SOLUTIONS TO EXERCISES FOR

MATHEMATICS 133 — Part 2

Fall 2007

NOTE ON ILLUSTRATIONS. Drawings for several of the solutions in this file are available in the following document:

`http://math.ucr.edu/~res/math133/math133solutions2figures.pdf`

I. Topics from linear algebra

I.4 : Barycentric coordinates

1. The general method is to take a typical point $\mathbf{x}$ and write $\mathbf{x}-\mathbf{a}$ as a linear combination

$$\mathbf{x}-\mathbf{a} \quad = \quad u\mathbf{b}-\mathbf{a} \ - \ v\mathbf{c}-\mathbf{a}$$

from which one obtains the barycentric coordinate expression

$$\mathbf{x} \quad = \quad (1-u-v)\mathbf{a} \ + \ u\mathbf{b} \ + \ v\mathbf{c} \ .$$

For all the problems below we have $\mathbf{b}-\mathbf{a} = (2,0)$ and $\mathbf{c}-\mathbf{a} = (1,1)$.

(a) In this case $\mathbf{x}-\mathbf{a} = (1,0) = \frac{1}{2}\cdot(2,0)+0\cdot(1,1)$. Hence $u = \frac{1}{2}$, $v = 0$, and t must be equal to $\frac{1}{2}$.

(b) In this case $\mathbf{x}-\mathbf{a} = (2,1) = \frac{1}{2}\cdot(2,0)+1\cdot(1,1)$. Hence $u = \frac{1}{2}$, $1 = 0$, and t must be equal to $-\frac{1}{2}$.

(c) In this case $\mathbf{x}-\mathbf{a} = (\sqrt{2}+1,\sqrt{2}) = \frac{1}{2}\cdot(2,0)+\sqrt{2}\cdot(1,1)$. Hence $u = \frac{1}{2}$, $v = \sqrt{2}$, and t must be equal to $\frac{1}{2}-\sqrt{2}$.

(d) In this case $\mathbf{x}-\mathbf{a} = (1,5) = -2\cdot(2,0)+5\cdot(1,1)$. Hence $u = -2$, $v = 5$, and t must be equal to -2.

(e) In this case $\mathbf{x}-\mathbf{a} = (3,-1) = 2\cdot(2,0)-1\cdot(1,1)$. Hence $u = 2$, $v = -1$, and t must be equal to 0.

(f) In this case $\mathbf{x}-\mathbf{a} = (\frac{1}{2},-\frac{1}{2}) = \frac{1}{2}\cdot(2,0)-\frac{1}{2}\cdot(1,1)$. Hence $u = \frac{1}{2}$, $v = -\frac{1}{2}$, and t must be equal to 1.$\blacksquare$

2. We shall first show that if $\{\,\mathbf{v}_0,\ \cdots\ ,\mathbf{v}_n\,\}$ is affinely independent then the associated set $\{\,\mathbf{v}_1-\mathbf{v}_0,\ \cdots\ ,\mathbf{v}_n-\mathbf{v}_0\,\}$ is linearly independent. — Suppose we have

$$\sum_{j=1}^{n} c_j(\mathbf{v}_j-\mathbf{v}_0) \quad = \quad \mathbf{0} \ .$$

If we add $\mathbf{v}_0$ to both sides and rearrange terms, we obtain

$$\mathbf{v}_0 \;=\; \mathbf{v}_0 + \sum_{j=1}^{n} c_j(\mathbf{v}_j - \mathbf{v}_0) \;=\; \left(1 - \sum_{j=1}^{n} c_j\right)\mathbf{v}_0 + \sum_{j=1}^{n} c_j\mathbf{v}_j \;.$$

Now the left hand side is an expression of $\mathbf{v}_0$ as a linear combination of $\{\,\mathbf{v}_0, \;\cdots\;, \mathbf{v}_n\,\}$ such that the coeeficients add up to 1, and therefore by the affine independence assumption we know that the corresponding coefficients on the left and right hand sides of the displayed equation(s) are equal. In particular, this means that $c_j = 0$ for all j; the latter in turn implies that the set $\{\,\mathbf{v}_1 - \mathbf{v}_0, \;\cdots\;, \mathbf{v}_n - \mathbf{v}_0\,\}$ is linearly independent.

