Isovariant mappings of degree 1 and the Gap Hypothesis

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Abstract Unpublished results of S. Straus and W. Browder state that two notions of homotopy equivalence for manifolds with smooth group actions — isovariant and equivariant — often coincide under a condition called the Gap Hypothesis; the proofs use deep results in geometric topology. This paper analyzes the difference between the two types of maps from a homotopy theoretic viewpoint more generally for degree one maps if the manifolds satisfy the Gap Hypothesis, and it gives a more homotopy theoretic proof of the Straus-Browder result.

AMS Classification 55P91, 57S17; 55R91, 55S15, 55S91

Keywords Blakers-Massey Theorem, deleted cyclic reduced product, diagram category, diagram cohomology, equivariant mapping, Gap Hypothesis, group action, homotopy equivalence, isovariant mapping, normally straightened mapping

In order to motivate and explain our results more thoroughly, we shall begin with various pieces of background material. Readers who would rather focus on the main results and their proofs may go directly to Section 2.

1 Background

Ever since the topological classification of surfaces was discovered, one basic theme in geometric topology has been the reduction of existence and classification questions for manifolds to problems in algebraic topology. A collection of techniques known as *surgery theory* has been particularly effective in this regard (compare [36] , pp. 375–378). For well over four decades topologists have also known that such techniques also have far reaching implications for manifolds with group actions (*cf.* [9] and [36], pp. 378–379). Not surprisingly, many of the most striking applications of surgery theory require some assumption on the manifolds, mappings or structures under consideration, and for group actions the following restriction has been employed quite often:

Standard Gap Hypothesis *isotropy subgroups* $H \supsetneqq K$ *and each pair of components* $B \subset M^H$, $C \subset M^K$ *such that* $B \subsetneqq C$ *we have* $\dim B + 1 \leq (\dim C)/2$.

A condition of this sort first appeared explicitly in the unpublished Berkeley doctoral dissertation of Sandor H. Straus [**?**], and the importance and usefulness of the restriction became apparent in work of T. Petrie ([32] and [33]; see also Dovermann-Petrie [16], Dovermann-Rothenberg [17], and Lück-Madsen [28]). The reason for this is that the Gap Hypothesis automatically implies that certain maps are **isovariant** (*cf.* [31]): A mapping of $f : X \to Y$ of spaces with actions of a group G is said to be isovariant if it is equivariant — so that $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$ — and for each x the isotropy subgroup G_x of all group elements fixing x is equal to the isotropy subgroup $G_{f(x)}$ of the image point (in general one can only say that the first subgroup is contained in the second). Specifically, surgery theory yields equivariant mappings of the form $G/K \times S^q \to G \cdot M^K$ where $q \leq \frac{1}{2} \dim M^K$, and one formulation of the crucial point is that such maps are equivariantly homotopic to isovariant maps by general position if the Gap Hypothesis holds; usually this is expressed in other terms (*e.g.*, the image of $\{H\} \times S^q$ lies in $G \cdot M^H$ and is disjoint from $G \cdot M^L$ for all isotropy subgroups properly containing a conjugate of H), but it is straightforward to verify that the usual version is equivalent to the one stated here in terms of isovariance. Applications of surgery to group actions that do not require the Gap Hypothesis frequently assume that the underlying maps of manifolds are isovariant or *almost isovariant* (*cf.* Browder-Quinn [11], the final section of Dula-Schultz [19], the second part of [39], and Weinberger's book [47]). The general notion of almost isovariance is defined precisely on page 27 of [19], and the most important special case is reproduced below. For the time being, we merely note that

(1) the sets of nonisovariant points ($G_x \neq G_{f(x)}$) for such a map may be pushed into very small pieces of the domain where they cause no problems,

(2) standard methods of homotopy theory extend directly to a suitably defined category of almost isovariant mappings (*cf.* [19] and [**?**]),

(3) results of [19] show that almost isovariant homotopy and isovariant homotopy are equivalent in many important cases (including all smooth actions of finite cyclic p-groups), and a standard conjecture (believed by most workers in the area) states that the same is true for arbitrary smooth actions of finite groups.

