EQUIVARIANT NORMAL INVARIANTS AND ISOVARIANT HOMOTOPY EQUIVALENCES

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ABSTRACT. An equivariant analog of the usual normal invariants in surgery is defined and applied to isovariant homotopy equivalences. In some cases one can use this invariant to show that certain equivariant homotopy equivalences are not representable by isovariant homotopy equivalences. A second application complements an earlier result on approximating equivariant homotopy equivalences by isovariant homotopy equivalences if a condition called the Gap Hypothesis holds; specifically, under a very mild strengthening of the latter condition we show that such isovariant approximations are unique up to isovariant homotopy.

One of the basic questions relating geometric and algebraic topology involves classifying suitable families of manifolds up to homotopy type. A closely related question of almost equal importance is to study various automorphism groups associated to manifolds in a given homotopy type, and the most basic problem of this sort is to study the group $\mathcal{E}(M)$ of homotopy classes of homotopy self-equivalences for a given compact manifold M. Such problems often have important consequences in numerous directions. For example, the mapping class groups of oriented surfaces (see [Bi] or [FM]) have many uses, and the structure of such groups and their applications have remained an active area of research for many decades (*cf.* [Fa]). In higher dimensions, there are also some strong global results on the structure of $\mathcal{E}(M)$ in many cases (for example, see Sullivan [Sull] or Wilkerson [Wilk]).

Many of methods and results in topology have natural extensions to spaces equipped with actions of a finite group, and in a reasonably broad range of cases one can effectively study classification and automorphism problems for manifolds with group actions by modifying basic techniques so that the associated group actions are taken into consideration. Such an approach eventually yields several appropriate notions of homotopic equivalence. The most standard is the concept of *equivariant homotopy equivalence* as described in standard references such as [????]. However, for many purposes a slightly stronger concept of **isovariant** homotopy equivalence seems to be more tractable; we recall that a mapping of $f : X \to Y$ of objects with actions of a group G is said to be isovariant if it is equivariant and for each x the isotropy subgroup G_x of (as defined in [Bredon]) is equal to the isotropy subgroup $G_{f(x)}$ of the image point (equivariance only implies that the first subgroup is contained in the second). In particular, it is often more convenient to classify manifolds with group action up to isovariant rather than equivariant homotopy equivalence.

Given that there are two natural notions of homotopic equivalence, the following question arises immediately:

Comparison problem. *Suppose that we are given a "reasonable" manifold M with and action of a finite group G (for example, a smooth action on a compact smooth manifold,* *possibly with boundary). Let $\mathcal{E}_G(M)$ be the group of all equivariant homotopy classes of equivariant homotopy self-equivalences of M, and let $\mathcal{E}_G^{\mathbf{ISO}}(M,\partial)$ be the group of all isovariant homotopy classes of equivariant homotopy self-equivalences of $(M,\partial M)$. How are these groups related?*

To avoid the bookkeeping problems which arise for group actions with many isotropy subgroups, for the remainder of this introduction we shall restrict our attention to actions that are **semifree** (the only isotropy subgroups are $\{1\}$ and G; for an arbitrary prime p, all actions of the cyclic group $\mathbb{Z}_p$ are semifree).

The first step in analyzing the Comparison Problem is simple. Purely formal considerations yield a natural forgetful homomorphism

$$j : \mathcal{E}_G^{\mathbf{ISO}}(M) \longrightarrow \mathcal{E}_G(M)$$

which sends an isovariant homotopy class of maps to its associated equivariant homotopy class of maps. In general, this map is neither injective nor surjective, but the results of [isogap] show that the mapping is surjective if $\partial M = \emptyset$ and a basic condition known as the **Gap Hypothesis** is satisfied (and the group action on M is smooth and semifree). One major purpose of this paper is to obtain the following stronger conclusions:

(1) The homomotphism j is also surjective when ∂M is nonempty.
(2) The mapping j is *also injective* if the Gap Hypothesis holds for $M \times [0,1]$, where we take the trivial group action on the unit interval $[0,1]$.

Statements of the main results

The central result of this paper is an extension of Theorem 1.1 in [isogap] to bounded manifolds.

Theorem I. *Let G be a finite group, let $(M,\partial M)$ and $(N,\partial N)$ be compact, bounded, smooth semifree G-manifolds that satisfy the Gap Hypothesis, and let $f : (M,\partial M) \to (N,\partial N)$ be an equivariant homotopy equivalence that restricts to an isovariant homotopy equivalence from ∂M to ∂N. Then f is equivariantly homotopic to an isovariant homotopy equivalence, and in fact one can choose the homotopy to be constant on ∂M.*

One important consequence of Theorem I is the following uniqueness property for the isovariant homotopy equivalences given by this result and Theorem 1.1 from [Sc9]:

Theorem II. *In the setting of the previous result, suppose that f and g are isovariant homotopy equivalences of pairs from $(M,\partial M)$ to $(N,\partial N)$ that are equivariantly homotopic, and assume further that the Gap Hypothesis holds for $M \times [0,1]$ and $N \times [0,1]$. Then there is an equivariant deformation of the equivariant homotopy between f and g to an isovariant homotopy, and this deformation is fixed on $M \times \{0,1\}$; furthermore, if the original equivariant homotopy is isovariant on $\partial M \times [0,1]$, then one can choose the equivariant deformation so that it is also fixed on $\partial M \times [0,1]$.*

Note that if $X \times [0,1]$ satisfies the Gap Hypothesis then we have

$$\dim(F \times [0,1]) + 1 = \dim F + 2 \leq \tfrac{1}{2}\dim(X \times [0,1]) \leq \tfrac{1}{2}(\dim X + 1)$$

so that

$$\dim F + 1 \leq [\tfrac{1}{2}(\dim X + 1)] - 1 = \tfrac{1}{2}(\dim X - 1) < \tfrac{1}{2}\dim X$$

and hence X also satisfies the Gap Hypothesis. Therefore we obtain the application menntioned earlier.

