SOLUTIONS TO EXERCISES FOR

MATHEMATICS 133 — Part 4

Fall 2007

NOTE ON ILLUSTRATIONS. Drawings for several of the solutions in this file are available in the following file:

`http://math.ucr.edu/~res/math133/math133solutions4figures.pdf`

II. Linear algebra and Euclidean geometry

II.5 : Euclidean parallelism

1. Follow the hint, and write $L = \mathbf{x} + V$ and $M = \mathbf{y} + W$. If L and M were parallel then we would have $V = W$, so since they are not we know that $V \neq W$. This implies that V and W are both **proper** vector subspaces of $U = V + W$. Now V and W each have one element bases given by the nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ respectively, so $V + W$ is spanned by $\{\,\mathbf{v},\ \mathbf{w}\,\}$, which means that $\dim U \leq 2$; since U properly contains V and W, it follows that its dimension is exactly 2.

Now let $P = \mathbf{x} + U$, so that $L \subset P$. We claim that $P \cap M = \emptyset$, and we shall do so by *reductio ad absurdum*. So assume M and P meet at the point $\mathbf{z}$. Then by the Coset Property we know that $M = \mathbf{z} + W$ and $P = \mathbf{z} + U$. Thus we have $\mathbf{x} = \mathbf{z} + b\mathbf{v} + c\mathbf{w}$ for suitable scalars b, c. Rearranging this, we obtain

$$\mathbf{x} \ - \ b\mathbf{v} \ = \ \mathbf{y} \ + \ c\mathbf{w}$$

and by the first sentence of this proof the latter yields a point on $L \cap M$. But we are given that $L \cap M = \emptyset$, so this is a contradiction. Therefore the line M and the plane P cannot have any points in common.∎

2. The lines $S \cap P$ and $S \cap Q$ both lie in the plane S. If the intersection were nonempty and X belonged to that intersection, then we would have

$$X \ \in \ (S \cap P) \ \cap \ (S \cap Q) \ = \ (S \cap P \cap Q) \ \subset \ P \cap Q$$

which contradicts the hypothesis that $P \cap Q = \emptyset$.∎

3. We shall do this problem using linear algebra. Following the hint, we first show that if S is a plane and $\mathbf{v} \notin S$, then there is a unique plane T such that $\mathbf{v} \in T$ and $S \cap T = \emptyset$.

Existence. Write $S = \mathbf{u} + W$ where W is a 2-dimensional vector subspace of $\mathbf{R}^3$, and let $T = \mathbf{v} + W$. Then $T \neq S$ because $\mathbf{v} \in T$ but $\mathbf{v} \notin S$, and therefore by the Coset Property for translates of the same subspace we know that S and T must be disjoint.

Uniqueness. Suppose we have an arbitrary 2-plane $\mathbf{y} + W'$ for some 2-dimensional vector subspace W'. By the Coset Property we have $\mathbf{y} + W' \ = \ \mathbf{v} + W'$. We claim that $\mathbf{v} + W$ and $\mathbf{v} + W'$ have a point in common if $W \neq W'$. The latter means that $W + W'$ properly contains either W or W'; this proper containment implies that $\dim W + W' \geq 3$, and since $W + W'$ is a vector subspace of $\mathbf{R}^3$ it follows that $W + W' = \mathbf{R}^3$. This in turn implies that $\dim W \cap W' = 1$. By the proof of the dimension theorem for subspaces of a vector space, it follows that there are vectors $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ such that the first one defines a basis for $W \cap W'$, the first two define a basis for W, and the first and last define a basis for W', so that all three form a basis for $\mathbf{R}^3$.

We may now write

$$\mathbf{v} - \mathbf{u} = t_1 \mathbf{y}_1 + t_2 \mathbf{y}_2 + t_3 \mathbf{y}_3$$

for suitable scalars t_i, and we may rewrite this equation as follows:

$$\mathbf{v} - t_3 \mathbf{y}_3 = \mathbf{u} + t_1 \mathbf{y}_1 + t_2 \mathbf{y}_2$$

The expression on the right hand side of this equation lies in $\mathbf{v} + W' = T$, whild the expression on the left hand side lies in $\mathbf{u} + W = S$, and thus we conclude that $S \cap T \neq \emptyset$. — This completes the proof of the assertion in the second sentence of this solution.