In fact, isovariant techniques play a central role in classification results for group actions when the Gap Hypothesis fails; in many cases where such machinery is not used explicitly, the work can readily be interpreted in these terms. A fundamentally important breakthrough in this area was due to Browder and Quinn [11] (see also the commentary on the latter in [21]), and a more general discussion of the situation in the smooth category — which also extends earlier work on the smooth classification of topologically linear actions on spheres — appears in Section II.1 of [39] (see also the final section of [19]). In the piecewise linear and topological categories, there is a distinct body of results which is largely based on techniques from controlled topology (*cf.* Weinberger's book [47], the survey articles by Hughes-Weinberger [21] and Cappell-Weinberger [12], and the doctoral dissertation of A. Beshears [**?**]). A full historical account of the topic is beyond the scope of this paper, but some additional references include the work of A. Assadi with Browder [2] and P. Vogel [3], the monograph by L. Jones on symmetries of disks [24], and the material on symmetries of aspherical manifolds in Weinberger's paper on higher Atiyah-Singer invariants [48].

Focus of this paper

In this paper we are particularly interested in the following unpublished result,

which is due to Straus [**?**] for actions that are semifree (the group acts freely off the fixed point set) and W. Browder [10] more generally. It implies a fairly strong, general and precise connection between almost isovariance and the Gap Hypothesis.

Theorem 1.1 *Let $f : M \to N$ be an equivariant homotopy equivalence of oriented closed smooth G-manifolds that satisfy the Gap Hypothesis. Then f is equivariantly homotopic to an almost isovariant homotopy equivalence.*

As noted ebove, in some cases the results of [19] allow one to replace "almost isovariant" by "isovariant" in the conclusion; in particular, this is true if the isotropy subgroups are normal and linearly ordered by inclusion. Although

Theorem 1.1 is a purely homotopy theoretic statement, the proofs in [**?**] and [10] require fairly deep results from Wall's nonsimply connected surgery theory [45], which in turn depends upon other deep geometric results such as the classification theory for immersions (*cf.* Phillips [34]) and the Whitney process for geometrically eliminating pairs of double points that cancel algebraically (*cf.* Milnor's monograph on the h-cobordism theorem [30]). It is natural to ask

whether one can prove Theorem 1.1 without relying so extensively on such a large amount of auxiliary material (this is a special case of the classical scientific maxim called **Ockham's Razor**). In particular, since one can construct a version of obstruction theory for isovariant maps and define obstructions to finding an isovariant deformation of a given equivariant map [19], it is natural to search for a proof that is related to this obstruction theory. More generally, one would also like to understand the obstructions to isovariance for arbitrary equivariant mappings of degree ± 1 from one smooth manifold to another. Some basic test cases are examples of equivariant degree one mappings mentioned in [10] which are not equivariantly homotopic to isovariant maps; results of K. H. Dovermann on isovariant normal maps [15] also provide some motivation. The main objective of this paper is to analyze the problem of deforming an

equivariant degree one map into an isovariant map when the Gap Hypothesis holds, to use this criterion to provide an essentially homotopy-theoretic proof of Theorem 1.1, and to see how the criterion applies to equivariant homotopy equivalences and other basic examples. In contrast to [**?**] and [10], our approach requires a minimum of input from geometric topology; namely, **nonequivariant** transversality and standard results on smooth embeddings in the general position range. For the sake of clarity we shall restrict attention to finite group actions that are semifree in the sense described above; if G is cyclic of prime order, then all actions are semifree. We shall also discuss some applications of Theorem 1.1 to cyclic reduced products that were first considered in [**?**] and a few positive and negative results just outside the range of the Gap Hypothesis (further information on the latter will appear in sequels to this paper).

Acknowledgments I am extremely grateful to Bill Browder for helpful conversations and correspondence regarding his results on the questions treated here, and especially for providing a detailed account of his counterexamples that appear in Section 4 of this paper. I would also like to thank Heiner Dovermann for various conversations involving his work. Comments on the referee for enhancing the exposition of the paper were also valuable and appreciated. The research in this paper was partially supported by National Science Foundation Grants DMS 86-02543, 89-02622 and 91-02711, and also by the Max-Planck-Institut für Mathematik in Bonn, and sources are also gratefully acknowledged.

