MA 571 Supplementary Exercises, Part II

November, 1988

Fundamental groups and covering spaces

1. Let X be an arcwise connected topological space. For each pair of points $a, b \in X$ define $\Pi_1 X(a, b)$ to be the set of path homotopy classes of curves from a to b, and for each triple of points a, b, c define a binary operation

$$\Phi_{a,b,c} : \Pi_1 X(a, b) \times \Pi_1 X(b, c) \to \Pi_1 X(a, c)$$

by $\Phi([\alpha], [\beta]) = [\alpha + \beta]$.

(a) Explain why Φ is well-defined and $\Pi_1 X(a, a)$ is the fundamental group of (X, a).

(b) Show that $(X, \Pi_1 X, \Phi)$ is a category in which all morphisms in each set $\Pi_1 X(a, b)$ are isomorphisms (*i.e.*, a *groupoid*). This category is known as the *fundamental groupoid* of X. [*Hint:* Show that the constant classes $[C_a]$ behave like identities and that $[\overline{\alpha}]$ is an inverse to $[\alpha]$).

2. Prove that $A = ([-1, 1] \times \{0\}) \cup (\{-1, 1\} \times [0, 1])$ is a strong deformation retract of $X = [-1, 1] \times [0, 1]$. [*Hint:* Let p be the point $(0, 2)$. For each $v = (x, y) \in X$ show that there is a unique point $a \in A$ such that $a = p + t(v - p)$ for some $t > 0$. Show that t and hence a are continuous functions of v; the formula for $t(v)$ is given by two different expressions depending on whether $2|x| + y \leq 2$ or $2|x| + y \geq 2$.]

3. Let $p : E \to X$ be a covering projection. Suppose that E is Hausdorff and X is T_3 or $T_{3\frac{1}{2}}$. Prove that E is also T_3 or $T_{3\frac{1}{2}}$ respectively.

4. Suppose we are given covering maps $p_i : E_i \to X$ $(i = 1, 2)$ and a factorization $p : E_1 \to E_2$ such that $p_1 = p_2 p$. Prove that p is also a covering map.

5. The *Klein bottle* K is usually defined to be the quotient space of $S^1 \times I$ modulo the equivalence relation R generated by $(z, 0) R (\overline{z}, 1)$, where $\overline{z}$ denotes the complex conjugate of z.

(a) Show that there is a covering space projection $f : S^1 \times S^1 \to K$ that is two-sheeted (the inverse image of each point has two elements). [*Hint:* View $S^1 \times S^1$ as $S^1 \times [0, 2]$ modulo the equivalence relation generated by $(z, 0) \sim (z, 2)$.]

(b) Prove that the universal covering space of K is given by $\mathbb{R}^2 \cong \mathbb{C}$ with the covering transformations generated by $A_n(z) = z + ni$ and $B_n(z) = \chi^n(z) + n$, where $n \in \mathbb{Z}$ and χ denotes complex conjugation. Show that

$$A_1 B_1 A_1^{-1} B_1^{-1} = A_1^2$$

and conclude from this that the fundamental group of K is infinite and nonabelian.

6. Let X be a Hausdorff space that is arcwise connected and semilocally simply connected, and let $f : X \to X$ be a homeomorphism. The *mapping torus* of f is defined to be the quotient space M_f obtained from $X \times I$ modulo the equivalence relation generated by $(x, 0) \sim (f(x), 1)$.

(a) For each positive integer n show that M_{f^n} is an n-sheeted covering space of M_f.

(b) Assume that f preserves a basepoint $x \in X$, and let

$$\varphi : \mathbb{Z} \to \operatorname{Aut} \pi_1(X, x)$$

be the homomorphism $\varphi(n) = (f_*)^n\, [= (f^n)_*]$. Prove that $\pi_1(M_f, [x, 0])$ is the semidirect product of $\pi_1(X, x)$ and $\mathbb{Z}$ by φ. [*Hint:* Show that $X \times \mathbb{R}$ is a $\mathbb{Z}$-regular covering space of M_f with covering space automorphisms given by $T^n(x, t) = (f^n(x), t + n)$.]