Conclusion of the argument. Suppose that $\mathbf{z} \in P \cap Q$. Then P and Q are two planes through $\mathbf{z}$ such that each is disjoint from S. Since this contradicts that result that we established above, it follows that there cannot be a point which lies on both P and Q.■

4. Again follow the hint; we shall use the notation introduced there.

The vector $Q(\mathbf{z})$ is perpendicular to $\mathbf{e}$ and $\mathbf{f}$ by the argument for deriving the Gram-Schmidt orthonormalization process (see Section I.1 of the notes). Since $\mathbf{v}$ and $\mathbf{w}$ are linear combinations of $\mathbf{e}$ and $\mathbf{f}$, it follows that $Q(\mathbf{z})$ is also perpendicular to $\mathbf{v}$ and $\mathbf{w}$. The Pythagorean principle for inner products now implies that

$$\left|\mathbf{z} - s\mathbf{e} - t\mathbf{f}\right|^2 = |Q(\mathbf{z})|^2 + |s - \langle \mathbf{z}, \mathbf{e}\rangle \mathbf{e}|^2 + |t - \langle \mathbf{z}, \mathbf{f}\rangle \mathbf{f}|^2$$

and this expression takes its minimum value for the values of s and t which make the last two summands equal to zero. Now this is precisely the condition under which $\mathbf{z} - s\mathbf{e} - t\mathbf{f}$ is perpendicular to (all linear combinations of) $\mathbf{e}$ and $\mathbf{f}$.

Finally since $\mathbf{e}$ and $\mathbf{f}$ span the same subspace of $\mathbf{R}^3$ as $\mathbf{v}$ and $\mathbf{w}$, it follows that the set of all vectors expressible in the form $\mathbf{z} - s\mathbf{e} - t\mathbf{f}$ is identical to the corresponding set of vectors expressible as $\mathbf{z} - a\mathbf{v} - b\mathbf{w}$, and therefore the minimum values for the (squares of the) lengths of vectors of the two types must also be equal. Combining this with the previous observations, we obtain the conclusion stated in the exercise.■

5. Write $\mathbf{x} = p\mathbf{a}$ and $\mathbf{y} = \mathbf{b} + q(\mathbf{c} - \mathbf{b})$, where p and q are scalars. Then we have

$$\mathbf{y} - \mathbf{x} = \mathbf{b} + q(\mathbf{c} - \mathbf{b}) + p\mathbf{a}$$

and since the original lines are skew lines we know that $\mathbf{a}$ and $\mathbf{c} - \mathbf{b}$ are linearly independent by (the solution to) Exercise 1. If we now apply Exercise 2, we see that the length of the displayed vector is minimized when the vector in question is perpendicular to $\mathbf{a}$ and $\mathbf{c} - \mathbf{b}$, which is the same as saying that the line $\mathbf{xy}$ is perpendicular to $\mathbf{0a}$ and $\mathbf{bc}$.■

6. It will suffice to show that $B - A$ and $D - C$ are nonzero multiples of each other (so that the lines AB and CD are either equal or parallel, and we know they are not equal), and similarly that $D - A$ and $B - C$ are nonzero multiples of each other.

By our hypotheses we have $A = \frac{1}{2}(W + X)$, $B = \frac{1}{2}(X + Y)$, $C = \frac{1}{2}(Y + Z)$, and $D = \frac{1}{2}(Z + W)$, so that

$$B - A = \tfrac{1}{2}(Y - W), \qquad D - C = \tfrac{1}{2}(W - Y)$$

and this shows AB is parallel to CD. Similarly, we have

$$D - A = \tfrac{1}{2}(Z - X), \qquad C - B = \tfrac{1}{2}(Z - X)$$

and this shows AD is parallel to BC.

