langle x, i^* J^{n+1}(\alpha \cup W_1)\rangle$ for all $x \in \underline{\underline{N}}_*(BO(n)^0)$. The result follows, since we know the structure of $\underline{\underline{N}}^*(BO(n)^0)$, and (by e.g. [C5]) $\underline{\underline{N}}_*(BO(n)^0)$.

Still using cap products, we can obtain a very precise relation between J_m and J_{m+n}. Let π_1 be the bundle induced from 9.5 for m + n as follows:

$$\begin{array}{ccc} E_1 & \longrightarrow & BO(m+n-1) \times BO(1) \\ \downarrow \pi_1 & & \downarrow \pi \\ BO(m) \times BO(n) & \longrightarrow & BO(m+n). \end{array}$$

Write q for the composite $E_1 \to BO(m+n-1) \times BO(1) \to BO(1)$, and $p:BO(m) \times BO(n) \to BO(m)$ for the projection.

9.10 Lemma $J_m p_* x = q_* (\pi_1^{\natural} x \cap \alpha)$ for $x \in \underline{\underline{N}}_*(\{BO(m) \times BO(n)\}^0)$, where the element $\alpha \in \underline{\underline{N}}^*(E_1^0)$ is induced from the Stiefel-Whitney cobordism class $W_n \in \underline{\underline{N}}^*(BO(n)^0)$ by means of the maps

$$E_1 \to BO(n) \times BO(1) \to BO(n),$$

where the first is obtained from π_1 and q, and the second is induced by $\otimes : \underset{\sim}{O}(n) \times \underset{\sim}{O}(1) \to \underset{\sim}{O}(n)$.

Proof Geometrically, E_1 is a bundle over $B\underset{\sim}{O}(m) \times B\underset{\sim}{O}(n)$ with fibre $P_{m+n-1}(\underset{\sim}{R})$, containing a subbundle E_2 with fibre $P_{m-1}(\underset{\sim}{R})$ and projection π_2, say. Given a singular manifold $X \to B\underset{\sim}{O}(m) \times B\underset{\sim}{O}(n)$ representing x, the construction 6.18 of the bundle transfers gives singular manifolds of E_1 and E_2 which yield in $\underline{\underline{N}}_*(B\underset{\sim}{O}(1)^0)$ representatives for $q_*\pi_1^{!}x$ and $J_m p_* x$ respectively.

By 5.9 applied over X, we obtain the required formula, where $\alpha : E_1 \to M\underset{\sim}{O}(n) \to M\underset{\sim}{O}$ is the Thom map of the normal bundle of E_2 in E_1. Write ξ and η for the universal line and vector bundles over $B\underset{\sim}{O}(1)$ and $B\underset{\sim}{O}(n)$. Then the normal bundle of E_2 in E_1 is $(\pi_1^*\eta \otimes q^*\xi)|E_2$. This time making strong use of 9.2, we see that the Thom map we require is homotopic to the classifying map $E_1 \to B\underset{\sim}{O}(n)$ of $\pi_1^*\eta \otimes q^*\xi$, followed by $B\underset{\sim}{O}(n) \subset M\underset{\sim}{O}(n)$. The latter gives W_n, by 8.6.]]]

9.11 Corollary $\pi_{1!}(q^*\beta \cup \alpha) = p^* J^m \beta$ for $\beta \in \underline{\underline{N}}^*(B\underset{\sim}{O}(1)^0)$.

Proof This dual result is obtained in the same way as 9.9.]]]

These results will enable us to carry out computations in VI.

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