FINITE SYMMETRIES OF $\mathbb{R}^4$ AND S^4

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In the theory of transformation groups, two classes of manifolds play a special role; namely, Euclidean spaces $\mathbb{R}^n$ and standard spheres S^{n-1}. There is a large body of a literature devoted to finite (compact Lie) group actions on these spaces (*cf.* [37]).

Classical results completely describe all finite group actions on $\mathbb{R}^2$ and S^2 (*e.g.*, see [10]), results from the late 1970s to early 1990s yielded strong partial results for $\mathbb{R}^3$ and S^3 (compare [2] and [19]), and breakthroughs during the past decade have completed work on the three-dimensional case (see ?????). However, there still are many basic questions left unanswered, beginning in dimension four.

One example is the following (Problem 11 in A. Edmonds' extensive survey of group actions on 4-manifolds [11]):

Problem 1. *Is every finite group that acts on* $\mathbb{R}^4$ *is isomorphic to a subgroup of* $\mathrm{O}(4)$*?*

Another question (Problem 9 in [11]) is:

Problem 2. *Is a finite group acting on* S^4 *(without fixed points) isomorphic to a subgroup of* $\mathrm{O}(5)$*?*

Unless explicitly stated otherwise, all group actions in this paper are assumed to be effective.

Our answers to these questions are contained in the following:

Theorem 1. *If G is a finite group that acts locally linearly on* $\mathbb{R}^4$*, then G is isomorphic with a subgroup of* $\mathrm{O}(4)$*.*

Theorem 2. *There are finite groups G that are not isomorphic to subgroups of* $\mathrm{O}(4)$ *and which act (topologically) on* $\mathbb{R}^4$*.*

Theorem 3. *There are finite groups G that are not isomorphic with subgroups of* $\mathrm{O}(5)$ *and which act locally linearly on* S^4 *without fixed points. Moreover, if a* *finite group G acts on S^4 locally linearly and orientation preservingly then G is isomorphic to a subgroup of* $\mathrm{SO}(5)$.

In order to make this paper relatively self contained we will include general comments of somewhat expository nature. We hope that these comments will also make the paper easier to read.

Remark. The results described in Theorem 1 and 2 had been known to the authors for quite some time. For example, in an attempt in [5] to construct fixed point free actions of a finite group on $\mathbb{R}^4$ the authors also determined that the alternating group A_5 is potentially only such group. This is essentially Theorem 1; however, this result was deleted from the final version of [5]. Similarly, see ([18],p. 454) for Theorem 2. On the other hand, Theorem 3 is new and will be the main focus of this paper. The proof combines some fairly standard methods for applying 4-dimensional topological surgery to group actions with computational results for Wall's surgery obstruction groups [42] in certain cases. More precisely, these concern suitable versions of the Rothenberg exact sequences relating different types of Wall groups, and the key quantitative input involves class computations for certain algebraic number fields, somewhat in the spirit of papers like [3], [9], [14] and [25].

A version of Theorem 1 for arbitrary acyclic 4-manifolds was proved independently in [13]; their approach is similar in spirit but more group-theoretic.

The paper is divided into three sections. Section 1 discuses locally linear and topological actions on $\mathbb{R}^4$, and Section 2 is devoted to constructing exotic topological and locally linear actions on S^4. A short third section discusses some further questions in several directions.

1. Symmetries of $\mathbb{R}^4$.

The following simple group-theoretic observation was made by the authors in [5] (*i.e.*, Assertion, p. 648).

Let $G \subset \mathrm{SO}(4)$ be a nontrivial finite group and let A_5 be the alternating group on 5 letters.

Assertion 1. *Either G contains a nontrivial normal p-subgroup or G is isomorphic to* A_5 *or* $\mathrm{A}_5 \times \mathrm{A}_5$.

Now, let G be any nontrivial finite group acting locally linearly and orientation preservingly on $\mathbb{R}^4$. If G has a normal p-subgroup $H \lhd G$, then $G \subset \mathrm{SO}(4)$. For $(\mathbb{R}^4)^H = \left\{ \begin{array}{l} \mathbb{R}^2 \\ \{pt\} \end{array} \right.$ by Smith theory. The group G/H acts on $\left\{ \begin{array}{l} \mathbb{R}^2 \\ \{pt\} \end{array} \right.$ and since every group action on $\mathbb{R}^2$ is conjugate a linear one then $(\mathbb{R}^4)^G \neq \emptyset$. Now local linearity of the action of G in $\mathbb{R}^4$ implies $G \subset \mathrm{SO}(4)$.

Our proof of Theorem 1 essentially boils down to proving a version of the above Assertion where instead condition $G \subset \mathrm{SO}(4)$ we assume G acts orientation preservingly and locally linearly on $\mathbb{R}^4$ and then conclude $G \subset \mathrm{SO}(4)$.

