

## ADDITIONAL PROPERTIES OF BETWEENNESS

**NOTE.** *The following should be inserted before the subheading Separation axioms on page 47 of the file geometrynotes1.pdf.*

Here are some additional facts about betweenness with vector proofs (however, many computational steps have been omitted).

**Example 1.** *Suppose that  $\mathbf{a} \neq \mathbf{b}$  and for  $i = 1, 2, 3$  we have  $\mathbf{x}_i = \mathbf{a} + t_i(\mathbf{b} - \mathbf{a})$  where  $t_i \in \mathbf{R}$ . If  $t_1 < t_2 < t_3$ , then we have the ordering relation  $\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$ .*

**SOLUTION.** The distance between  $\mathbf{x}_i$  and  $\mathbf{x}_j$  is equal to  $|\mathbf{x}_i - \mathbf{x}_j|$ , and if we substitute the expressions for the vectors in question we see that

$$|\mathbf{x}_i - \mathbf{x}_j| = |t_i - t_j| \cdot |\mathbf{b} - \mathbf{a}|.$$

Therefore we have

$$\begin{aligned} |\mathbf{x}_1 - \mathbf{x}_3| &= (t_3 - t_1) \cdot |\mathbf{b} - \mathbf{a}| = (t_3 - t_2) \cdot |\mathbf{b} - \mathbf{a}| + (t_2 - t_1) \cdot |\mathbf{b} - \mathbf{a}| = \\ &|\mathbf{x}_2 - \mathbf{x}_3| + |\mathbf{x}_1 - \mathbf{x}_2| \end{aligned}$$

and therefore by the definition of betweenness we have  $\mathbf{x}_1 * \mathbf{x}_2 * \mathbf{x}_3$ .■

**Example 2.** *Suppose that  $\mathbf{a} * \mathbf{b} * \mathbf{c}$ . Then we have the inclusion of closed rays  $[\mathbf{bc} \subset [\mathbf{ac}$ .*

**SOLUTION.** By hypothesis we have  $\mathbf{b} = \mathbf{a} + t(\mathbf{c} - \mathbf{a})$  for some scalar  $t$  satisfying  $0 < t < 1$ . If  $\mathbf{x} \in [\mathbf{bc}$ , then  $\mathbf{x} = \mathbf{b} + u(\mathbf{c} - \mathbf{b})$  for some scalar  $u$  satisfying  $0 < u < 1$ . Substituting the expression for  $\mathbf{b}$  in the preceding equation, we find that

$$\mathbf{x} = \mathbf{a} + (t + u - tu) \cdot (\mathbf{c} - \mathbf{a}).$$

To show that  $\mathbf{x} \in [\mathbf{ac}$ , we need to check that the coefficient of  $\mathbf{c} - \mathbf{a}$  is positive. However, we have  $t + u - tu = t(1 - u) + u$ ; the first term on the right hand side is a product of positive factors and hence is positive, and since  $u > 0$  it follows that the entire expression is positive.■

**Example 3.** *Suppose that  $\mathbf{a} * \mathbf{b} * \mathbf{c}$  and  $\mathbf{b} * \mathbf{x} * \mathbf{c}$ . Then we have  $\mathbf{a} * \mathbf{x} * \mathbf{c}$ .*

**SOLUTION.** This is similar to the preceding example. By assumption we have

$$\mathbf{b} = \mathbf{a} + t(\mathbf{c} - \mathbf{a}) \quad \mathbf{x} = \mathbf{b} + u(\mathbf{c} - \mathbf{b})$$

where  $0 < t, u < 1$ . It follows that

$$\mathbf{x} = \mathbf{a} + (t + u - tu) \cdot (\mathbf{c} - \mathbf{a}).$$

The desired conclusion will follow if we can verify that the coefficient of  $\mathbf{c} - \mathbf{a}$  lies between 0 and 1. Now

$$(1 - t)(1 - u) = 1 - u - t + tu$$

and since the left hand side lies between 0 and 1 it follows that the same is true for the right hand side. But this means that

$$u + t - tu = 1 - (1 - u - t + tu)$$

must also lie between 0 and 1.■