7. Let M be the unique line through D which is equal or parallel to AC. Then $B \notin M$, and since AC has points in common with AB and BC it follows that M also has points X and Y in common with these two lines.

There are three cases, depending upon whether

(*i*) $D \in AC$,
(*ii*) D and B lie on the same side of AC,
(*iii*) D and B lie on opposite sides of AC.

In the first case it suffices to show that $D \in (AC)$, and in the other cases it we need to show that $X \in (BA$ and $Y \in (BC$. Note that $X \neq A$ and $Y \neq C$ in the last two cases because M must be parallel to AC.

In the first case, we have $D \in AC$ and D cannot be A or C. Now $A * C * D$ would imply A and D lie on opposite sides of BC, and $D * A * C$ would imply C and D lie on opposite sides of AB, so the condition $D \in \mathrm{Int}\, \angle ABC$ forces the conclusion $A * D * C$, so that $D \in (AC)$.

In the second case, by the Crossbar Theorem we know that $(BD$ and (AC) have a point E in common. We claim that $B * D * E$ holds. Since $E \in (BD$, the other possibilities are $D = E$ or $B * E * D$. The first of these is impossible because we have a pair of parallel lines such that one contains D and the other contains E, and the second contradicts our assumption that B and D lie on the same side of AC. Therefore the line M contains a point between A and E. If we apply this and Pasch's Theorem to ΔABE and ΔCBE, we find that M must contain points on (AB) and (BC) because the line $AC = AE = CE$ is parallel to M. These points on M must be X and Y respectively, so we know that $X \in (BA) \subset (BA$ and $Y \in (BC) \subset (BC$.

The third case is similar, and once again we have the point E, but this time we claim that $B * E * D$ holds. As before, we cannot have $D = E$, but now $B * D * E$ would imply that D and B were on the same side of AC, so we are forced to conclude that $B * E * D$. In this case we know that the line AC contains the point E betwween B and D. If we apply this and Pasch's Theorem to ΔXBD and ΔYBD, we find that AC must contain points on (BX) and (BY) because the line $M = DX = DY$ is parallel to AC. These points on AC must be A and C respectively, so we know that $A \in (BX)$ and $C \in (BY)$. Therefore we have $B * A * X$ and $B * C * Y$, which means that $X \in (BA$ and $Y \in (BC$.■

III. Basic Euclidean concepts and theorems

III.1 : Perpendicular lines and planes

1. Assuming that the three planes P, Q and T have one point in common but do not have a line in common, define lines L, M, N such that $L = P \cap Q$, $M = P \cap T$ and $N = Q \cap T$. Then we have $L \cap M \cap N = P \cap Q \cap T$, and we know it contains at least one point. However, if the planes do not have a line in common, then it follows that $L \neq M$, for otherwise they would. Now the lines

L and M have at most one point in common, so the same is true for the subset $L \cap M \cap N$. Since this subset contains at least one point and cannot contain more than one point, it must consist of exactly one point.■

2. As usual, follow the hint. The formula from Section I.2 states that

$$\big(\mathbf{v} \times \mathbf{w}\big) \cdot \big(\mathbf{y} \times \mathbf{z}\big) \quad = \quad (\mathbf{v} \cdot \mathbf{y})(\mathbf{w} \cdot \mathbf{z}) \;-\; (\mathbf{v} \cdot \mathbf{z})(\mathbf{w} \cdot \mathbf{y}) \; .$$

In our situation $\mathbf{w} = \mathbf{z} = \mathbf{u}$, while $\mathbf{v} = \mathbf{a}$ and $\mathbf{y} = \mathbf{b}$; we know that $\mathbf{u}$ is a unit vector which is perpendicular to $\mathbf{a}$ and $\mathbf{b}$. If we make these substitutions we find that

$$\big(\mathbf{a} \times \mathbf{u}\big) \cdot \big(\mathbf{b} \times \mathbf{u}\big) \quad = \quad \mathbf{a} \cdot \mathbf{b}$$

and also that $|\mathbf{a}| = |\mathbf{a} \times \mathbf{u}|$ as well as $|\mathbf{b}| = |\mathbf{b} \times \mathbf{u}|$. Combining these, we see that the cosine of the angle $\angle(\mathbf{x} + \mathbf{a})\mathbf{x}(\mathbf{x} + \mathbf{b})$ is equal to the cosine of the angle $\angle(\mathbf{a} \times \mathbf{u})\mathbf{0}(\mathbf{b} \times \mathbf{u})$.■