Proof of Theorem 1. Our first simple observation is the following:

Observation 1. *The only finite simple group which acts orientation preservingly on* $\mathbb{R}^4$ *is* A_5.

This observation follows (somewhat inelegantly) from the direct inspection of all finite non-abelian simple groups in the Atlas of Finite Groups (cf. [8]). The point here is that each such group except A_5 has a subgroup (solvable) which is too large to act effectively on $\mathbb{R}^4$. (Note that an action of a solvable group on $\mathbb{R}^4$ always has a fixed point). For example, in A_6 one can take the normalizers of Sylow 3-subgroups.

Suppose then that G is NOT simple.

Let $H \neq \{e\}$ be a maximal normal proper subgroup of G (*i.e.*, G/H is simple).

Case 1. H is a non-abelian simple group (hence $H \cong \mathrm{A}_5$).

We have then an extension

$$1 \to H \to G \to G/H \to 1 \tag{1}$$

In order to classify such extensions (cf. [4] p.105) let $\psi : G/H \to Out(\mathrm{A}_5) \cong \mathbb{Z}_2$ be a homomorphism. Then $\psi : G/H \to \mathbb{Z}_2$ is trivial except when $G/H \cong \mathbb{Z}_2$ (note that G/H is a simple group).

The set of extensions (1) with fixed ψ is classified by $H^2(G/H; Z[H])$ where $Z[H]$ is the center of H (cf. [4], p. 105). Consequently there is only one extension

$$G \cong H \times G/H \quad \text{for} \quad G/H \ncong \mathbb{Z}_2$$

and two extensions $G \cong H \times \mathbb{Z}_2$ and $G \cong \mathrm{S}_5$ for $G/H \cong \mathbb{Z}_2$.

Neither of there groups can act locally linearly on $\mathbb{R}^4$.

Case 2. H is simple abelian.

In this case $(\mathbb{R}^4)^G \cong \left((\mathbb{R}^4)^H\right)^{G/H} = \{pt\}$. Consequently $G \subset \mathrm{SO}(4)$.

Case 3. H is not simple.

Repeating the argument from Case 1 and Case 2 with H replacing G one easily concludes $G \subset \mathrm{SO}(4)$.

Case 4. Suppose G has an orientation reversing element.

Let $K \lhd G$ be the normal subgroup of orientation preserving elements, so that $G/H \cong \mathbb{Z}_2$.

Then either $(\mathbb{R}^4)^K \neq \emptyset$ and hence $G \subset \mathrm{SO}(4)$ or $K \cong \mathrm{A}_5$ and we have an extension

$$1 \to K \to G \to \mathbb{Z}_2 \to 1.$$

This however was handled in Case 1 and hence the proof of Theorem 1 is concluded. $\square$

Before giving a proof of Theorem 2 we start again with general comments.

Let G be a finite group acting freely on some sphere S^{n-1}, where $n \geq 2$ and $(n-1)$ is odd. Then it follows that G has a free resolution of period n, and its Tate cohomology $\widehat{H}^*(G, \mathbb{Z}_2)$ is periodic of period n.

Now given a finite group G of period n one can ask if G can act freely on a finite CW-complex (manifold) X with $X \cong S^{n-1}$.

It turns out that there is a finiteness obstruction $\sigma_n(G)$, (introduced by R. Swan in [39]) for existence of such action. This obstruction takes value in a certain quotient of $\widetilde{K}_0(Z[G])$, i.e. in $\widetilde{K}_0(Z[G])/T_G$ (cf. [9]).

Computations of these finiteness obstructions is in general very complicated and quite technical task. For various groups it was carried successfully however by R. J. Milgram and I. Madsen (cf. [28], [25]). Quite extensive calculations were done for the class of groups $Q(2^n a, b, c)$. where a, b, c are coprime integers, $n \geq 2$ and $Q(2^n a, b, c)$ is given by the semi-direct product

$$1 \to \mathbb{Z}/n \times \mathbb{Z}/a \times \mathbb{Z}/b \times \mathbb{Z}/c \to Q(2^n a, b, c) \to Q(2^n) \to 1$$

in which $Q(2^n)$ is the quaternionic 2-group (*cf.* [9]). In particular for $\pi = Q(8p, q, 1)$, there are conditions on p, q (*cf.* [28], [25]) which imply $\sigma_4(\pi) = 0$.

Let $\widetilde{X} \simeq S^3$ be a finite complex with a free action of π on $\widetilde{X}$ be given. Then $\widetilde{X}/\pi = X$ is a finite 3-dimensional Poincaré complex and hence it is equipped with a Spivak normal bundle (*i.e.*, a homotopy spherical fibration; *cf.* [42], [26]). Let $f : X \to BSG$ be the classifying map for this fibration (*cf.* [26]).