2 Statements of main results

Suppose that M and N are compact, oriented, semifree unbounded smooth G-manifolds satisfying the Gap Hypothesis such that all components of the fixed point sets, and suppose that $f : M \to N$ is a G-equivariant map of degree 1. Let $\{N_\alpha\}$ denote the set of components of the fixed point set N^G, where we may as well assume that α runs through the elements of $\pi_0(N^G)$, and suppose that the associated map f^G of fixed point sets defines a $1-1$ correspondence between the components of M^G and N^G; for each α let

$$M_\alpha \;=\; f^{-1}(N_\alpha) \cap M^G$$

and let $f_\alpha : M_\alpha \to N_\alpha$ denote the partial map of fixed point sets determined by f. Denote the equivariant normal bundles of M_α and N_α in M and N by ξ_α and ω_α respectively, and let $S(\nu)$ generically represent the unit sphere bundle of the vector bundle ν (with the associated group action if ν is a G-vector bundle).

Theorem 2.1 *Suppose we are given the setting above such that* $\dim M_\alpha = \dim N_\alpha$ *for each* α.

(*i*) *If* f *is homotopic to an isovariant map, then for each* α *the map* f_α *has degree* ± 1, *and* $S(\xi_\alpha)$ *is equivariantly fiber homotopy equivalent to* $S(f^*\omega_\alpha)$.

(*ii*) *If the two conditions in the preceding statement hold, then* f *is equivariantly homotopic to a map that is isovariant on a neighborhood of the fixed point set.*

(*iii*) *If* f *is isovariant on a neighborhood of the fixed point set, then* f *is equivariantly homotopic to an isovariant map if and only if* f *is equivariantly homotopic to a map* f_1 *for which the set of nonisovariant points of* f_1 *is contained in a tubular neighborhood of* M^G.

Theorem **??** will follow immediately from Theorem 2.1 and the specialization of the the latter to equivariant homotopy equivalences.

Theorem 2.2 *In the setting of the previous result, suppose that* f *is an equivariant homotopy equivalence. Then* f *is equivariantly homotopic to an isovariant homotopy equivalence.*

Note. If f is an equivariant homotopy equivalence, then some of the assumptions in the setting and statement of Theorem 2 are redundant because an equivariant homotopy equivalence determines a 1–1 correspondence of fixed point set components and the dimensions of corresponding components are also equal because the components are homotopy equivalent. *Application to deleted reduced prod*

Given a topological space X and an integer n, the n-fold cyclic reduced prod-

uct is defined to be the quotient of the product space X^n (*i.e.*, n copies of X) modulo the action of $\mathbb{Z}_n$ on the latter by permuting coordinates, and the deleted cyclic reduced product is the subset of the latter obtained by removing the image of the diagonal $\Delta_X^{[n]}$ consisting of those points whose coordinates are all equal. In his thesis [**?**] Straus used his version of Theorem 1.1 to obtain the following homotopy invariance property for such spaces:

Theorem 2.3 *Let M and N be closed smooth manifolds of dimension ≥ 2, let p be an odd prime, and suppose that M and N are homotopy equivalent. Let $\mathbb{Z}_p$ act smoothly on the p-fold self products $\Pi^p M$ and $\Pi^p N$ (where $\Pi^p X = X \times \cdots \times X$, p factors) by cyclically permuting the coordinates, and let $\mathbf{D}^p(M)$, $\mathbf{D}^p(N)$ be the invariant subsets sets given by removing the diagonals from $\Pi^p M$ and $\Pi^p N$. Then the deleted reduced cyclic products $\mathbf{D}^p(M)/\mathbb{Z}_p$ and $\mathbf{D}^p(N)/\mathbb{Z}_p$ are homotopy equivalent.*

As also noted in [**?**], this result does not extend to compact bounded manifolds, and in fact closed unit disks of different dimensions generate simple counterexamples. The results of [38] imply that Theorem 5 extends to simply connected manifolds if $p = 2$, but recent results of R. Longoni and P. Salvatore [27] imply that the result does not extend to 3-dimensional lens spaces when $p = 2$. Further results on the relationship between $\mathbf{D}^2(M)/\mathbb{Z}_2$ and $\mathbf{D}^2(N)/\mathbb{Z}_2$ for homotopy equivalent manifolds appear in a paper by P. Löffler and R. J. Milgram [26]. Since statement and proof of Theorem 4 have apparently not appeared

previously in print, we shall outline the (fairly straightforward) argument for the sake of completeness.

