SOLUTIONS TO EXERCISES FOR

MATHEMATICS 133 — Part 3

Fall 2006

NOTE ON ILLUSTRATIONS. Drawings for several of the solutions in this file are available in the file (s)

`http://math.ucr.edu/∼res/math133/math133solutions3figures.∗`

where the extension $*$ is one of `doc`, `ps`, or `pdf`.

II. Linear algebra and Euclidean geometry

II.1 : Measurement axioms

1. First of all, f is 1–1. Define k_X so that $X = A + k_X(B - A)$. Then $f(X) = f(Y)$ implies $k_X d(A,B) = k_Y d(A,B)$, and since $d(A,B)$ is positive this means $k_X = k_Y$. Also, if r is an arbitrary real number and $k = r/d(A,B)$, then it follows that f maps $X = A + k(B - A)$ to r. Therefore f is onto. Finally, to verify the statement on distances, note that the distance from X to Y is equal to

$$|X - Y| = \Big|[A + k_X(B - A)] - [A + k_Y(B - A)]\Big| = \Big|[k_X(B - A)] - [k_Y(B - A)]\Big| =$$

$$\big|(k_X - k_Y)(B - A)\big| = |(k_X - k_Y)| \cdot |(B - A)| = |(k_X - k_Y)| \cdot d(A,B) =$$

$$|(k_X d(A,B) - k_Y)d(A,B)| = |f(X) - f(Y)|$$

which is the identity to be shown.■

2. Follow the hint, and let $h = g \circ f^{-1}$. By construction, h is a 1–1 onto map from the real numbers to themselves such that $|u - v| = |h(u) - h(v)|$ for all u and v. If $k(t) = h(t) - h(0)$, then elementary algebra shows that we also have $|u - v| = |k(u) - k(v)|$ but also $k(0) = 0$. Therefore we have $|k(t)| = |k(t) - k(0)| = |t - 0| = |t|$ for all t. In particular, this means that for each t we have $k(t) = \varepsilon_t \cdot t$, where $\varepsilon_t = \pm 1$. We claim that ε_t is the same for all $t \neq 0$; for $t = 0$ the value of ε does not matter. But suppose that we had $k(u) = u$ and $k(v) = -v$, where $u, v \neq 0$. Then we could not have $|u - v| = |k(u) - k(v)|$; If u and v have the same sign, then the right hand side is greater than the left, and if they have opposite signs, then the right hand side is less than the left (why?). Therefore $k(t) = \varepsilon \cdot t$ where $\varepsilon = \pm 1$, and hence also $h(t) = k(t) + h(0) = \pm t + h(0)$, which is the form required in the exercise.■

3. Write things out using barycentric coordinates. We have $X - A = (2, -6)$, while $B - A = (-5, -9)$ and $C - A = (4, -9)$. Thus we need to solve

$$(2, -6) = y(-5, -9) + z(4, -9) = (-5y + 4z, -9y - 9z)$$

and if we do so we obtain $z = \frac{12}{81}$ and $y = \frac{6}{81}$; using $x + y + z = 1$ we also obtain $x = \frac{7}{9}$. Thus all three barycentric coordinates are positive and the point lie in the interior.

To work the second part, we have $X - A = (10, k - 10)$, and we need to consider the system

$$(10, k-10) \;=\; y(-5,-9) \;+\; z(4,-9) \;=\; (-5y+4z, -9y-9z)$$

and determine those values of k for which $z > 0$ and $x = 1 - y - z > 0$. Solving for the barycentric coordinates, we find

$$z \;=\; \frac{150-5k}{81}\,, \qquad y \;=\; \frac{50-4k}{81}\,, \qquad x \;=\; \frac{9k-109}{81}\,.$$

The point will lie in the interior if and only if x and z are positive, which is the same as saying that the numerators $150 - 5k$ and $9k - 109$ should both be positive. This happens if and only if $12\frac{1}{9} < k < 30$.■

4. In this case we need to work the first part of the problem when X is either $(30, 200)$ or $(75, 135)$, so that $X - A$ is either $(23, 190)$ or $(68, 125)$. The barycentric coordinate z in both cases are negative and thererore neither point lies in the interior of the angle.■

5. We now have $X - A = (-1, 0)$, while $B - A = (3, -6)$ and $C - A = (-1, 20)$. Thus we need to solve

$$(-1,0) \;=\; y(3,-6) \;+\; z(-1,-20) \;=\; (3y-z, -6y-20z)$$

and if we do so we obtain $z = -\frac{1}{9}$ so that the point cannot lie in the interior of the angle.