By considering p-Sylow subgroups π_p of π and using the fact that there are manifold models for each lifting $\widetilde{X}_{(p)} \cong S^3/\pi_p$ one can conclude as in ([9], [25]) that $f : X \to BSG$ lifts to $f : X \to BSO$ where BSO is the classifying space for oriented bundles. The existence of such lifting leads to the existence of a normal map

$$f : (M^3, \nu_{M^3}) \to (X, \xi_X)$$

where ν is the stable normal bundle, and ξ_X is the Spivak normal bundle. Rather intricate and quite lengthy computations (see [3], [9], [25]) show that the surgery obstruction $\lambda(f) \in L_3^h(\pi)$ is trivial for each of the pairs

$$(p, q) = (3, 313), (3, 433), (3, 601), (7, 113), (5, 461), (7, 809), (11, 1321), (17, 103)\ .$$

In dimension 3 this means (*cf.* [16]) that there is a manifold N^3 and a map $k : N^3 \to X$ which is a $Z[\pi]$-homology equivalence (in other words: there is a free action of π on some integral homology 3-sphere $\widetilde{N}^3$).

Proof of Theorem 2. Let π be the group $Q(8p, q)$, where (p, q) is any of the pairs mentioned earlier. Let

$$k : M^3 \to X$$

be a $\mathbb{Z}[\pi]$-homology equivalence. Consider the map

$$h = k \times \mathrm{id} : M^3 \times I \to X \times I$$

and let $\lambda(h) \in L_0^h(\pi)$ be the surgery obstruction for changing h to a homotopy equivalence without modifying anything on the boundaries. Now let $\mathcal{F} : Z[\pi] \to Z[\pi]$ be the identity homomorphism and $\Gamma_0(\mathcal{F})$ be the Cappell-Shaneson homological surgery obstruction group as in [6]. The natural homomorphism $j_* : L_0^h(\pi) \to \Gamma_0(\mathcal{F})$ is an isomorphism (see [6] p.288) and clearly $j_*(\lambda(h)) = 0$ so that $\lambda(h) = 0$ in $L_0^h(\pi)$. Let $\overline{h} : (W^4; M, M) \to (X \times I, X, X)$ be a homotopy equivalence. Form a two ended open manifold

$$W_0 := \ldots \cup W^4 \cup_M W^4 \cup_M W^4 \cup \ldots$$

by stacking together copies of W^4.

Obviously $\pi_1(W_0) \cong \pi$ and the universal cover $\widetilde{W_0}$ of W_0 is a manifold properly homotopy equivalent to $S^3 \times \mathbb{R}$ and hence homeomorphic to $S^3 \times \mathbb{R}$ by [12].

Compactifying $\widetilde{W_0}$ by adding points on the respective ends, one gets an action of π on S^4 with two fixed points, whereas compactification at one of the two ends yields an action of π on $\mathbb{R}^4$ with one fixed point. Since π is not isomorphic to a subgroup of $\mathrm{O}(4)$ the proof of Theorem 2 is complete. □

The techniques used in the proof of Theorem 1 can be modified to show that the existence of an orientation preserving locally linear action of a finite group G on S^4 implies $G \subset \mathrm{SO}(5)$. A detailed and independent argument for this claim is contained in [27].

Consequently we have

Corollary 1. *If G is a finite group acting locally linearly and orientation preservingly on S^4 then G is isomorphic to a subgroup of* $\mathrm{SO}(5)$*. However, there is an orientation preserving topological action of a finite group π on S^4 such that π is not isomorphic to a subgroup of* $\mathrm{SO}(5)$*.*

Although the actions in the corollary are not locally linear, they obviously define homotopy stratifications in the sense of [33].

2. Symmetries of S^4.

Let $k : N^3 \to X$ be the $Z[\pi]$-homotopy equivalence discussed earlier, and let E_X be the total space of a twisted I-bundle over X (= the unit disk bundle of a real line bundle).

Now if $Q(8)$ is the quaternionic group given by

$$Q(8) = \{x, y | x^4 = 1, x^2 = y^2, yxy^{-1} = x^{-1}\}$$

let $\omega : Q(8) \to \mathbb{Z}_2$ be a nontrivial orientation homomorphism given by

$$\omega(x) = +1,\ \omega(y) = -1.$$

It induces a corresponding homomorphism

$$\omega : Q(8p, q) \to \mathbb{Z}_2$$

which in turn yields a specific line bundle, and hence a specific choice of E_X.

Next, let E_M be the pull back of E_X by k. Then there is a degree one map

$$h : (E_M, \partial) \to (E_X, \partial)$$

and h is a $Z[\pi]$ homotopy equivalence of manifolds with boundary. Once again $\lambda(h) \in L_0^h(\pi, \omega)$ is trivial (*i.e.*, $L_0^h(\pi, \omega) \cong \Gamma_0(\mathcal{F}, \omega)$).

Consequently we can replace the manifold E_M by a manifold (W^4, ∂) homotopy equivalent (rel ∂) to (E_X, ∂).