Conversely, suppose $\{\,\mathbf{v}_1 - \mathbf{v}_0, \;\cdots\;, \mathbf{v}_n - \mathbf{v}_0\,\}$ is linearly independent, and suppose that $\mathbf{x}$ is an affine combination $\sum_j \; t_j\mathbf{v}_j$, where $\sum_j t_j = 1$. We then have

$$\mathbf{x} - \mathbf{v}_0 \;=\; \left(\sum_{j=0}^{n} t_j\mathbf{v}_j\right) - \mathbf{v}_0 \;=\; \left(\sum_{j=0}^{n} t_j\mathbf{v}_j\right) - \left(\sum_{j=0}^{n} t_j\;\mathbf{v}_0\right) \;=\; \sum_{j=1}^{n} t_j(\mathbf{v}_j - \mathbf{v}_0).$$

If we now take an arbitrary expression of $\mathbf{x}$ as an affine combination $\sum_j \; u_j\mathbf{v}_j$, where $\sum_j u_j = 1$, then the same sort of argument implies that

$$\mathbf{x} - \mathbf{v}_0 \;=\; \sum_{j=1}^{n} u_j(\mathbf{v}_j - \mathbf{v}_0)$$

and by the linear independence of $\{\,\mathbf{v}_1 - \mathbf{v}_0, \;\cdots\;, \mathbf{v}_n - \mathbf{v}_0\,\}$ we therefore know that $u_j = t_j$ for all $j \geq 1$. But then we also have

$$t_0 \;=\; 1 - \sum_{j=1}^{n} t_j \;=\; 1 - \sum_{j=1}^{n} u_j \;=\; u_0$$

so that all the corresponding coefficients t_j and u_j are equal and hence the set $\{\,\mathbf{v}_0, \;\cdots\;, \mathbf{v}_n\,\}$ is affinely independent.■

3. We shall follow the hint. The lines **ab** and **cd** are given by $\mathbf{a}+V$ and $\mathbf{c}+V$ respectively since $\mathbf{b}-\mathbf{a}$ and $\mathbf{d}-\mathbf{c}$ are nonzero scalar multiples of each other, and similarly the lines **ad** and **bc** are given by $\mathbf{a}+W$ and $\mathbf{b}+W$ respectively. By the Coset Property, if either of these pairs of lines has a point in common, then the two lines in the pair must be the same. However, this is impossible in both cases. Since **a**, **d** and **b** are noncollinear, it follows that the lines **ab** and **cd** cannot be equal, and likewise the lines **ad** and **bc** cannot be equal. Hence it follows that **ab** is parallel to **cd** and **ad** is parallel to **bc**.■

4. By the results of the previous problem we know that $C = B + D - A$. Since E is a midpoint we know that $E = \frac{1}{2}(A+B)$. Since F lies on DE and AC we may write

$$tA + (1-t)C \;=\; F \;=\; uE + (1-u)D$$

and if we substitute for C and E in the left and right sides we obtain the following equation:

$$(2t-1)A + (1-t)B + (1-t)D \;=\; tA + (1-t)C \;=\; \frac{u}{2}A + \frac{u}{2}B + (1-u)D$$

Since A, B and D are noncollinear and the coefficients on both sides of the equation add up to 1, we may set the corresponding coefficients equal and conclude that $2t-1 = \frac{1}{2}u$, $1-t = \frac{1}{2}u$, and $1-t = 1-u$. Solving these equations, we see that $t = u = \frac{2}{3}$, so that $F - C = \frac{2}{3}(A-C)$ and $F-D = \frac{2}{3}(E-D)$. Further algebraic manipulation shows that $A-F = (A-C)-(F-C)$ is equal to $\frac{1}{3}(A-C)$ and likewise $E-F = (E-D)-(F-D)$ is equal to $\frac{1}{3}(E-D)$. The distance relationships follow by taking the lengths of the vectors on both sides of the resulting two equations.■