To work the second part, we have $X - A = (21, k - 8)$, and we need to consider the system

$$(21, k-8) \;=\; (3y-z, -6y-20z)$$

and determine those values of k for which $z > 0$ and $x = 1 - y - z > 0$. Solving for the barycentric coordinates, we find

$$z \;=\; \frac{3k+118}{54}\,, \qquad y \;=\; \frac{k+412}{54}\,, \qquad x \;=\; \frac{50k-476}{54}\,.$$

The point will lie in the interior if and only if x and z are positive, which is the same as saying that the numerators for x and z should both be positive. The inequality $50k - 476 > 0$ implies $k > 0$ and hence we see that z is positive whenever x is positive, so that the condition for the point to lie in the interior of the angle is simply $k > 238/25$.■

6. The original file had a misprint; the open segment should be (AC). Suppose that $X \in (AC)$, so that $A * X * C$ is true. By theorems on plane separation, this implies that A and X lie on the same side of BC, and C and X lie on the same side of AB. But these are the two criteria for a point to lie in the interior of $\angle ABC$, and therefore we know that X lies in the interior of this angle.■

7. It looks as if the open ray $(DE$ meets the triangle in exactly one point. See the illustration in the file of figures.■

8. The segment (AC) should be corrected to (AX). With this correction, proceed as follows: If Y lies on (AX), then $A * Y * X$ is true, so that X and Y lie on the same sides of AB and AC. However, $B * X * C$ implies that X and B lie on the same side of AC and X and C lie on the same side of AB. Therefore we also know that Y and B lie on the same side of AC and Y and C lie on the same side of AB. All that remains is to show that Y and A lie on the same side of BC. But this follows because $A * Y * X$ and $X \in BC$.■

9. By the Protractor Postulat there is a point E on the side of BC opposite A such that $|\angle EBC| = |\angle ABC|$, and by a consequence of the Ruler Postulate there is a point $D \in (AE$ such that $d(D,B) = d(A,B)$. By construction the distance equation holds, and the angle measurement equation holds because $[AD = [AE$. Finally D is on the side of BC opposite A because $D \in (BE$, while E and A lie on opposite sides and all points of $(BE$ lie on a single side of BC.■

10. The interior of the triangle is contained in the interior of $\angle BAC$, so by the Crossbar Theorem we know that $(AD$ meets (BC) in some point E. It will suffice to show that we have the order relationship $A*D*E$ (take $A = X$ and $E = Y$). But this follows because A and D are on the same side of BC, which implies either $A*D*E$ or $D*A*E$. The second of these is incompatible with the known condition $E \in (AD$, so therefore the first must be true and we have shown what we needed.■

II.4 : Synthetic axioms of order and separation

1. Follow the hint. By **SSS** we have $\Delta BDC \cong \Delta BDE$, so that $|\angle EDB| = |\angle CDB|$. Since D lies on the segment (CE) it is in the interior of $\angle EBA = \angle ABC$, and therefore by additivity we have

$$|\angle ABC| \;=\; |\angle EDB| + |\angle CDB| \;=\; 2 \cdot |\angle DBC| \;=\; 2 \cdot |\angle DBA| \;.$$

This shows that the ray $(BD$ bisects $\angle ABC$ and proves existence.

To prove uniqueness, suppose that $(BG$ is an arbitrary bisector ray. Since $(BG$ lies in the interior of $\angle BAC$, it follows that it lies on the same side of AB as C. By hypothesis its measure is $\frac{1}{2}|\angle ABC|$, which is the same as $|\angle ABD|$. Therefore the uniqueness part of the Protractor Postulate implies that $[BG = [BD$.■

2. Take a 3–4–5 right triangle with a right angle at B and $d(A,B) = 3$. Then $\Delta ABC \cong \Delta BCA$ is false because $3 = d(A,B) \neq 4 = d(B,C)$.■

3. By the Isosceles Triangle Theorem and the identities $\angle DAB = \angle CAB$ and $\angle EBA = \angle CBA$ we have

$$|\angle DAB| \;=\; |\angle CAB| \;=\; |\angle CBA| \;=\; |\angle EBA|$$

and the midpoint conditions together with the isosceles triangle assumption imply $d(A,D) = \frac{1}{2}d(A,C) = \frac{1}{2}d(A,C) = d(B,E)$. Since $d(A,B) = d(B,A)$, by **SAS** we have $\Delta DAB \cong \Delta EBA$,■

4. An affine transformation T has the property

$$T\big(s\mathbf{a} + (1-s)\mathbf{b}\big) \;=\; sT(\mathbf{a}) + (1-s)T(\mathbf{b}$$

so a point $\mathbf{c}$ is between two points $\mathbf{a}$ and $\mathbf{b}$ of K if and only if its image $T(\mathbf{c})$ is between the two image points $T(\mathbf{a})$ and $(\mathbf{b})$. Therefore $\mathbf{c}$ is between two points of K if and only if its image is between two points of the image of K under the affine transformation. Therefore, if $\mathbf{c}$ is not between two points of K, then its image cannot be between two points in the image of K.■