In particular, $h_0 = h|_{\partial W^4} : \partial W^4 \to \partial E_X$ is a $Z[\pi]$-homotopy equivalence where $\tau \subset \pi$ is a subgroup of index two; in fact, $\tau \cong Q(4p, q)$.

Now applying the argument used in the proof of Theorem 2 to

$$h_0 \times \mathrm{id} : \partial W^4 \times I \to \partial E_X \times I$$

one obtains a manifold (N^4, ∂) homotopy equivalent to $E_X \times I(\text{rel } \partial)$. Let

$$\overline{N} = N^4 \cup_M N^4 \cup_M \ldots .$$

and form $M^4 = W^4 \cup_M \overline{N}$. Then M^4 is a one ended manifold with the universal covering $\widetilde{M^4} \approx_{top} S^3 \times \mathbb{R}$.

As in the preceding section, a two point compactification of $\widetilde{M^4}$ gives an action of π on S^4; the induced group action on the two "points at infinity" is given by the homomorphism ω described above. This action is fixed point free (it is pseudo-free with two singular points having isotopy group τ).

Consequently we have

Corollary 2. *There is a finite group G acting topologically on S^4 without fixed points such that G is not isomorphic to a subgroup of* $\mathrm{O}(5)$.

This corollary gives a topological solution to Problem 2. Now our goal is to transform the topological action in the corollary into a locally linear one.

We shall first describe a strategy for proving this, and we shall give a detailed argument afterwards.

Let (p, q) be the pair of primes such that $Q(8p, q)$ acts freely on a homology 3-sphere Σ. Write $G = Q(8p, q)$ and $\tau = Q(4p, q)$ where $\tau \subset G$ is a subgroup of index two.

Let $h : \Sigma^3/G \to X^3/G$ be a $\mathbb{Z}[G]$ homology equivalence, where X^3/G is a finite Poincaré complex with $\widetilde{X^3} \cong S^3$.

Consider the exact homology surgery sequence in dimension 3 (*e.g.*, see [16], [18]):

$$L_0^h(G) \xrightarrow{\gamma} S_H(X^3/G) \xrightarrow{\eta} [X^3/G, G/Top] \xrightarrow{\Theta_3^h} L_3^h(G)$$

The transfer to the 2-fold cover $X/\tau \to X/G$ gives a commutative diagram

$$\begin{array}{ccccccc}
L_0^h(\tau) & \xrightarrow{\gamma} & S_H(X^3/\tau) & \xrightarrow{\eta} & [X^3/\tau, G/Top] & \xrightarrow{\Theta_3^h} & L_3^h(\tau) \\
\uparrow{\scriptstyle tr_*} & & \uparrow{\scriptstyle tr_*} & & \uparrow{\scriptstyle tr_*} & & \uparrow{\scriptstyle tr_*} \\
L_0^h(G) & \xrightarrow{\gamma} & S_H(X^3/G) & \xrightarrow{\eta} & [X^3/G, G/Top] & \xrightarrow{\Theta_3^h} & L_3^h(G)
\end{array}$$

Since $X^3/\tau \cong S^3/\tau$ and S^3/τ is a manifold we have a base point in $S_H(X^3/\tau)$.

Let $tr_*(h) = \widetilde{h} : \Sigma^3/\tau \to S^3/\tau$ be a $\mathbb{Z}[\pi]$-homology equivalence and let $[\widetilde{h}] \in S_H(X^3/\tau)$. We claim that $\eta[\widetilde{h}] = 0$ in $[X^3/\tau, G/Top]$. Indeed, $[X^3/\tau, G/Top] \cong H_1(\tau, \mathbb{Z}_2) \cong \mathbb{Z}_2$ and since Θ_3^h is a monomorphism (see [14]), our claim follows.

This means that there is an element $\widetilde{x} \in L_0^h(\tau)$ with $[\widetilde{h}] = \gamma\widetilde{x}$.

Hypothesis 1. *Suppose that there is an element $x \in L_0^h(G)$ such that $tr_*(x) = \widetilde{x}$.*

If this is true and we act on $[h] \in S_H(X^3/G)$ by $(-x)$, then we get a new $\mathbb{Z}[G]$-homology equivalence:

$$\bar{h} : \Sigma'/G \to X^3/G$$

with the lifting $\widetilde{\bar{h}} : \Sigma'/\tau \to S^3/\tau$ such that $\widetilde{\bar{h}}$ is H-cobordant to the $\mathrm{id}_{S^3/\tau}$.

Let $\overline{W^4}$ be such an H-cobordism

$$(\overline{W^4}; \Sigma'/\tau; S^3/\tau) = (\overline{W^4}, \partial_0, \partial_1).$$

Now in our construction of one ended manifold M^4 we simply take

$$M^4 = W^4 \cup_{\partial_0} \overline{W^4} \cup_{\partial_1} (S^3/\tau) \times [0, \infty).$$

Obviously, $\widetilde{M^4} \approx_{top} S^3 \times \mathbb{R}$ and the action of G on two point compactification is locally linear.