5. By assumption we know that the barycentric coordinate expression for each point $\mathbf{p}_j$ is given by the formula

$$\mathbf{p}_j \;\; = \;\; t_j\mathbf{a} \; + \; \mathbf{u}_j\mathbf{b} \; + \; \mathbf{v}_j\mathbf{c} \; .$$

Suppose that the three points are collinear, so that $\mathbf{p}_3$ lies on the line $\mathbf{p}_1\mathbf{p}_2$. Then we have

$$\mathbf{p}_3 \;\; = \;\; w\mathbf{p}_2 \; + \; (1-w)\mathbf{p}_1$$

for some scalar w. Combining this with the previous expansions for $\mathbf{p}_1$ and $\mathbf{p}_2$, we obtain the following expression for $\mathbf{p}_3$:

$$t_3\mathbf{a} \; + \; \mathbf{u}_3\mathbf{b} \; + \; \mathbf{v}_3\mathbf{c} \;\; = \;\; \mathbf{p}_3 \;\; = \;\; w\big(t_1\mathbf{a} \; + \; \mathbf{u}_1\mathbf{b} \; + \; \mathbf{v}_1\mathbf{c}\big) \; + \; (1-w)\big(t_2\mathbf{a} \; + \; \mathbf{u}_2\mathbf{b} \; + \; \mathbf{v}_2\mathbf{c}\big) \;\; =$$

$$[wt_2 + (1-w)t_1]\mathbf{a} \; + \; [wu_2 + (1-w)u_1]\mathbf{b} \; + \; [wv_2 + (1-w)v_1]\mathbf{c}$$

The coefficients of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in the last expression add up to 1, and thus we may equate the barycentric coordinates in the first and last expressions in the chain of equations displayed above. We may rewrite these equations for barycentric coordinates in vector form as follows:

$$\big(t_3, u_3, v_3\big) \;\; = \;\; w \cdot \big(t_2, u_2, v_2\big) \; + \; (1-w) \cdot \big(t_3, u_3, v_3\big)$$

Therefore the row of the matrix

$$A \;\; = \;\; \begin{pmatrix} t_1 & u_1 & v_1 \\ t_2 & u_2 & v_2 \\ t_3 & u_3 & v_3 \end{pmatrix}$$

are linearly dependent, and hence the determinant of this matrix is equal to zero.

Conversely, suppose that the determinant of the matrix A is zero. Then the rows are linearly independent, so there are scalars $x,\ y,\ z$ not all zero such that

$$x \cdot \big(t_3, u_3, v_3\big) \; + \; y \cdot \big(t_2, u_2, v_2\big) \; + \; z \cdot \big(t_3, u_3, v_3\big) \;\; = \;\; \mathbf{0} \; .$$

Since the three coordinates of the expression on the left hand side are all equal to zero, we have the following equations:

$$xt_1 \; + \; yt_2 \; + \; zt_3 \;\; = \;\; 0$$

$$xu_1 \; + \; yu_2 \; + \; zu_3 \;\; = \;\; 0$$

$$xv_1 \; + \; yv_2 \; + \; zv_3 \;\; = \;\; 0$$

These equations in turn imply the vector equation

$$x\mathbf{p}_1 \; + \; y\mathbf{p}_2 \; + \; z\mathbf{p}_3 \;\; = \;\; \mathbf{0} \; .$$

If we add the three scalar equations we obtain

$$0 \;\; = \;\; (xt_1 \; + \; yt_2 \; + \; zt_3) \; + \; (xu_1 \; + \; yu_2 \; + \; zu_3) \; + \; (xv_1 \; + \; yv_2 \; + \; zv_3) \;\; =$$

$$x(t_1 \ + \ u_1 \ + \ v_1) \ + \ y(t_2 \ + \ u_2 \ + \ v_2) \ + \ z(t_3 \ + \ u_3 \ + \ v_3)$$

and since $t_1 \ + \ u_1 \ + \ v_1 \ = \ t_2 \ + \ u_2 \ + \ v_2 \ = \ t_3 \ + \ u_3 \ + \ v_3 \ = \ 1$ the preceding equations imply that $x + y + z = 0$.