3. Follow the hint, and write $P = \mathbf{x} + W$, where W is a 2–dimensional vector subspace. Let $\mathbf{e}$ and $\mathbf{f}$ form an orthonormal basis for W, let U and V be the 1–dimensional subspaces they span, and take L and M to be the lines $\mathbf{x} + U$ and $\mathbf{x} + V$. Since $\mathbf{e}$ and $\mathbf{f}$ are perpendicular, the lines L and M will also be perpendicular. Suppose that we have a third line in the plane, say $\mathbf{x} + T$, which is perpendicular to both L and M, and let $\mathbf{g}$ be a nonzero vector in T; note that T must be contained in W. It will follow that $\{\, \mathbf{e}, \mathbf{f}, \mathbf{g} \,\}$ is a set of nonzero mutually perpendicular vectors and hence is linearly independent. This is impossible; since W is 2–dimensional, every linearly independent subset of it contains at most two vectors. Therefore a third perpendicular cannot exist.■

4. We use similar ideas to those of the preceding exercise. Clearly we can form another line $\mathbf{x} + T$ in this case, where T is spanned by $\mathbf{e} \times \mathbf{f}$. If we are given any line $\mathbf{x} + S$ perpendicular to L and M, then S is spanned by a vector $\mathbf{h}$ which is perpendicular to $\mathbf{e}$ and $\mathbf{f}$; since all such vectors are scalar multiples of the cross product, it follows that $\mathbf{x} + S$ must be the previously described line $\mathbf{x} + T$.■

III.2 : Basic theorems on triangles

(Not complete. The remainder will be posted in another file.)

1. Suppose that $\Delta ABC \cong \Delta DEF$. Then $d(A, B) = d(D, E)$ and

$$d(B, G) \quad = \quad \tfrac{1}{2}\, d(B, C) \quad = \quad \tfrac{1}{2}\, d(E, F) \quad = \quad d(E, H)$$

and since $\angle CBA = \angle GBA$ and $\angle FED = \angle HED$ we also have

$$|\angle GBA| \quad = \quad |\angle CBA| \quad = \quad |\angle FED| \quad = \quad |\angle HED| \; .$$

Therefore we have $\Delta GBA \cong \Delta HED$ by **SAS**.

Conversely, suppose that $\Delta ABG \cong \Delta DEH$. Then as before we have $d(A, B) = d(D, E)$, and

$$d(B, C) \quad = \quad 2\, d(B, G) \quad = \quad 2\, d(E, H) \quad = \quad d(E, F)$$

and the reasoning in the previous paragraph yields

$$|\angle CBA| \quad = \quad |\angle GBA| \quad = \quad |\angle HED| \quad = \quad |\angle FED| \; .$$

Therefore we have $\Delta CBA \cong \Delta FED$ by **SAS.**■

2. Since $d(A,B) = d(A,C)$, $d(A,D) = d(A,D)$ and the bisection condition implies

$$|\angle DAB| \;\; \tfrac{1}{2}|\angle CAB| \;\; = \;\; |\angle DAC|$$

it follows that $\Delta DAB \cong \Delta DAC$ by **SAS**. Therefore we also have $d(A,D) = d(B,D)$, so that D must be the midpoint of $[BC]$. Combining the latter with $d(A,B) = d(A,C)$, we see that AD is the perpendicular bisector of $[BC]$, so that $AD \perp BC$.■

3. By the Isosceles Triangle Theorem we have $|\angle PTS| = |\angle PST|$. Therefore by the Supplement Postulate for angle measure we have

$$|\angle PLT| \;=\; 180 - |\angle PTS| \;=\; 180 - |\angle PST| \;=\; |\angle PSR|$$

so that $\Delta PLT \cong \Delta PSR$ by **SAS** and the hypothesis $d(R,S) = d(L,T)$. The triangle congruence implies that $d(P,L) = d(P,R)$.