Sketch of proof If $f : M \to N$ is a homotopy equivalence then $\Pi^p f : \Pi^p M \to \Pi^p N$ is an equivariant homotopy equivalence of closed smooth $\mathbb{Z}_p$-manifolds. All actions of $\mathbb{Z}_p$ are semifree if p is prime, so this condition holds

automatically; the Gap Hypothesis also holds because $p > 2$. Therefore Theorem 1 implies that $\Pi^p f$ is equivariantly homotopic to an isovariant homotopy equivalence, and the latter in turn yields an equivariant homotopy equivalence from $\mathbf{D}^p(M)$ to $\mathbf{D}^p(N)$. The induced map of orbit spaces is the desired homotopy equivalence from $\mathbf{D}^p(M)/\mathbb{Z}_p$ to $\mathbf{D}^p(N)/\mathbb{Z}_p$. □

Further results

The results of [10] also include a uniqueness statement (up to isovariant homotopy) if $M \times [0,1]$ and $N \times [0,1]$ satisfy the Gap Hypothesis. One can also use the methods of this paper together with some additional geometric and homotopy theoretic input to prove such a uniqueness result. The necessary machinery to do so will be developed in a subsequent paper.

Since the results of [10] also apply to actions that are not necessarily semifree, it is natural to ask whether the methods of this paper extend. The answer is yes, but a proof would require the introduction of a considerable amount of extra machinery that would take time to develop and might obscure the main ideas, and this is a major reaon for sticking with the semifree case. A brief discussion of some tools needed to carry out such extensions appears at the end of Section 3, and we plan to pursue this further in another sequel to this article.

Overview of the paper

We shall begin Section 2 by proving that the conditions in Theorem 2 are

necessary for a map f as above to be properly homotopic to an isovariant map. The proof of sufficiency in Theorem 2 splits into two steps, which are carried out in Sections 3 and 4. To motivate the first step, observe that an equivariant map of semifree G-manifolds is automatically isovariant on the fixed point set, so a natural starting point is to determine whether the given map can be equivariantly deformed to a map that is isovariant on a neighborhood of the fixed point set. If this is possible and we have a map with this additional property, the next step is to determine whether such a map can be further deformed to another one which is isovariant everywhere. Section 5 contains the proofs of Theorems 1.1 and 2.2. Finally, in Section 6 we shall discuss a variety of questions related to the main results. Some concern the interaction between equivariance and isovariance when the Gap Hypothesis holds, while others involve borderline situations in which the Gap Hypothesis inequalities fail, but only by a small amount. There are obvious questions about the extent to which Theorem 1.1 extends to such examples, and answers to such questions

turn out to have close connections to issues in equivariant surgery theory in such borderline cases. Sometimes the latter yields generalizations of Theorem 1.1 to situations where the conclusion is not particularly obvious from a homotopy theoretic viewpoint, and in other cases homotopy theory has implications for equivariant surgery theory.

3 Preliminary adjustments and necessity

ESUME INSERTIONS HERE

The `latex` document class file `gtart.cls` is intended for formatting articles for submission to *Geometry & Topology* and *Algebraic & Geometric Topology*. You are welcome to adapt it for any other purpose, but if you do adapt the file, then *please change its name*. If you use a modified version for formatting an article submitted to *Geometry & Topology* or *Algebraic & Geometric Topology*, then please submit the modified file as well.

To use `gtart.cls` collect a copy of the file [**?**] from the `gt/info/macros` subdirectory; install it on your system where `tex` can find it (eg in your working directory) and start your file with the line:

```
\documentclass{gtart}
```

Then type your file using standard `latex` structure. If you wish to use `amslatex` commands, add the following line:

```
\usepackage{amsmath,amssymb}
```

and then type your file exactly as if you were using `amsart` style.

If you are using `gtart.cls` to format an article for *Geometry & Topology* or *Algebraic & Geometric Topology* then:

> *Do not use any commands which alter page height or width or line-spacing or the headline or footline* eg `\texheight`, `\textwidth` or `\baselinestretch` or `\pagestyle`.

In use `gtart.cls` is virtually transparent and all the usual commands from `latex` article class and `amsart` class, respectively, are available. However there are differences in the layout of the title material for *Geometry & Topology* and *Algebraic & Geometric Topology* articles over the standard layout in `latex`; the commands are much the same, but you may need to adjust them to get a reasonnable layout. One point to note is that, with `gtart.cls`, the command

`\maketitle` must come after the abstract, keywords and AMS classification numbers. For details here see section 2 below.