Consequently, with the notation introduced above, our main effort goes into proving the following:

Theorem 4. *For some choice of (p,q) there is an element $x \in L_0^h(G)$ such that $tr_*(x) = \widetilde{x}$.*

Proof of Theorem. In order to prove the above theorem we will need quite detailed knowledge of methods and techniques involved in computation of both $L_0^h(G)$ and $L_0^h(\tau)$.

We start with a brief review of Wall's approach to computing surgery obstruction groups for finite fundamental groups [41] and further results due to Madsen [25]. First we consider the case of $G = Q(8p,q)$. Since the $SK_1(G) = 0$ (see [32]) the relevant part of the Rothenberg exact sequence connecting L_*^h and L_*^s groups reduces to the following (*cf.* [41], [25]):

$$0 \longrightarrow L_0'(G) \xrightarrow{j} L_0^h(G) \xrightarrow{\delta} Wh'(G) \otimes \mathbb{Z}_2 \longrightarrow \cdots$$

Here $L_*'(G) \cong L_*^s(G)$ and L_*' are the intermediate L_*-groups from [41].

Using [41] and [25], we can compute the group $L_0'(G)$ as follows: Let $F_{p,q} = \mathbb{Q}[\xi_p + \xi_p^{-1}, \xi_q + \xi_q^{-1}]$ be the algebraic number field and $\mathcal{O}_{p,q} = \mathbb{Z}[\xi_p + \xi_p^{-1}, \xi_q + \xi_q^{-1}]$ be its ring of integers. Similarly, let $\mathcal{O}_p$ and $\mathcal{O}_q$ denote the algebraic integers in the fields $F_p = \mathbb{Q}[\xi_p + \xi_p^{-1}]$ and $F_q = \mathbb{Q}[\xi_q + \xi_q^{-1}]$ respectively. Let $\hat{A}_{\mathfrak{c}}$ be the $\mathfrak{c}$-adic completion of $\mathcal{O}_{p,q}$ at a prime ideal $\mathfrak{c}$ in $\mathcal{O}_{p,q}$. As usual $\hat{A}_{\mathfrak{c}}^\times$ are the units of $\hat{A}_{\mathfrak{c}}$. Let $F^{(2)} \subset F_{p,q}^\times$ consist of all elements with even valuation at all finite primes. Let Γ be the ideal class group of $F_{p,q}$(*i.e.*, the ideal class group of $\mathcal{O}_{p,q}$).

One part of the torsion group $T_G(pq)$ of $L_0'(G)$ (denoted by coker ψ_1 in [41], [25]) is determined by the exact sequence of [41], p.74:

$$F^{(2)}/(F^\times)^2 \xrightarrow{\Phi'} \oplus \left(\hat{A}_{\mathfrak{p}}^\times/(\hat{A}_{\mathfrak{p}}^\times)^2\right) \longrightarrow T_G(pq) \longrightarrow \Gamma/\Gamma^2 \longrightarrow 0\ .$$

Here the summation runs through all primes $\mathfrak{p}$ over $n = pq$.

There is a corresponding exact sequence for the part of the torsion of $L_0'(\tau)$, where the algebraic number field is $F_{pq} = \mathbb{Q}[\xi_{pq}, \xi_{pq}^{-1}]$ with $Z_{pq} = \mathbb{Z}[\xi_{pq}, \xi_{pq}^{-1}]$ as its ring of integers. In fact, for both groups $L_0'(G)$ and $L_0'(\tau)$ the crucial information is contained in the following exact sequence (*cf.* [25], [41])

$$0 \to \operatorname{coker}^F \psi_1 \to L_0'(H)(pq) \to \ker_0^F \psi_0 \to 0 \tag{2}$$

where H is either G or τ, and F is the corresponding field introduced earlier.

The groups $\operatorname{coker}^F \psi_1$ and $\ker_0^F \psi_0$ are defined in [41] and [25]; in the preceding sequence, (pq) stands for the last component of $L_0'(H)$ in the decompositions

$$L_0'(G) \cong L_0'(Q(8)) \oplus \tilde{L}_0'(Q(8p)) \oplus \tilde{L}_0'(Q(8q)) \oplus \tilde{L}_0'(G)(pq)$$
$$L_0'(\tau) \cong L_0'(\mathbb{Z}_4) \oplus \tilde{L}_0'(Q(4p)) \oplus \tilde{L}_0'(Q(4q)) \oplus \tilde{L}_0'(\tau)(pq).$$

The transfer map induces a homomorphism of corresponding sequences (2) for G and τ. In fact, the transfer homomorphism is essentially given by the inclusion of centers (*i.e.*, inclusion of corresponding fields; see [25] p. 215, discussion after Theorem 4.18.

