SOLUTIONS TO EXERCISES FOR

MATHEMATICS 133 — Part 5

Fall 2007

NOTE ON ILLUSTRATIONS. Drawings for several of the solutions in this file are available in the following files:

`http://math.ucr.edu/~res/math133/math133solutions4figures.pdf`

`http://math.ucr.edu/~res/math133/math133solutions5figures.pdf`

III. Basic Euclidean concepts and theorems

III.2 : Basic theorems on triangles

(Solutions not posted previously)

6. Since the angle sum of a triangle is 180 degrees, by the Isosceles Triangle Theorem and $\angle BAC = \angle DAE$ we have

$$|\angle ABC| \ = \ \tfrac{1}{2}(180 \ - \ |\angle BAC|) \ = \ \tfrac{1}{2}(180 \ - \ |\angle DAE|) \ = \ |\angle ADE| \ .$$

Therefore the Corresponding Angles criterion implies that $BC||DE$.■

9. If Y is an arbitrary point on L then $d(B,Y) = d(C,Y)$ because L is the perpendicular bisector of $[BC]$. It follows that $d(A,Y) + d(Y,B) = d(A,Y) + d(Y,C)$. The right hand side is minimized when Y is between A and C, and this happens precisely when Y is the point X where AC meets L; note that this point is between A and C because A and C lie on opposite sides of L. There is only one point $X \in L$ with these properties, so we know that $d(A,Y) + d(Y,B) = d(A,Y) + d(Y,C) > d(A,C)$ for all other points Y on the line L.■

10. The points X, Y, Z are not collinear because a line cannot intersect all three open sides of a triangle. Also, the betweenness hypotheses imply

$d(A,B) = d(A,X) + d(X,B)$,
$d(B,C) = d(B,Y) + d(Y,C)$, and
$d(A,C) = d(A,Z) + d(Z,C)$.

Finally, the strong form of the Triangle Inequality (for noncollinear triples) implies that

$d(X,Y) < d(B,X) + d(B,Y)$,
$d(Y,Z) < d(C,Y) + d(C,Z)$, and
$d(X,Z) < d(A,X) + d(A,Z)$.

If we add these we obtain

$$d(X,Y) \ + \ d(Y,Z) \ + \ d(X,Z) \ < \ d(B,X) \ + \ d(B,Y) \ + \ d(C,Y) + d(C,Z) \ + \ d(A,X) \ + \ d(A,Z)$$

and using the betweennes identities we see that the right hand side is equal to $d(A,B)+d(B,C)+d(A<C)$; thus we have shown the inequality stated in the exercise.■

12. Let E be the midpoint of $[AB]$. Then by the final result in Section I.4 we know that $D-E=\frac{1}{2}(C-B)$. Since $E \in (AB)$ and $AD \neq AB$ we know that $A,\ D,\ E$ are noncollinear, and thus by the Triangle Inequality for noncollinear points we have

$$d(A,D) \;<\; d(D,E) \;+\; d(A,E) \;=\; \tfrac{1}{2}\big(\, d(A,C) \;+\; d(A,B)\,\big)$$

which is the inequality stated in the exercise.■

13. Apply the theorem on angle sums of a triangle to the four triangles described in the hint to obtain the following equations:

$$|\angle CAB| \;+\; |\angle ABC| \;+\; |\angle BCA| \;=\; 180$$

$$|\angle XAB| \;+\; |\angle ABX| \;+\; |\angle BXA| \;=\; 180$$

$$|\angle CAX| \;+\; |\angle AXC| \;+\; |\angle XCA| \;=\; 180$$

$$|\angle CXB| \;+\; |\angle XBC| \;+\; |\angle BCX| \;=\; 180$$

Adding the last three equations, we obtain

$$|\angle XAB| \;+\; |\angle ABX| \;+\; |\angle BXA| \;+\; |\angle CAX| \;+\; |\angle AXC| \;+$$

$$|\angle XCA| \;+\; |\angle CXB| \;+\; |\angle XBC| \;+\; |\angle BCX| \;=\; 540\ .$$