We know that at least one of $x, \ y, \ z$ is nonzero. Suppose that $x \neq 0$; then it follows that $-x^{-1}y - x^{-1}z = 1$ and $\mathbf{p}_1 = -x^{-1}y\mathbf{p}_2 - x^{-1}z\mathbf{p}_3$, so that $\mathbf{p}_1$ lies on the line containing $\mathbf{p}_2$ and $\mathbf{p}_3$. Similarly, if $y \neq 0$ then it follows that $\mathbf{p}_2$ lies on the line containing $\mathbf{p}_1$ and $\mathbf{p}_3$, and finally if $z \neq 0$ then it follows that $\mathbf{p}_3$ lies on the line containing $\mathbf{p}_1$ and $\mathbf{p}_2$. In all three cases the points $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are collinear.■

6. Following the hint, we shall use the criterion of the previous exercise. With the hypotheses in Menelaus' Theorem, we obtain the following criterion for D, E and F to be collinear:

$$0 \ = \ \begin{vmatrix} t & (1-t) & 0 \\ 0 & u & (1-u) \\ (1-v) & 0 & v \end{vmatrix} \ = \ tuv \ + \ (1-t)(1-u)(1-v)$$

Since the vanishing of the right hand side is equivalent to the condition in the Theorem of Menelaus, this proves the latter.■

7. In this problem the hypothesis on D is equivalent to the equation $t = -1$, while the hypothesis on E is equivalent to the equation $u = \frac{1}{2}$. Therefore the equation in the Theorem of Menelaus becomes

$$\begin{aligned} u \ &= \ -(-1)(1-u)\tfrac{1}{2}v \\ (-1)\tfrac{1}{2}v \ &= \ -2\tfrac{1}{2}(1-v) \ = \ v-1 \end{aligned}$$

which simplifies to $v \ = \ \frac{2}{3}$. Substituting this into the equation for F in Exercise 6, we get $F = \frac{1}{3}A + \frac{2}{3}C$.■

8. The hypotheses guarantee that none of the numbers $t, \ u, \ v$ is equal to 0 and 1. Furthermore, if we write the intersection point G in the forms

$$xB \ + \ (1-x)E \ = \ yC \ + \ (1-y)F$$

then the condition $G \neq B, C$ implies that neither x nor y is equal to 1.

The hypotheses and elementary algebra also yield expressions for G as an affine combination of $A, \ B, \ C$ by the following chain of equations:

$$G \ = \ xB \ + \ (1-x)E \ = \ xB \ + \ (1-x)uC \ + \ (1-x)(1-u)A$$

$$G \ = \ yC \ + \ (1-y)F \ = \ yC \ + \ v(1-y)A \ + \ (1-x)(1-y)B$$

It follows that the corresponding barycentric coordinates in the right hand expressions equal, so that we have the following relationships among the various coefficients:

$$y \ = \ (1-x)u \qquad x \ = \ (1-v)(1-y) \qquad v(1-y) \ = \ (1-x)(1-u)$$

As noted in the hint, the lines AD, BE and CF are concurrent if and only if G lies on AD, or equivalently the points A, G and D are collinear. By the formula of Exercise 5, this happens if and only if we have

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & t & (1-t) \\ (1-x)(1-u) & x & (1-x)u \end{vmatrix} \ = \ 0 \ .$$

Evaluating the determinant, we see that concurrence is equivalent to the equation $tu(1-x) = (1-v)(1-y)(1-t)$. We have already seen that $v(1-y) = (1-x)(1-u)$, and we know that all the factors are nonzero (hence both sides are nonzero); therefore the equation in the previous sentence is equivalent to

$$tuv(1-x)(1-y) \quad = \quad (1-t)(1-u)(1-v)(1-x)(1-y)$$

and since $(1-x)$ and $(1-y)$ are nonzero it follows that the equation displayed above is equivalent to $tuv = (1-t)(1-u)(1-v)$, which is the criterion stated in the theorem.■