Claim. *The transfer homomorphism tr_* is onto $T_\tau(pq)$ if the class numbers of $\mathcal{O}_{pq}$, $\mathcal{O}_p$ and $\mathcal{O}_q$ are odd.*

Proof of Claim. In this case we have a diagram

$$\begin{tikzcd}
F^{(2)}/(F^\times)^2 \arrow[r, "\Phi'"] & \oplus\left(\hat{A}_{\mathfrak{p}}^\times/(\hat{A}_{\mathfrak{p}}^\times)^2\right)_\tau \arrow[r] & T_\tau(pq) \arrow[r] & 0 \\
F^{(2)}/(F^\times)^2 \arrow[r, "\Phi'"] \arrow[u, "tr_*"'] & \oplus\left(\hat{A}_{\mathfrak{p}}^\times/(\hat{A}_{\mathfrak{p}}^\times)^2\right)_G \arrow[r] \arrow[u, "tr_*"'] & T_G(pq) \arrow[r] \arrow[u, "tr_*"'] & \cdots
\end{tikzcd}$$

Since the numbers $g_{pq}(F_{pq})$ and $g_{pq}(F_{p,q})$ are equal (see [25], p. 216) (these are numbers of primes over pq in the corresponding fields) and since $\hat{A}_{\mathfrak{p}}^\times/(\hat{A}_{\mathfrak{p}}^\times)^2$ has 2-rank 1 (compare [41], pp. 51–52), it follows that tr_* is an isomorphism

$$tr_* : \oplus\left(\hat{A}_{\mathfrak{p}}^\times/(\hat{A}_{\mathfrak{p}}^\times)^2\right)_G \to \oplus\left(\hat{A}_{\mathfrak{p}}^\times/(\hat{A}_{\mathfrak{p}}^\times)^2\right)_\tau$$

and the claim follows. □

The torsion free part Σ_G of $L_0'(G)(pq)$ and Σ_τ of $\tilde{L}_0'(\tau)(pq)$ are determined by the multisignature (*cf.* [41], [25]). The transfer homomorphism $tr_* : \Sigma_G \to \Sigma_\tau$ turns out to be an isomorphism. This follows from computations in [22]; more specifically, see Example 5.15 for Σ_G and Example 5.19 for Σ_τ.

This shows that $tr_* : \tilde{L}_0'(G)(pq) \to \tilde{L}_0'(\tau)(pq)$ is onto in the case of odd class number for $\mathcal{O}_{pq}$.

Now consider the transfer map $tr_* : \tilde{L}_0'(Q(8p)) \to \tilde{L}_0'(Q(4p))$. In this case, we claim the following hold:

(a) The torsion group of $\tilde{L}_0'(Q(4p))$ is trivial.
(b) The transfer tr_* restricted to the torsion free parts $tr_* : \Sigma_G \to \Sigma_\tau$ is onto.

To see (a) note that for each prime p and q the class numbers of $\mathcal{O}_p$ and $\mathcal{O}_q$ are odd by our assumption, and since the map $\Phi' : F^{(2)}/(F^\times)^2 \to \oplus \left(\hat{A}_{\mathfrak{p}}^\times/(\hat{A}_{\mathfrak{p}}^\times)^2\right)_\tau$ is onto (see [41], p. 74, discussion after the corresponding exact sequence in the middle of the page), the 2-rank of the group Γ is the rank of the torsion group, and hence the torsion group is trivial.

To see (b) one appeals to the computations in [22]. Specifically, take Example 5.16 for Σ_G and Example 5.15 for Σ_τ.

The case of $tr_* : \tilde{L}'_0(Q(8q)) \to \tilde{L}'_0(Q(4q))$ follows by the same considerations.

Now consider the transfer map $tr_* : L'_0(Q(8p)) \to L'_0(\mathbb{Z}_4)$. It is NOT onto, and in fact its image has index two on one single copy of $\mathbb{Z}$, i.e. one of $8\mathbb{Z}$ goes to $4\mathbb{Z}$, where $L'_0(\mathbb{Z}_4) \cong 8\mathbb{Z} \oplus 8\mathbb{Z} \oplus 4\mathbb{Z}$ (*cf.* [41]). The transfer however is onto the remaining $8\mathbb{Z} \oplus 8\mathbb{Z}$. Fortunately in the exact surgery sequence

$$\xrightarrow{\Theta_4} L_0^h(\tau) \longrightarrow S_H(X^3/\tau) \longrightarrow [X^3/\tau, G/Top] \longrightarrow L_3^h(\tau)$$

the copy of $4\mathbb{Z}$ in $L_0^h(\tau) \cong L'_0(\tau)$ is in the image of Θ_4.

Subject to our condition on the parity of the class number of $\mathcal{O}_{p,q}$ we proved the following:

Fact. *If Ooze $\subset L'_0(\pi)$ denotes the "oozing subgroup" of surgery obstructions realized by normal maps of closed manifolds with fundamental group π, then the transfer map $tr_* : L'_0(G)/Ooze \to L'_0(\tau)/Ooze$ is onto.*

(See [36], p. 25, and [14] for background information on the oozing subgroup.)