Corollary III. *Suppose that M is a compact smooth semifree G-manifold where G is a finite group acting semifreely on M. Let $\mathcal{E}_G(M)$ be the group of all equivariant homotopy classes of equivariant homotopy self-equivalences of M, let $\mathcal{E}_G^{\mathbf{ISO}}(M)$ be the group of all isovariant homotopy classes of equivariant homotopy self-equivalences of $(M.\partial M)$, and let $j : \mathcal{E}_G^{\mathbf{ISO}}(M) \to \mathcal{E}_G(M)$ be the forgetful homomorphism. Then j is surjective if M satisfies the Gap Hypothesis and j is bijective if $M \times [0,1]$ satisfies the Gap Hypothesis.*

For the most part, the proofs of Theorems I and II are analogous to the proof of Theorem 1.1 in [isogap2006], the main difference arising from some compatibility issues that must be addressed. WE shall do this by means of an equivariant version of the normal invariants which play a fundamental role in surgery theory (*cf.*,[rourke]). These invariants have some important consequences for the comparison of equivariant and isovariant homotopy equivalences. In addition to their role in proving our main results, they also yield a necessary condition for equivariant homotopy equivalences to be (equivariantly) homotopic to isovariant homotopy equivalences, and in Section 3 we shall use this criterion to equivariant homotopy self-equivalences $f : M \to M$ of suitable G-manifolds M such that f is not equivariantly homotopic to an isovariant homotopy equivalence. It is interesting to consider the similarities and differences between these examples and some others considered in [Kw]; the latter describes examples of equivariant homotopy equivalences $f : N \to M$ that are isovariant but are not isovariant homotopy equivalences, and the examples here cannot be deformed to isovariant maps even though the domain and codomain are identical (hence isovariantly homotopy equivalent by identity mappings, but not by the given maps).

The extra complication in proving Theorem I for bounded manifolds results can be described as follows: One crucial step in proving of Theroem 1.1 in [Sc9] is to deform an equivariant homotopy equivalence so that it is isovariant near the fixed point set; in fact, one chooses the deformed map so that it satisfies a homotopy theoretic analog of transversality near the fixed point set that is called *normal straightening*. Suppose now that we are given an equivariant homotopy equivalence F of manifolds with boundary and that the restriction of F to the boundary determines an isovariant homotopy equivalence ∂F. The results of [DuS] imply that ∂F is isovariantly homotopic to a map that is normally straightened near the fixed point set, and the results of [Sc9] imply the map F is equivariantly homotopic to a map of pairs that is normally straightened near the fixed point set. Thus we obtain two deformations of ∂F to maps that are normally straightened near the fixed point set.

Question. *Can one choose these deformations so that they determine compatible normal straightenings near the fixed point set of the boundary?*

Here is one way of making the notion of compatibility more precise:

Sharpened question. *Are the restrictions of the maps to neighborhoods of the fixed point set isovariantly homotopic?*

If it is possible to answer the sharpened question affirmatively, then the balance of the proof of Theorem I can be completed using arguments in [Sc9]. However, the deformation to a normally straightened map in the latter involves the arbitrary choice of an equivariant fiber homotopy equivalence, so the method provides no way of ensuring a positive answer to the compatibility question. In fact it is very easy to construct examples for which the induced maps near the fixed point sets are not at all compatible (in particular, their restrictions to neighborhoods of the fixed point sets need not be isovariantly homotopic).

In order to overcome such difficulties we must introduce some means for guaranteeing compatibility. We shall do this using equivariant analogs of the normal invariants which arise in ordinary surgery theory (compare Rourke [Rk], p. 140, as well as Chapter 3 of Lück [L] and Browder's book [Br1]). As in the nonequivariant case, such maps can be defined and studied directly by homotopy theoretic methods without any need to discuss surgery problems as such. Using such invariants we shall describe **canonical choices** for normal straightenings with the desired compatibility properties. These invariants lie in equivariant analogs of some standard homotopy functors which arise in surgery theory (*cf.* [MaMi]). Although the constructions of these objects parallel the nonequivariant case, clear statements and proofs are difficult — and in some cases impossible — to find in the literature, so in Section 1 we shall describe the sets in which our equivariant normal invariants are defined.

The remaining sections of this paper are orgainized as follows. Equivariant normal invariants are defined in Section 2, and in Section 3 we shall use the equivariant normal invariant to give examples of equivariant homotopy self-equivalences that cannot be approximated by isovariant homotopy equivalences. Section 4 concentrates on examples satisfying the Gap Hypothesis and gives the proofs of Theorems I and II (as well as Corollary III). The final Section 5 compares the proof of Theorems I and II in this paper and [isogap2006] with the surgery-theoretic approach their proofs due to S. Straus and W. Browder.

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1. Equivariant fiber retraction structures

TO BE CONTINUED.