We are given the betweenness conditions $R*S*T$, $R*S*L$ and $R*T*L$, and from these we conclude that $L*T*S$ is also true. Combining $L*T*S$ and $R*S*T$ with the distance equations, we find that

$$d(L,S) \;=\; d(L,T) + d(T,S) \;=\; d(R,S) + d(S,T) \;=\; d(R,T)$$

and if we combine this with the previously obtained relations we see that $\Delta RTP \cong \Delta LSP$ by **SSS.**■

4. Since $C*B*D$ and $A*B*E$ hold, it follows that A and E lie on opposite sides of CD. Therefore we shall have $AC||DE$ if $|\angle ACD| = |\angle CDE|$; since $B \in (CD)$, the latter is equivalent to $|\angle ACB| = |\angle BDE|$.

To prove the final statement, note first that the common midpoint condition implies that $d(A,B) = d(B,E)$ and $d(C,B) = d(B,D)$. By the Vertical Angle Theorem we also have $|\angle ABC| = |\angle DBE|$, and therefore by **SAS** we have $\Delta ACB \cong \Delta DBE$. The desired equation $|\angle ACB| = |\angle BDE|$ is an immediate consequence of this.■

5. We shall follow the hint and first verify the betweenness relationships $A*C*E$ and $F*C*G$. First of all, $B*C*D$ and $C \in AE$ imply that B and D lie on opposite sides of AE. Next, $A*F*B$ and $D*G*E$ imply that B and F lie on the same side of AE and also that D and G lie on the same side of AE. This means that F and G must lie on opposite sides of AE, and since $C \in AE \cap FG$ this means that $F*C*G$ must be true. Furthermore, $A*F*B$, $D*G*E$ and $F*C*G$ imply that A and E lie on opposite sides of BC.

We can now use the Vertical Angle Theorem to conclude that $|\angle BCA| = |\angle DCE|$, and therefore $\Delta BCA \cong \Delta DCE$ by **SAS**. By the Alternate Interior Angle Theorem, we also know that AB is parallel to BE.

To conclude tha proof, we can now use the Alternate Interior Angle Theorem again to conclude that $|\angle FAC| = |\angle GEC|$, and another application of the Vertical Angle Theorem implies that $|\angle FCA| = |\angle GCE|$. Since we are given that $d(A,C) = d(C,E)$, it follows that $\Delta FAC \cong \Delta GCE$ by **ASA.**■

7. Let α, β, γ and δ be the measures of $\angle BAC$, $\angle ABC$, $\angle ACB$, and $\angle ADB$. We know that $d = 130$, but for the time being it is simpler to file this away for future use.

By the theorem on angle sums of a triangle we have

$$\tfrac{1}{2}\alpha + \tfrac{1}{2}\beta + \delta \;=\; 180 \;=\; \alpha + \beta + \gamma$$

and if we subtract half the second equation from the first and afterwards multiply both sides by two we obtain

$$2\,\delta \;-\gamma \;\;=\;\; 180\ .$$

If we substitute $\delta = 130$ and solve for γ we find that $\gamma = 80$.■

8. By the Vertical Angle Theorem we have $|\angle ABD| = |\angle CBF|$, and since $|\angle ADB| = 90 = |\angle BCF|$, we may apply the "Third Angles Are Equal" theorem to conclude that $|\angle DAB| = |\angle BFC|$.■

11. We know that $D = \frac{1}{2}(A + B)$ and $E = \frac{1}{2}(A + C)$, so that $d(D, E)$ is equal to $|D - E| = \left|\frac{1}{2}(B - C)\right|$; since the latter is equal to $\frac{1}{2}|B - C|$, the conclusion of the exercise follows.■