Back to Theorem 4

Let $h : \Sigma^3/G \to X^3/G$ be a $\mathbb{Z}[G]$-homology equivalence and let $\widetilde{h} : \Sigma^3/\tau \to X^3/\tau$ be the lifting.

Let $\rho = \rho(h)$ be the Whitehead torsion of h, and similarly let $\widetilde{\rho} = \rho(\widetilde{h})$, where $\rho \in Wh(G)$ and $\widetilde{\rho} \in Wh(\tau)$. Obviously the transfer map $tr_* : Wh(G) \to Wh(\tau)$ sends ρ to $\widetilde{\rho}$:

$$tr_*(\rho) = \widetilde{\rho}.$$

Now let $[h] \in S_H(X^3/G)$. Then the exact sequence

$$\cdots \longrightarrow L_0^h(\tau) \xrightarrow{\gamma} S_H(X^3/G) \xrightarrow{\eta} [X^3/G, G/Top] \xrightarrow{\Theta_3^h} L_3^h(G)$$

together with $[X^3/G, G/Top] \cong H_1(G;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the computation of Θ_3^h implies that there is an element $r \in L_0^h(G)$ such that $\gamma(r) = [h]$.

Consider now the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \overline{L}_0'(G) \arrow[r, "i"] \arrow[d, "tr_*"] & \overline{L}_0^h(G) \arrow[r, "j"] \arrow[d, "tr_*"] & Wh'(G)\otimes\mathbb{Z}_2 \arrow[r] \arrow[d, "tr_*"] & \cdots \\
0 \arrow[r] & \overline{L}_0'(\tau) \arrow[r, "i"] & \overline{L}_0^h(\tau) \arrow[r, "j"] & Wh'(\tau)\otimes\mathbb{Z}_2 \arrow[r] & \cdots
\end{tikzcd}$$

where $\overline{L}$ are the quotients $L/Ooze$. Let $[\widetilde{h}] \in S_H(X^3/\tau)$ so that $[\widetilde{h}] = \gamma(\widetilde{x})$ for some $\widetilde{x} \in \overline{L}_0^h(\tau)$.

Suppose $j(\widetilde{x})$ is nontrivial; *i.e.*, $j(\widetilde{x}) = \widetilde{\rho}\otimes\mathbb{Z}_2$. Obviously there is an element $\rho\otimes\mathbb{Z}_2$ in $Wh'(G)\otimes\mathbb{Z}_2$ with $tr_*(\rho\otimes\mathbb{Z}_2) = \widetilde{\rho}\otimes\mathbb{Z}_2$. Let $r \in \overline{L}_0^h(G)$ be an element with $j(r) = \rho\otimes\mathbb{Z}_2$.

Consider $(\widetilde{x} - tr_*(r)) \in \overline{L}_0^h(\tau)$.

Then $j(\widetilde{x} - tr_*(r)) = 0$ and hence there is an element $\widetilde{l} \in \overline{L}_0'(\tau)$ with $i(\widetilde{l}) = (\widetilde{x} - tr_*(r))$ and since $tr_* : \overline{L}_0'(G) \to \overline{L}_0'(\tau)$ is onto then there is an element $l \in \overline{L}_0'(G)$ with $tr_*(l) = \widetilde{l}$. Consider the element

$$(i(l) + r) \in \overline{L}_0^h(G).$$

Then

$$\begin{aligned} tr_*(i(l)+r) &= tr_*(i(l)) + tr_*(r) = i(tr_*(l)) + tr_*(r) \\ &= i(\widetilde{l}) + tr_*(r) = \widetilde{x} - tr_*(r) + tr_*(r) = \widetilde{x} \end{aligned}$$

This completes the proof of Theorem 4 modulo the claim about the parity of the corresponding class number. The next step involves examples where the hypothesis on class numbers is satisfied.

Claim. *The groups $Q(8p,q)$ for $(p,q) = (17,103), (3,313), (3,433)$ act locally linearly on S^4.*

It turns out that for each pair (p,q) the class number h_{pq}^+ of $\mathbb{Z}[\xi_{pq}, \xi_{pq}^{-1}]$ is odd. We give an argument for $(3,313)$ which works for the remaining pairs (p,q).

Let h_{pq} be the class number of $\mathbb{Q}(\xi_{pq})$ (*i.e.*, of $\mathbb{Z}_{pq}$) and let h_{pq}^- be the relative class number; *i.e.*, $h_{pq} = h_{pq}^+ \cdot h_{pq}^-$. Put $(p,q) = (3,313)$. The extension $\mathbb{Q}(\xi_{pq})/\mathbb{Q}(\xi_q)$ has a cylic Galois group whose order is a power of 2, and hence by Iwasawa's Theorem

II in [17] (compare also Theorem 10.4 in [43], p.186) we know that

$$2|h_{pq} \qquad \Rightarrow \qquad 2|h_q \; .$$

However the parity of h_q is the same as that of h_q^- (cf. [43]) and h_q^- is odd for $q = 313$ (*cf.* [24]). Consequently h_{pq} is odd and hence h_{pq}^+ must be odd as well.