9. In this problem we have $D = \frac{1}{2}(B+C)$ and $E = uC + (1-u)A$, $F = uB + (1-u)A$. Since G lies on the lines AD, BE, and CF, we have an equation of the form

$$G \quad = \quad sA \ + \ \frac{1-s}{2}B \ + \ \frac{1-s}{2}C \quad = \quad rB \ + \ u(1-r)C \ + \ (1-u)(1-r)A$$

for suitably chosed real numbers r and s. Both the second and the third expression are affine combinations of G in terms of A, B and C, and therefore the corresponding coefficients must be equal. Thus we have

$$\frac{1-s}{2} \quad = \quad r \ \text{ and } \ u(1-r) \ \frac{1-s}{2} \quad = \quad r \ .$$

The second string of equations yields

$$\frac{r \quad = \quad u}{1+u}$$

while the first yields

$$s \quad = \quad \frac{1-u}{1+u} \ .$$

If we substitute these quantities into the expression for G and also use the equation for D in the first sentence of this solution, we conclude that

$$G \quad = \quad \frac{1-u}{1+u}A \ + \ \frac{u}{1+u}D \ .■$$

10. By hypothesis, for each $i = 0, \ \cdots, m$ we have $\mathbf{w}_i = \sum_j \ c_{i,j}\mathbf{v_j}$ where the sum runs from $j = 0$ to $j = k$ and $\sum_j \ c_{i,j} = 1$ for each i. Suppose now that we may write $\mathbf{y}$ as an affine combination $\sum_i \ t_i\mathbf{w}_i$, where the sum runs from $i = 0$ to m and $\sum \ t_i = 1$. Then we have

$$\mathbf{y} \quad = \quad \sum_{i=0}^{m} \left(\sum_{j=0}^{k} \ c_{i,j}\mathbf{v}_j \right) \quad = \quad \sum_{i,j=(0,0)}^{(m,k)} \ t_i c_{i,j}\mathbf{v}_j \quad = \quad \sum_{j=0}^{k} \left(\sum_{i-0}^{m} \ t_i c_{i,j} \right) \mathbf{v}_j$$

and to show this linear combination of the $\mathbf{v}_j$'s is an affine combination we need to show that

$$\sum_{j=0}^{k} \left(\sum_{i=0}^{m} \ t_i c_{i,j} \right) \quad = \quad 1 \ .$$

But the left hand side is equal to

$$\sum_{i,j=(0,0)}^{(m,k)} \ t_i c_{i,j} \quad = \quad \sum_{i=0}^{m} \left(\sum_{j=0}^{k} \ c_{i,j} \right)$$

and since each sum $\sum_j \; c_{i,j} = 1$ the right hand side reduces to $\sum_i t_i$, which we know is equal to 1.∎

II. Linear algebra and Euclidean geometry

II.1 : Approaches to Euclidean geometry

1. Two planes are needed. We have noncoplanar lines AB, AC and AD, and they lie in the union of planes ABC and ACD. They cannot lie in one plane because we are assuming the lines are not coplanar.∎

2. Again two planes are needed. If the points are $A,\ B,\ C,\ D,\ E$ then the planes ABC and CDE contain all three points.

It is natural to ask what happens if we have more than five points. If there are six points, then two planes are still enough (the plane of the first three points and the plane of the last three points), while if there are seven points then three planes are needed (if two planes contain a set of seven points, at least one will contain four of them). More generally, the minimum number of planes needed to contain k points, no three of which are collinear, is equal to $\mathbf{INT}\big((n+2)/3\big)$.∎

3. The lines **ab** and **cd** are not coplanar since **a, b, c, d** are not coplanar. If the lines intersected, then they would be coplanar, and of course they are not parallel since they are not coplanar.∎

4. Suppose that two of the lines are the same, say $\mathbf{xp}_i = \mathbf{xp}_j$. If we call this line M, then M contains $\mathbf{x}$, $\mathbf{p}_i$ and $\mathbf{p}_j$. Now L contains the last two of these three points, therefore $L = M$. Hence we also have $\mathbf{x} \in M = L$; but this contradicts the basic condition $\mathbf{x} \notin L$. The problem with the logic arises from our assumption that $\mathbf{xp}_i = \mathbf{xp}_j$, and thus these lines must be sdistinct.