5. (a) This is just a simple partial derivative calculation involving polynomials.■

(b) We have $T_1(\mathbf{x}) = A_1\mathbf{x} + \mathbf{b}_1$ and $T_2(\mathbf{x}) = A_2\mathbf{x} + \mathbf{b}_2$ where A_1 and A_2 are invertible matrices and $\mathbf{b}_1$ and $\mathbf{b}_2$ are vectors. The composite $T_1 \circ T_2$ sends $\mathbf{x}$ to $A_1A_2\mathbf{x} + A_1\mathbf{b}_2 + \mathbf{b}_1$. Therefore A_1A_2 gives the matrix part of $T_1 \circ T_2$.■

(c) $D(T)$ is the identity if and only if for each i the partial derivatives of the i^{th} coordinate function is equal to the partial derivative of the standard function x_i. But the latter holds if and only if the i^{th} coordinate function has the form $x_i + b_i$ for some constant b_i, and this is precisely the condition for T to be a translation.■

(d) This can be done directly, but we shall do it using the ideas described above. We have

$$D\left(S^{-1}\circ T\circ S\right) \quad = \quad D(S^{-1})D(T)D(S)$$

and te relation $S-1\circ S =$ identity implies $D(S^{-1})D(S) = I$, so that $D(S^{-1}) = D(S)^{-1}$. If we now assume T above is a translation, this gives us

$$D\left(S^{-1}\circ T\circ S\right) \quad = \quad D(S^{-1})D(T)D(S) \quad = \quad D(S^{-1})\,I\,D(S) \quad = \quad I$$

which shows that $S^{-1}TS$ must be a translation.■

6. Follow the hints. By construction every $DS(t)$ is equal to the diagonal matrix whose entries in order are 1 and -1. The product of this matrix with itself is the identity, and therefore D applied to $S(a)S(b)$ is the identity. By the preceding exercise, $S(a)S(b)$ must be a translation. We can find the translation vector fairly directly by evaluating at $(0, 0)$, and if we do so we fine that the twofold composite sends $(0, 0)$ to $(0, 2a - 2b)$.

Applying this to the threefold composite, we obtain

$$S(a)S(b)S(c)(x_1, x_2) \quad = \quad S(a)(x_1, x_2 + 2b - 2c) \quad = \quad (x_1, 2a + 2c - 2b - x_2)$$

which means that the threefold composite is $S(d)$ where $d = a + c - b$.■

7. The most direct way to do this is to prove that there are nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ such that A sends $\mathbf{a}$ to itself, A sends $\mathbf{b}$ to $-\mathbf{b}$, and the vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular. We can then get the desired orthonormal vectors by letting $\mathbf{u}$ and $\mathbf{v}$ be $\mathbf{a}$ and $\mathbf{b}$ multiplied by the reciprocals of their repsective lengths.

We can find nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ as above if and only if the equations $(A + I)\mathbf{x} = \mathbf{0}$ and if and only if the equations $(A - I)\mathbf{x} = \mathbf{0}$ have nontrivial solutions, which is the same as showing that the determinants of $A \pm I$ are equal to zero. Direct computation shows that

$$0 \quad = \quad \det(A - kI) \quad = \quad k^2 - \cos^2\theta - \sin^2\theta \quad = \quad k^2 - 1$$

and hence the determinant is zero if $k = \pm 1$. This yields the vectors $\mathbf{a}$ and $\mathbf{b}$.

It is possible to solve directly for these vectors and show they are perpendicular by direct computation, but we shall give a conceptual proof which does not require finding the vectors explicitly. Since A is orthogonal we have

$$\langle \mathbf{u}, \mathbf{v}\rangle \quad = \quad \langle A\mathbf{u}, A\mathbf{v}\rangle \quad = \quad \langle \mathbf{u}, -\mathbf{v}\rangle \quad = \quad -\langle \mathbf{u}, \mathbf{v}\rangle\ .$$

The right hand side is the negative of the left hand side, and this can only happen if both sides are zero. Therefore the two vectors we want are perpendicular to each other.■

8. Follow the hint. To show that $A - I$ is invertible, compute its determinant; this turns out to be $1 + 2\cos\theta$, which is zero if and only if θ is an integral multiple of 2π. We have excluded these choices for θ, and hence the matrix will always be invertible in our situation.

Applying this to the question in the exercise, we need to show there is a unique $\mathbf{z}$ such that $T(\mathbf{z}) = A\mathbf{z} + \mathbf{b}$, or equivalently there is a unique solution to the equation $(A - I)\mathbf{z} = -\mathbf{b}$. Since $A - I$ is invertible, there is indeed a unique solution to this equation, and this suffices to prove the exercise.■