7. Let $p : E \to X$ be a covering map, and let $f : Y \to X$ be continuous. Define the *pullback*

$$Y \times_X E := \{(e, y) \in Y \times E |\ f(y) = p(e)\}.$$

Let $p_{(Y,f)} = \text{proj}_Y | Y \times_X E$.

(a) Prove that $p_{(Y,f)}$ is a covering map. Also prove that f lifts to E if and only if there is a map $s : Y \to Y \times_X E$ such that $p_{(Y,f)} s = 1_Y$.

(b) Suppose also that f is the inclusion of a subspace. Prove that there is a homeomorphism $h : Y \times_X E \to p^{-1}(Y)$ such that $ph = p_{(Y,f)}$.

Notation. If the condition in (b) holds we sometimes denote the covering space over Y by $E|Y$.

In Exercises 8–11, assume that all spaces are Hausdorff and locally arcwise connected.

8. Let $p : E \to X$ be a covering map, and let $f : A \to X$ be a subspace inclusion, where A, E, X are all connected. Denote the pullback covering by $E|A$.

(a) Show that A is evenly covered if the induced map of fundamental groups $\pi_1(A, a_0) \to \pi_1(X, a_0)$ is the trivial homomorphism.

(b) Show that if the induced map of fundamental groups $\pi_1(A, a_0) \to \pi_1(X, a_0)$ is onto, then $E|A$ is also connected.

9. Let X be the figure eight space. Find a connected two sheeted covering of X, and find an upper bound for the number of connected, regular, four sheeted coverings of X.

10. Suppose that X has a simply connected covering space $\tilde{X}$ and that $A \subset X$ such that the induced map of fundamental groups $\pi_1(A, a_0) \to \pi_1(X, a_0)$ is 1–1. Prove that the components of $\tilde{X}|A$ are simply connected.

11. Suppose that G is a locally arcwise connected Hausdorff topological group and that the identity has a simply connected neighborhood.

(a) Prove that the universal covering space of G is a topological group such that the projection $p : \tilde{G} \to G$ is a continuous open homomorphism.

(b) Let H be a connected topological group, and let A be a discrete normal subgroup of H. Prove that A is contained in the center of H.

(c) Show that the kernel of the map p in (a) is a central subgroup if G is connected.

12. Let X be a connected, locally arcwise connected, semilocally simply connected space and that $f : X_2 \to X_1$ and $g : X_1 \to X$ are coverings. Prove that gf is a covering.

13. Let Let X be a connected, locally arcwise connected space, and let $p : E \to X$ be a connected covering map that is nontrivial (not a homeomorphism). Let $Y = X \times X \times X \times ...$ be the countable product of X with itself, let E^n denote the product of n copies of E with itself, and for an arbitrary space Y let $Y_n = E^n \times Y$. Define $p_n : Y_n \to Y$ by

$$(e_1, ... e_n; x_1, x_2, ...) \to (p(e_1), ... p(e_n); x_1, x_2, ...)$$

Prove that p_n is a covering map. Let $\tilde{Z} = \amalg_{n \geq 1} Y_n$ and let Z be the countably infinite sum of Y with itself. Let $q = \amalg_n p_n : \tilde{Z} \to Z$ and let $r : Z \to Y$ be the obvious projection. Prove that r and q are covering maps but that the composite rq is not a covering map.

14. Let X be a connected, locally arcwise connected, semilocally simply connected, second countable space. Prove that $\pi_1(X, x)$ is countable for all $x \in X$, and prove that every connected covering space of X is second countable.

15. Suppose that $E \to X$ is a covering projection, where X is connected, locally arcwise connected, semilocally simply connected, and separable metric. Prove that E is metrizable.