To prove that there are infinitely many lines through $\mathbf{x}$, by the preceding argument it is enough to show that L contains infinitely many points; specifically, such an infinite family can be found by taking all points of the form $\mathbf{p_1} + n(\mathbf{p}_2 - \mathbf{p}_1)$.∎

5. Follow the hint. Let M be a line distinct from $L_1,\ \cdots\ , L_n$; this line exists by the previous result. We then have

$$M \cap \big(L_1 \cup\ \cdots\ \cup L_n\big) \;\;=\;\; (M \cap L_1) \cup\ \cdots\ \cup (M \cap L_n)\ .$$

Now each set $M \cap L_j$ has either one point or no points, and thus the displayed set contains at most n points. Now M contains infinitely many points, so that there is some point $y \in M$ not in the displayed set. We claim that $y \notin L_j$ for all j. This follows because $y \in M$ and $y \notin M \cap L_j$ for all j. The only way this can happen is if y does not lie on any of the lines L_j.∎

6. Again follow the hint, taking the points $\mathbf{p}_j$, a point $\mathbf{x}$ distinct from all of them, and a line L through $\mathbf{x}$ which is not equal to any of the lines $\mathbf{xp}_j$. We claim that L does not contain any of the points $\mathbf{p}_j$. Suppose that some $\mathbf{p}_k \in L$. Since L contains both this point and $\mathbf{x}$, it follows that $L = \mathbf{xp}_k$. But this contradicts our choice of L, so we have a contradiction. The problem arises from our assumption that L was one of the lines $\mathbf{xp}_j$, and hence L must be distinct from all of these lines.∎

7. We first show that the planes **abu** and **abv** are distinct. If they are equal, let Q be the common plane they represent. Since **u** and **v** lie on this plan, it follows that the line **uv** is contained in Q. Now the line **uv** contains the point **c**, and therefore Q contains the points **a, b, c**. The latter means that Q is equal to the plane $P = \mathbf{abc}$. Since we also have $\mathbf{z} \in \mathbf{uv}$, it follows that we also have $\mathbf{z} \in Q = P$. This contradicts our assumption that **z** was not on P, and thus the assumption that $\mathbf{abu} = \mathbf{abv}$ must be false. Therefore these two planes must be distinct.

To see that there are infinitely many planes through **ab**, note that there are infinitely many points $\mathbf{u}_j$ on the line **cz** and by the preceding paragraph the planes $\mathbf{abu}_j$ must be distinct.■

8. Let L be a line in P, and let $B,\ C,\ D$ be distinct points on L. Then there is a point A which does not lie on L. By Exercise 4, the lines $AB,\ AC,\ AD$ are distinct. Each of these lines contains a third point, so let $E \in AB$, $F \in AC$ and $G \in AD$ be the third points. These three points are distinct; for example, if $E = F$ then the lines AB and AC would be equal, and since this common line contains B and C it must be equal to L; this cannot happen since $A \in AB$ but $A \notin L$. Likewise (interchanging the roles of $B,\ C,\ D$) we can show that $E \neq G$ and $F \neq G$. By construction we know that A is not equal to E, F or G. We claim that B is not equal to any of these three points either; by construction we cannot have $B = E$. If, say, $B = F$, then we know that the lines AB and $AF = AC$ are equal, so that $C \in AB$and $A \in BC = L$. This contradicts our choice of A, so we cannot have $B = F$. Similarly, we cannot have $B = G$ (switch the roles of C and D in the argument we just finished). Finally, if we switch the roles of B and either C or D, then we may similarly conclude that C and D are also distinct from $E,\ F,\ G$. Putting this all together, we conclude that the points $A,\ B,\ C,\ D,\ E,\ F,\ G$ are all distinct.■

II.2 : Synthetic axioms of order and separation

1. Let $\mathbf{a} = (a_1,\, a_2\, a_3)$, $\mathbf{b} = (b_1,\, b_2\, b_3)$, and $\mathbf{c} = (c_1,\, c_2\, c_3)$. Since the three points are collinear we have $\mathbf{b}_2 = \mathbf{a} + t(\mathbf{c} - \mathbf{a}$ for some scalar t. We need to show that under the conditions of the exercise we have $0 < t < 1$.