`gtart.cls` uses the `amsthm` package to set theorem style. If you are not already familiar with the facilities provided by this package then read section 3. The package allows you to use theorem-like environments in a very flexible way.

Important note `gtart.cls` can only be used with `latex2e`. If you are still using the original `latex` (now known as `latex2.09`) then you are strongly advised to upgrade your installation; `latex2e` is a much improved format.[1]

The base style for `gtart.cls` is `latex` 11pt article style and essential use is made of the `amsthm` package (part of the `amslatex` distribution). This package needs one other package from the `amslatex` distribution, namely `amsgen`. Thus in order to use `gtart.cls` you need `latex2e` and the two files `amsthm.sty` and `amsgen.sty`. You will probably find that these two files are already installed as part of the `latex2e` installation that you are using. If not, copies of both files can be collected from the `gt/info/macros` subdirectory [**?**].

Using `gtart.cls` to reformat existing articles

`gtart.cls` is also intended to act as a "drop-in" replacement for both `latex {article}` and `amslatex {amsart}` styles, ie to reformat an existing article written in either of these styles.

To use `gtart.cls` to reformat an article written in `latex {article}` style, collect a copy and install it as above. Then replace the
`\documentclass[XX]{article}`
declaration at the start of your file (where `[XX]` is eg `[12pt]`) with the line:

`\documentclass{gtart}`.

To use `gtart.cls` as a replacement for `{amsart}` class collect and install a copy and then replace `\documentclass{amsart}` at the start of your file with the *two* lines:

```
\documentclass{gtart}
\usepackage{amsmath,amssymb}.
```

If you are using `gtart.cls` to format an article for *Geometry & Topology* or *Algebraic & Geometric Topology* then:

[1] If you cannot upgrade, then use the template `gtlatex.old` [**?**], which contains basic instructions on formatting articles for *Geometry & Topology* and *Algebraic & Geometric Topology* using `latex2.09`.

Delete all commands which alter page height or width or line-spacing
eg `\texheight`, `\textwidth` or `\baselinestretch`.

Apart from these changes, you may find that the title page or header material needs adjustment and you may wish to take advantage of the facilities provided by the `amsthm` package. For details here, see sections 2 and 3 respectively.

If you are using `gtart.cls` to reformat an existing article then you will probably have to deal with overful `\hbox`'s caused by the new line widths. *Please do not redefine* `\textwidth` *to overcome this problem.* Deal with overful `\hbox`'s by using optional hyphens, reordering the wording, replacing 1–line displays with 2–line displays etc. For details consult a `tex` or `latex` primer.

There are several useful abbreviations that are provided by `gtart.cls` which we list below. All these commands can redefined if they clash with your own `\newcommands`. There are however a few new commands which must not be redefined (or the final format of the article will be incorrect). Here are lists of both:

Commands which *must not* be redefined

`\stdskip` : the standard vertical skip,
`\stdspace` : the standard horizontal space,
`\ppar` : principal paragraph break (roughly the same as `\par\medskip`,)
`\sqr` : the square macro used to generate the end-of-proof symbol.

The journal titles :
`\gt` : *Geometry & Topology*,
`\gtm` : *Geometry & Topology Monographs*,
`\agt` : *Algebraic & Geometric Topology.*

And also the ingredients of the header material namely `\title`, `\author` etc (see next section).

Commands which you may find useful but which may be redefined

`\nocolon` : which suppresses the colon in the `{figure}` environment (use it before `\caption`'s which contain no text),

`\qua` : abbreviation for `\stdspace` — three-quarters of a quad,

`\sh` : for subheadings, which produces a subheading in the correct style without a number. Eg the above heading was typed:
`\sh{Commands which ... redefined}`

`\rk` : for remarks, notes etc, which produces a subheading in the correct style without a number such that the text follows on directly. Eg the note on page 2 was typed:
`\rk{Important note} "gtart.cls" can only be used ...`

`\sq` : abbreviation for `\sqr55` (the end-of-proof symbol) which can also be called via `\qed`,
`\co` : abbreviation for `\colon\thinspace` which is a colon with correct spacing for maps,

`\np` : abbreviation for `\newpage`,
`\nl` : abbreviation for `\hfil\break` ie new line,
`\cl` : abbreviation for `\centerline`.