Using [24], one can check that h_q^- is odd for $q = 433$ as well.

For $p = 3$, we have $h_p = 1$ (cf. [43]). □

3. Final Remarks.

A. *Automorphism groups of S^4.* It would be interesting to see if the "exotic" actions of finite groups $G \not\subset \begin{cases} \mathrm{SO}(5) \\ \mathrm{O}(5) \end{cases}$ on S^4 can be used to show that

$$Top_+(S^4) \not\cong \mathrm{SO}(5) \qquad \text{and} \qquad Top(S^4) \not\cong \mathrm{O}(5).$$

Note that for S^3 we have $Top(S^3) \cong \mathrm{O}(4)$ (*e.g.*, see [15]).

The existence of "exotic" free actions of certain finite groups on S^5 was used in [40] to assert that $Top_+(S^5) \not\cong \mathrm{SO}(6)$ and $Diff_+(S^5) \not\cong \mathrm{SO}(6)$. However, it seems (*cf.* [31] pp. 74–75) that there are serious problems with arguments used in [40], so further study of the the following still seems worthwhile:

Problem 3. *Find the lowest value of n for which*

$$Top_\pm(S^n) \not\cong \begin{cases} \mathrm{SO}(n+1) \\ \mathrm{O}(n+1) \end{cases} \quad \textit{and/or } Diff_\pm(S^n) \not\cong \begin{cases} \mathrm{SO}(n+1) \\ \mathrm{O}(n+1) \end{cases} .$$

THIS NEEDS TO BE REWORKED IN VIEW OF OUR DISCUSSIONS.

B. *Smoothability questions.* It is a natural to ask whether the locally linear action in Theorem 3 is equivariantly smoothable or whether its product with some $\mathbb{R}^k$ (with trivial group action) is smoothable. One possible approach to study the smoothability question could involve a use of gauge theory; we hope to return to such problems in a future paper. The methods of [23] may be useful for studying the stable smoothability question.

C. *Smooth actions of exotic finite groups on $\mathbb{Z}$-homology 3-spheres.* Integral homology spheres play important roles in several areas of geometric topology, frequently as close approximations to standard spheres; for example, consider the role of Seifert homology 3-spheres in the theory of S^1-actions on 3-manifolds [35], the work of E. Brieskorn on isolated singularities of complex hypersurfaces (*cf.* [30]), or the view of $\mathbb{Z}$-homology spheres as *Doppelgänger* in Chapter 2 of [44]. In particular, the main results of this paper rely heavily on the existence of free $Q(8p, q)$-actions on homology 3-spheres, and one can ask more general existence and classification questions about smooth finite group actions on homology 3-spheres which do not have linear models (*i.e.*, orthogonal actions on S^3). A few special cases involving exotic actions of A_5 (*e.g.*, examples with a single fixed point) were considered in [20] and the second section of [38]. Here are specific questions involving exotic groups which act freely on $\mathbb{Z}$-homology 3-spheres:

Problem 4. *Let G be a finite group which acts freely (and smoothly) on a closed $\mathbb{Z}$-homology 3-sphere, and let $C \subset G$ denote the unique central subgroup of order 2 (which exists by [29]). Is there an effective smooth (possibly nonfree) action of the quotient group G/C on a closed $\mathbb{Z}$-homology 3-sphere? If so, can they be realized as invariant submanifolds of locally linear group actions on S^4?*

The standard group isomorphism $S^3/\mathbb{Z}_2 \cong SO(3)$ implies the answers are yes if the finite group G supports a free linear action on S^3.

D. *Pseudofree smooth actions of exotic finite groups on higher dimensional spheres.* More generally, if we are given a free smooth action of some finite group G on a sphere S^n and a nontrivial homomorphism $\omega : G \to \mathbb{Z}_2$, one can ask whether the action extends to a pseudofree action (smooth or locally linear) on S^{n+1} which is free on the complement of a two point set $\{P, Q\}$, looks like the product action

$$g \cdot (\mathbf{v}, t) \ = \ (g \cdot \mathbf{v}, \omega(g)\, t)$$

on $S^{n+1} - \{P, Q\} \cong S^n \times (-1, 1)$, and permutes P and Q via the homomorphism ω, where the codomain is identified with the symmetric group on two letters. Note that the conditions imply that the kernel of ω admits a free orthogonal action on S^n.

The periodic groups studied in this paper are particularly difficult to work with, and one might ask if strong positive results on this problem can be obtained for other, more tractable classes of examples.

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