If we specialize the formula for $\mathbf{b}_2$ to the first coordinate we find that $b_1 = a_1 + t(c_1 - a_1)$. Solving for t, we obtain the following equation:

$$t \quad = \quad \frac{(b_1 - a_1}{c_1 - a_1}$$

Since $0 < b_1 - a_1 < c_1 - a_1$ it follows that $0 < t < 1$ as required.■

2. Following our usual procedure, we need to find scalars s and t such that $0 < t < 1 < s$ and

$$tA \ + \ (1-t)B \quad = \quad sC \ + \ (1-s)D \ .$$

Since $C = \mathbf{0}$, it is easier to work this problem using ordinary coordinates instead of barycentric coordinates. Thus we are looking for a point whose coordinates are $(t, 1-t) = \big(2(s-1),\, (s-1)\big)$. Setting the first and second coordinates equal, we have $t = 2s - 2$ and $1 - t = s - 1$. If we solve this system of equations we obtain the values $t = \frac{2}{3}$ and $s = \frac{4}{3}$. Thus there is a point X such that X is between A and B, and also C is between D and X.■

3. The betweenness conditions in the problem yield the equations $C = A + u(B - A)$ and $E = A + v(B - A)$ where $u,\ v > 1$. Therefore we want to find a point X such that

$$X \quad = \quad sB \ + \ (1-s)E \quad = \quad tD \ + \ (1-t)C$$

where $0 < s,\ t < 1$. We shall use the same method as in many other problems. Expand the second and third expressions in terms of A, B, and D, note that the coefficients add up to one in both cases, and use the uniqueness of barycentric coordinates to conclude that the coefficients of A, B, and D are equal.

Here is what we get if we expand the two expressions:

$$X \;=\; sB + (1-s)[A + v(D-A)] \;=\; (1-s)(1-v)A + sB + (1-s)vD$$

$$X \;=\; tD + (1-t)[A + u(B-A)] \;=\; (1-t)(1-u)A + (1-t)uB + tD$$

The coefficients of A, B, and D in both right hand expressions add up to 1, and therefore by the uniqueness of barycentric coordinates we know that the corresponding coefficients are equal. Therefore we have

$$s = (1-t)u, \qquad (1-s)(1-v) = (1-t)(1-u), \qquad t = (1-s)v$$

and these yield the following system of two linear equations for s and t:

$$s + tu = u$$

$$sv + t = v$$

If we solve this system we obtain the following values:

$$s = \frac{u-uv}{1-uv}, \qquad t = \frac{v-uv}{1-uv}$$

To complete the proof, we need to check that s and t are between 0 and 1; the key point is to recall that u and v are both greater than 1. The latter inequalities imply that the numerators and denominators of the expressions for s and t are both negative, and thus it follows that the quotient expressions for s and t are both positive. To see that these numbers are less than 1, multiply the numerators and denominators by -1 to obtain

$$s = \frac{uv-u}{uv-1}, \qquad t = \frac{uv-v}{uv-1}$$

and note that $0 < uv-u,\ uv-v < uv-1$ because u and v are both greater than 1.■

4. The equation of L is $4y = x+10$, and this may be rewritten in the form $0 = 4y-x-10 = f(x,y)$. Two points (a,b) and (c,d) lie on the same side of L if and only if the signs of $f(a,b)$ and $f(c,d)$ are both positive or both negative.