4 Header material

Geometry & Topology article style has a separate title page as first page of each article (always an odd numbered page) which contains the title, authors names and addresses, abstract, AMS classification numbers and keywords. The journal adds other information (publication date, page range etc) to this page when the article is published.

Algebraic & Geometric Topology article style has a shorter header containing just the title, authors names, abstract (set small), AMS classification numbers and keywords. The journal adds publication date and page range at publication. Authors addresses and the received date are given at the end of the article.

The layout of the *Geometry & Topology* header material is controlled by the `\maketitlepage` macro. To generate the title page, replace the normal `latex` command `\maketitle` with `\maketitlepage` which must come *after* the abstract, AMS classification numbers and keywords.

When the article is published, the journal will take the same information into a modified `\maketitlepage` macro which then adds the other information and the journal logo, and creates the published title page.

The layout of the *Algebraic & Geometric Topology* header material is controlled by the `\makeagttitle`. To generate the header, replace the normal `latex` command `\maketitle` with `\makeagttitle`, which again must come *after* the abstract, AMS classification numbers and keywords. You can also force `latex` to use *Geometry & Topology* or *Algebraic & Geometric Topology* style for the header material and also to produce running heads and feet similar to those

which appear on the published article. This allows you to check the final appearance of your article. To do this type `\gtart` or `\agtart` near the start of the file (after `\documentclass{gtart}`). You do not need to replace your `\maketitle` command if you do this.

The information for the header material for both journal layouts is supplied in exactly the same way, which we now discuss.

Header information can be suplied in a number of equivalent ways, including both `latex {article}` style and `amsart` style. Here is a complete sample:

```
\begin{document}

\title[A short spoof paper]{A short spoof paper\\
with a two-line title}
\authors{Albert Einstein\\Leonardo da Vinci}

\address{IAS, Princeton}
\secondaddress{Renaissance\\Venice}
\email{ae@ias.princeton.edu}
\secondemail{ldv@ren.ven.hist}

\begin{abstract}

This is a short spoof paper typed on the template
{\tt gtlatex.tem} purely in order to illustrate its use.
As this is a rather short summary, here is some nonsense to
spin it out.

We prove that colours come in fifteen primary hyperbands; each
hyperband being supported by a James product \cite{James} and
controlled by a colour changing macro \cite{Knuth}.

\end{abstract}

\primaryclass{00-01, 00-02}
\secondaryclass{68-00, 68-01}
\keywords{Spoof, template, colour, hyperband, James product}

\maketitlepage
```

You can see the result by looking at the title page of [**?**] which is one of our sample files.

We now examine this sample in detail. First a general point; the control sequence `\\` is used as standard "separator" — to separate lines in `\title` and

\address, and to separate multiple authors, email addresses and URL's (see below). Turning now to the items in order. The title has an an optional parameter (in square brackets) which is the "short title" to be used for running heads. If this is not given, then the full title is used instead. You can also supply the short title separately thus:

```
\title{A short spoof paper\\with a two-line title}
\shorttitle{A short spoof paper}
```

The authors and addresses can be supplied singly or combined and in any order. Thus the following produces exactly the same effect:

```
\author{Albert Einstein}
\address{IAS, Princeton}
\email{ae@ias.princeton.edu

\secondauthor{Leonardo da Vinci}
\secondaddress{Renaissance\\Venice}
\secondemail{ldv@ren.ven.hist}
```

Web addresses (URL's) can also be given:

```
\url{http://www.ias.edu/~al\\http://www.ven.hist/~lvd}
```

and there are a variety of aliases. Eg \author, \authors, \secondauthor are all aliases thus

```
\secondauthor{Leonardo da Vinci}\author{Albert Einstein}
```

produces exactly the same result as

```
\authors{Leonardo da Vinci\\Albert Einstein}
```

with Leonardo first ! Similarly \addresses and \address are aliases, as are \emails, \email, \emailaddr and \emailaddress and \url, \urls, \urladdr and \urladdress. However \secondaddress, \secondemail and \secondurl behave slightly differently. In each case they produce the word "and" to separate the adresses. They are specifically intended for when there are precisely two authors (as in the example above). If there are three or more authors then either supply the information in one set or in three similar sets. Thus:

```
\authors{John Major\\Margaret Thatcher\\Harold Wilson}
\addresses{First\\Address\smallskip Second\\Address etc}
\emails{jm@11.downing.st\\mt@11.downing.st\\hw@11.downing.st}
```

or

```
\author{John Major}\address{First\\Address}\email{jm@11.d.s}
\author{M Thatcher}\address{First\\Address}\email{mt@11.d.s}
\author{H Wilson}\address{First\\Address}\email{hw@11.d.s}
```