Since X lies in the interior of ΔABC it lies in the interiors of all the angles $|\angle CAB|$, $|\angle ABC|$, $|\angle BCA|$ and therefore we have

$$|\angle CAB| \;=\; |\angle CAX| \;+\; |\angle XAB|$$

$$|\angle ABC| \;=\; |\angle ABX| \;+\; |\angle XBC|$$

$$|\angle BCA| \;=\; |\angle BCX| \;+\; |\angle XCA|$$

If we substitute this into the previous equation we obtain

$$|\angle CAB| \;+\; |\angle ABC| \;+\; |\angle BCA| \;+\; |\angle AXB| \;+\; |\angle AXC| \;+\; |\angle XBC| \;=\; 540$$

and if we now use $|\angle CAB|+|\angle ABC|+|\angle BCA| \;=\; 180$ and subtract 180 from both sides we obtain

$$|\angle AXB| \;+\; |\angle AXC| \;+\; |\angle XBC| \;=\; 360$$

which is the equation stated in the exercise.■

14. Let $x=d(A,B)=d(D,E)$. Then by the Pythagorean Theorem we have $d(E,F)=\sqrt{x^2-d(D,F)^2}$ and $d(B,C)=\sqrt{x^2-d(A,C)^2}$. If $d(E,F)<d(B,C)$, then the formulas in the preceding sentence imply $d(A,C)<d(D,F)$.■

15. By Exercise 12, we know that $d(A,C)<d(E,C)<d(B,C)$; since the larger angle is opposite the longer side, it follows that $|\angle CEA|<|\angle CAE|$. On the other hand, the Exterior Angle Theorem implies that $|\angle CEB|>|\angle CAE|$, so that $|\angle CEB|>|\angle CEA|$. Since we also have

$|\angle CEB|+|\angle CEA| = 180$, it follows that $|\angle CEB| > 90 > |\angle CEA|$. Therefore $\angle CEA$ is an **ACUTE** angle.■

16. Suppose we are given numbers $a \leq b \leq c$; then these numbers are consistent with the strong Triangle Inequality if and only if $c < a + b$. So if this fails, then there is no triangle whose sides have the given lengths. In Section III.6 we show that, conversely, if the conditions in the first sentence hold, then one can realize the numbers as lengths of the sides of some triangle.

(a) Since $1 + 2 = 3$, these numbers do not satisfy the strong Triangle Inequality and hence cannot be the lengths of the sides of a triangle.■

(b) Since $4 < 5 < 6$ and $4 + 5 = 0 > 6$, these numbers are consistent with the Triangle Inequality.■

(c) Since $1 \leq 15 = 15$ and $1 + 15 > 15$, it follows that these numbers are consistent with the Triangle Inequality.■

(d) Since $1 + 5 < 8$, these numbers do not satisfy the strong Triangle Inequality and hence cannot be the lengths of the sides of a triangle.■

17. The strong Triangle Inequality implies that if there is a number x such that 10, 15, x are the lengths of the sides of a triangle, then $x + 10 > 15$ and $x < 10 + 15$. There is also the inequality $x + 15 > 10$, but it is weaker than the first one. Therefore the conditions on x are that $5 < x < 25$.■

18. We know that $(n + 1)^2 = n^2 + (2n + 1)$, so if we can write $(2n + 1) = m^2$, it follows that $n^2 + m^2 = (n + 1)^2$. Thus there is a right triangle whose sides have lengths n, m and $n + 1$ by the Pythagorean Theorem.

Since there are infinitely many odd positive integers who are perfect squares, it follows that there are infinitely many choices of n and m such that the preceding holds.

To find all $n < 100$ which satisfy this condition, it is necessary to find all $n < 100$ such that $2n + 1$ is a perfect square. In other words, we need to find all odd positive integers $m \geq 3$ such that $m^2 < 200$, and this is the set of all odd positive integers ≤ 13. We can retrieve n because it is equal to $\frac{1}{2}(m^2 - 1)$. The first few cases are then given as follows:

$$\begin{aligned} 3^2 + 4^2 &= 5^2 \\ 5^2 + 12^2 &= 13^2 \\ 7^2 + 24^2 &= 25^2 \\ 9^2 + 40^2 &= 41^2 \\ 11^2 + 60^2 &= 61^2 \\ 13^2 + 84^2 &= 85^2 \end{aligned}$$

Of course, this could be continued indefinitely.■

19. Apply the Law of Cosines to each triangle. Let $y = d(A, C) = d(A, D)$ and $z = d(A, B)$. Then we have

$$d(B, C)^2 = y^2 + z^2 - 2yz \cos|\angle CAB|$$

$$d(B, D)^2 = y^2 + z^2 - 2yz \cos|\angle DAB|$$

It follows that $d(B, C) < d(B, D)$ if and only if $\cos|\angle DAB| > \cos|\angle CAB|$, and since the cosine function is strictly increasing between 0 and 180, the latter holds if and only if $|\angle DAB| > |\angle CAB|$, Therefore we have $d(B, C) < d(B, D)$ if and only if $|\angle DAB| > |\angle CAB|$, which is the conclusion of the Hinge Theorem.■