If we substitute the values for the coordinates of the two points, we obtain $f(4,-2) = 4(-2) - 4 - 10 = -22$ and $f(6,8) = 4 \cdot 8 - 6 - 10 = 16$. Since one value is positive and the other is negative, it follows that the two points lie on opposite sides of the line L.■

5. In this case the line L is defined by the equation $3x-y-7 = 0$. We now have $f(8,5) = 12$ and $f(-2,4) = -17$, so once again the two points lie on opposite sides of the line L.■

6. Suppose that X and Y lie on the line M. Since $L \cap M = \emptyset$, it follows that neither X nor Y lies on L. Suppose that X and Y lie on opposite sides of L. Then by the Plane Separation Property it follows that the segment (XY) meets L in some point W. Since (XY) is a subset of the line M, it follows that $W \in L \cap M$; but this contradicts our original assumption that $L \cap M = \emptyset$. Therefore the only possibility is that X and Y lie on the same side of L.■

7. The analogous result in 3–dimensional space is as follows:

Let P and Q be parallel (hence disjoint) planes in space. Then all points of Q lie on the same side of P.

Here is the analogous proof: Suppose that $X$and Y lie on the plane Q. Since $P \cap Q = \emptyset$, it follows that neither X nor Y lies on P. Suppose that X and Y lie on opposite sides of P. Then by the Plane Separation Property it follows that the segment (XY) meets P in some point W. Since X and Y lie in Q, we know that Q contains the entire line XY, and hence it also contains the segment (XY). Therefore it follows that $W \in P \cap Q$; but this contradicts our original assumption that $P \cap Q = \emptyset$. Therefore the only possibility is that X and Y lie on the same side of P.■

8. Suppose the conclusion is false, so that there is some triangle ΔABC and line L such that L contains points on each of the open segments (AB), (BC) and (AC). We shall call these points X, Y and Z respectively.

Now the line L cannot be equal to any of the lines AB, BC or AC; to see that $L \neq AB$, note that L also contains a point of (AC) and hence we would also have $L = AZ = AC$, which in turn would imply that A, B and C would be collinear. Similar considerations show that L is not equal to either of the lines BC or AC.

We also claim that L cannot contain any of the points A, B, C. For example, if $A \in L$, then $X, Z \in L$ would imply $A = AX = AY$, and since $AX = AB$ and $AZ = AC$ would imply that L contains all of A, B, C and this is impossible since these three points are not collinear. Interchanging the roles of the three vertices, we also find that neither B nor C can lie on L.

Since $X \in L$ and $A * X * B$ our results on order and separation imply that A and B lie on opposite sides of L (compare the proof of Pasch's Theorem). Similarly $Y \in L$ and $B * Y * C$ imply that B and C lie on opposite sides of L, and $Z \in L$ and $A * Z * C$ imply that A and C lie on opposite sides of L. On the other hand, the first two conclusions imply that both A and C both lie on the side of L which does not contain B, so that they must lie on the same side of L, and thus we have a contradiction. The source of this contradiction was our original assumption that there was a triangle ΔABC and line L such that L contains points on each of the open segments (AB), (BC) and (AC). Therefore this cannot happen, and the statement in the exercise must be true for every line and triangle in the same plane.■

9. By assumption we have

$$\mathbf{y} \;=\; \sum_j b_j \mathbf{w}_j \qquad \text{where all } b_j \;\geq\; 0 \quad \text{and} \quad \sum_j b_j \;=\; 1\;.$$

Furthermore, for each j we also know that

$$\mathbf{w}_j \;=\; \sum_k a_{j,k} \mathbf{v}_k \qquad \text{where all } a_{j,k} \;\geq\; 0 \quad \text{and} \quad \sum_k a_{j,k} \;=\; 1 \quad \text{for all } j.$$

Therefore we have

$$\mathbf{y} \;=\; \sum_{j,k} b_j \, a_{j,k} \, \mathbf{v_k}$$

and if we set $c_k = \sum_j b_j \, a_{j,k}$ it follows immediately that $c_k \geq 0$ for all j and

$$\sum_k c_k = \sum_{j,k} b_j \, a_{j,k} = \sum_j b_j \left(\sum_k a_{j,k} \right) .$$

Since the summations inside the parentheses are all equal to 1, it follows that the summation of interest is equal to $\sum_j b_j$; but we know the latter is equal to 1. Therefore it follows that $\mathbf{y}$ is a convex combination of the vectors $\mathbf{v}_k$ in the set S.■