Important note `\secondaddress`, `\secondemail` and `\secondurl` must not be used *except* as the final (usually the second) address etc. The macros will generate some very odd looking `tex` errors if you ignore this advice!

The author and address layout works well for one or two authors and (if addresses are short) for three authors. After that you will probably need to take some control "by hand". For example here is an excerpt from the title page information for a five-author paper which was formatted using `gtart.cls`:

```
\title{Ordering the braid groups}
\authors{Roger Fenn, Michael T~Greene, Dale Rolfsen\\
Colin Rourke, Bert Wiest}
\shortauthors{Fenn, Greene, Rolfsen, Rourke and Wiest}

\address{%
\rm Email:\stdspace \tt R.A.Fenn@sussex.ac.uk\ \
Michael.Greene@uk.radan.com\\rolfsen@math.ubc.ca\ \
cpr@maths.warwick.ac.uk\\ bertold@maths.warwick.ac.uk}
```

Here the five authors are displayed on two lines and only email addresses are given on the title page, with formatting controlled by hand. You can see the result at [**?**].

The abstract must be given inside the `abstract` environment as in the above example and finally AMS classification numbers (both primary and secondary) and keywords should be given. For compatibility with `amsart` style, `\subjclass` is an alias for `\primaryclass`, but at this point the file will probably have to be edited by hand, to separate out the primary and secondary classification numbers.

Notes (1) *Geometry & Topology* and *Algebraic & Geometric Topology* styles do not put thanks or dedications with the header material. Instead, you are recommended to collect these together with any other acknowlegdements at the end of your introduction.

If you use the commands `\thanks` or `\dedicatory`, you will get a helful message to this effect, and your thanks or dedications will not be printed.

(2) If you use multiple `\address` declarations then the addresses must contain only text and no control sequences other than `\\` or primitive commands. Thus if you want to format the addresses yourself, then make one single address including commands to specify your layout (as in the five-author example above). A similar remark applies to `\email` and `\url` declarations. Again you will get some very odd looking `tex` errors if you ignore this advice!

(3) To print author addresses (at the end of the paper for the *Algebraic & Geometric Topology* style) use the command `\Addresses`.

(4) There is a variant of the `\maketitlepage` macro, namely `\makeshorttitle` alias `\maketitle`. This typesets the heading material as a conventional head with author addresses and the abstract set small. If you are using these macros for typesetting an article not necessarily intended for *Geometry & Topology* or *Algebraic & Geometric Topology* then you will probably prefer to use this variant which produces a more conventional preprint. This set of instructions used `\makeshorttitle`.

5 Theorem style

Theorem style is set by the `amsthm` package. If you are familiar with the facilities offered by this package, then skim through this section and start reading again at section 3.

Using `amsthm`, the `\newtheorem` declarations are exactly the same as in other `latex` styles with one exception. There is a variant `\newtheorem*` which provides a theorem-like environment without numbering the theorem. This is useful for remarks, named theorems (eg Zorn's Lemma) etc. `\newtheorem` declarations are controlled by the current "theorem style". Thus if you type:

```
\theoremstyle{definition}
```

then subsequent `\newtheorem` declarations will define theorem-like environments in the "definition" style. There are two predefined theorem styles provided by `gtart.cls`:

(1) `\theoremstyle{plain}` which has heading bold and left justified, optional note bracketed in roman type and statement in slanted type.

(2) `\theoremstyle{definition}` alias `\theoremstyle{remark}` which has heading bold and left justified, optional note unbracketed in slanted type and statement in roman type.

`\theoremstyle{plain}` is recommended for theorems, lemmas, corollaries etc and is the default style, whilst `\theoremstyle{definition}` is recommended for definitions, remarks etc.