III.3 : Convex polygons

1. We shall use the theorem stating that the line joining the midpoints of two sides of a triangle is parallel to the third side (see Section I.4). Applying this to ΔABD and ΔCBD, we conclude that $PS||BD$ and $QR||BD$. Therefore it follows that either $PS||QR$ or else $PS = QR$. Suppose that the latter is true; we know that PS and A lie on the same side of BD, while QR and C lie on the same side of BD. Thus $PS = QR$ implies that A and C lie on the same side of BD. However, this is impossible because we know that A and C lie on opposite sides of BD (the diagonal segments of a convex quadrilateral have a point in common). Therefore we have $PS||QR$.

A similar argument holds for PQ and RS. Both of these lines are parallel to AC by applying the triangle theorem to ΔABC and ΔADC, showing that $PQ||AC$ and $RS||AC$. We can now argue as in the previous paragraph that $PQ \neq RS$, so that the lines PQ and RS are parallel. It follows that $P,\ Q,\ R,\ S$ form the vertices of a parallelogram.■

2. By the preceding result we know that $P,\ Q,\ R,\ S$ form the vertices of a parallelogram, and it follows that (PR) and (QS) meet at their common midpoint.■

3. Following the hint, we shall use vector methods. The parallelogram condition implies that $C = B + D - A$ (see the exercises for Unit I), and the midpoint conditions imply that $E = \frac{1}{2}(A+B)$ and $F = \frac{1}{2}(C+D)$. To show that $E,\ B,\ F,\ D$ form the vertices of a parallelogram, it will suffice to show that $F = B + D - E$.

If we substitute the expression $C = B + D - A$ in the midpoint equation for F, we find that $F = D + \frac{1}{2}(B-A)$, and if we substitute the expression for E in terms of A and B into $B+D-E$, we find that the latter is also equal to $D + \frac{1}{2}(B-A)$. Combining these equations, we find that $F = B + D - E$ as desired, so that the four points in the given order form the vertices of a parallelogram.

4. First of all, we know that C lies in the interior of $\angle DAB$. Next, by the Isosceles Triangle Theorem we know that $|\angle DAC| = |\angle DCA|$. Now $AB||CD$ and the Alternate Interior Angle Theorem imply that $|\angle DCA| = |\angle CAB|$, and thus we have $|\angle DAC| = |\angle CAB|$, which means that $[AC$ bisects $\angle DAB$.■

5. We know that $\angle ADE = \angle ADB$ and $\angle CBF = \angle CBD$. Since $AD||BC$, the Alternate Interior Angle Theorem implies that $|\angle ADE| = |\angle CBF|$. Since $A,\ B,\ C,\ D$ form the vertices of a parallelogram, it follows that $d(A,D) = d(B,C)$; combining these observations with the assumption that $d(B,F) = d(D,E)$, we conclude that $\Delta ADE \cong \Delta CBF$. Therefore we have that $|\angle AED| = |\angle CFB|$, and by the Supplement Postulate for angle measurement we then also have

$$|\angle AEF| \ = \ 180 - |\angle AED| \ = \ 180 - |\angle CFB| \ = \ |\angle CFD| \ .$$

Now A and C lie on opposite sides of $EF = BD$, and if we combine this with the displayed equation and the Alternate Interior Angle Theorem we conclude that AE must be parallel to CF.■

6. Before proving this result, for the sake of completeness we include a verification that the diagonals (AC) and (BD) of a parallelogram $ABCD$ meet in their common midpoint, which we shall call E. The fastest say to do this is algebraically, using the fact that $C = B + D - A$ and then checking directly that $\frac{1}{2}(A+C) = \frac{1}{2}(B+D)$.

Suppose now that we have a parallelogram $ABCD$ which is a rhombus. Then $d(A,B) = d(C,B)$ and $d(A,D) = d(C,D)$ imply that BD is the perpendicular bisector of $[AC]$ and hence $AC \perp BD$.