Here are some sample `\newtheorem` declarations:

```
\newtheorem{thm}{Theorem}[section]        % (1)
```

```
\newtheorem{lem}[thm]{Lemma}              % (2)
\newtheorem*{zlem}{Zorn's Lemma}          % (3)
\theoremstyle{definition}
\newtheorem{defn}[thm]{Definition}        % (4)
\newtheorem*{rem}{Remark}                 % (5)
```

In order these define:

(1) a standard theorem environment with numbering of the style **7.3** (result number 3 in section 7),

(2) a standard lemma environment with numbering consecutive to theorems,

(3) a special unnumbered lemma called **Zorn's lemma**,

(4) a numbered definition environment with numbering consecutive to theorems and lemmas,

(5) an unnumbered environment for remarks.

Note that (1), (2) and (3) are in default style `theoremstyle{plain}` and this does not need to be declared.

Here are some samples in order:

Theorem 5.1 (Sample (1)) *This is the first theorem.*

Lemma 5.2 (Sample (2)) *This is the first lemma.*

Zorn's Lemma *Sample (3)* Here is Zorn's lemma.

Definition 5.3 *Sample (4)* This is the first definition.

Remark *Sample (5)* This completes the samples.

These were typed:

```
\begin{thm}[Sample (1)]
This is the first theorem.\end{thm}
\begin{lem}[Sample (2)]
This is the first lemma.\end{lem}
\begin{zlem}[Sample (3)]
Here is Zorn's lemma.\end{zlem}
\begin{defn}[Sample (4)]
This is the first definition.\end{thm}
\begin{rem}[Sample (5)]
This completes the samples.\end{rem}
```

You can define your own theorem styles using a `\newtheoremstyle` declaration. Here is an example which you may find useful:

```
\newtheoremstyle{bold}{14pt plus6pt minus6pt}%
{6pt plus3pt minus3pt}{\bf}{}{\bf}{}{1em}%
{\thmname{#1}\thmnumber{#2}\thmnote{\bf\stdspace[#3]}}
```

which defines a theorem style similar to `remark` except that both the optional theorem note and the theorem statememts are set in bold type (with the note enclosed in square brackets) and there is no space between theorem name and number but there is a quad (1em) of space after the theorem head.

The nine parameters are in order: name `{bold}`, space before theorem, space after theorem,[2] font for theorem body (`\bf`), indent (zero), font for theorem head (`\bf`), punctuation (nothing), space after the theorem head (1em) and finally a detailed specification of how to set out the theorem head and optional note. The above sample specifies theorem name first, then the theorem number (with no space between) then (finally and optionally) a `\stdspace` (standard space of 0.75em) followed by the note square bracketed in bold type. For more examples, read the source code for `gtart.cls` where the theorem styles `plain`, `remark` and `definition` are defined.

Here is a example of the use of the `bold` style to provide subsection headings with numbering consecutive with theorems etc:

5.4 [Sample (6)] Sample subsection head

which was typed:

```
\theoremstyle{bold}
\newtheorem{sshead}[thm]{}
\begin{sshead}[Sample (6)]Sample subsection head\end{sshead}
```

The point is that the `\newtheorem` declaration provides an empty theorem name so the number comes directly at the left margin.

Proof style

The `amsthm` package also provides a flexible environment for proofs. If you type:

```
\begin{proof}
This is a sample proof.
\end{proof}
```

this is the result:

[2] Note the `\parskip` (6pt) gets added to space after, so these spaces are nearly the same!

Proof This is a sample proof. □

But you can change the name of the proof without altering the layout by inserting an optional proof name thus:

```
\begin{proof}[Proof of sample theorem]
This is a sample proof.
\end{proof}
```

with this result:

Proof of sample theorem This is a sample proof. □

6 Bibliography style

Geometry & Topology has an established style for setting out the bibliography. References are numbered in alphabetical order. Authors are set in bold face. The principal title of the reference is set in italic face and all then rest is set in standard face. See any of our sample files for examples. If you use `bibtex` then you can automatically set your bibliography in house style by using the *Geometry & Topology* bibliography style file `gtart.bst` [**?**].

Algebraic & Geometric Topology has a more relaxed bibliography style but authors are still recommended to use `gtart.bst` if possible.

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