Conversely, suppose that $AC \perp BD$. Since E is the midpoint of both $[AC]$ and $[BD]$, it follows that AC is the perpendicular bisector of $[BD]$ and BD is the perpendicular bisector of $[AC]$. The first conclusion implies that $d(B, A) = d(D, A)$, and since we have $d(A, B) = d(C, D)$ and $d(B, C) = d(A, D)$ for every parallelogram it follows that all four sides of $ABCD$ have the same length.∎

7. Since ΔEAB is an equilateral triangle, we have $|\angle EAB| = |\angle EBA| = |\angle AEB| = 60$. The point E is assumed to lie in the interior of the square, so we then have

$$90 \;=\; |\angle DAB| \;=\; |\angle EAB| \,+\, |\angle EAD| \;=\; 60 \,+\, |\angle EAD|$$

$$90 \;=\; |\angle CBA| \;=\; |\angle EBA| \,+\, |\angle EBC| \;=\; 60 \,+\, |\angle EBC|$$

and therefore we have $|\angle EAD| = |\angle EBC| = 30$. Since the sides of a square have equal length, it follows that $\Delta EAD \cong \Delta EBC$ by **SAS**. This means that $|\angle AEB| = |\angle CEB|$ and $d(D, E) = d(C, E)$. The latter in turn implies $\angle EDC| = |\angle ECD|$.

In order to compute the measures of the angles in the preceding sentence, we need one more property of the figure. Since ΔABE is equilateral and $ABCD$ is a square, it follows that $d(A, D) = d(A, B) = d(A, E)$ and $d(B, C) = d(A, B) = d(B, E)$, so that ΔAED and ΔBEC are isosceles and hence $|\angle AED| =$ —∠ADE—*and*—∠BEC—$= |\angle BCE|$.

To simplify the algebra, let $x = |\angle EDC|$ and $y = |\angle EDA|$. The preceding observations then imply that $x + y = 90$ and $30 + 2y = 180$. If we solve these equations for x and y we obtain $y = 75$ and $x = 15$, and therefore it follows that $|\angle EDC| = |\angle ECD| = 15$.∎

8. By the assumption in the exercise we know that (AC) and (BD) meet at some point E. Since we have $A * E * C$ and $B * E * D$, it follows from the theorems on order and separation that

C and D lie on the same side of AB,
A and B lie on the same side of CD,
B and C lie on the same side of AD, and
A and D lie on the same side of BC.

Therefore the four points A, B, C and D (taken in the alphabetical ordering) form the vertices of a convex quadrilateral.∎

9. Follow the hint and use the conclusion of the preceding exercise. If the four points form the vertices of a convex quadrilateral (taken in the alphabetical ordering), then (AC) and (BD) have a point E in common by the proposition in the notes. The point E then lies in the interior of $\angle ABC$, and since we have $B * E * D$ it follows that the open ray $(BE = (BD$ also lies in the interior of $\angle ABC$. Furthermore, since $B * E * D$ holds and $E \in AC$, it follows that B and D lie on opposite sides of AC.

Conversely, suppose now that D lies in the interior of $\angle ABC$ and D and B lie on opposite sides of AC. The first of these implies that the open ray $(BD$ meets (AC) in some point E, and the second implies that (BD) meets AC in some point F. Since both E and F lie on the intersection of the (distinct!) lines AC and BD and these lines have at most one point in common, it follows that $E = F$. Finally, since $E \in (AC)$ and $F \in (BD)$, it follows that (AC) and (BD) have a point in common, which must be $E - F$.∎

10. By the preceding exercise the points form the vertices of a convex quadrilateral (taken in the alphabetical ordering) if and only if D lies in the interior of $\angle ABC$ and B and D lie on opposite sides of AC. The first of these implies x and z are positive, and the second implies that y is negative.

Conversely, suppose we have the conditions on the barycentric coordinates in the preceding sentence. Since y is negative, it follows that B and D lie on opposite sides of AC, and since the other two barycentric coordinates are positive it follows that D lies in the interior of $\angle ABC$.■

11. The assumptions are equivalent to saying that $C - D$ is a nonzero multiple of $B - A$, so write $C - D = k(B - A)$, where $k \neq 0$. We then have

$$D \;=\; kA \;-\; kB \;+\; C$$

and since the coefficients on the right hand side add up to 1 they give the barycentric coordinates of D with respect to A, B and C. By the preceding exercise, we know that the four points form the vertices of a convex quadrilateral (taken in the alphabetical ordering) if and only if k is positive.■

12. By the preceding exercise we know that $C - D = k(B - A)$, where $k > 0$. As before we have $D = kA - kB + C$, and the conditions $y = d(A,B)$ and $x = d(C,D)$ also imply $x = ky$. The midpoints G and H of $[AD]$ and $[BC]$ are then given by $H = \frac{1}{2}(B + C)$ and

$$G \;=\; \frac{(1+k)}{2}A \;-\; \frac{k}{2}B \;+\; \frac{1}{2}C \;.$$

It follows that

$$H \;-\; G \;=\; \frac{(1+k)}{2}\big(B \;-\; A\big)$$

so that GH is parallel to AB and CD, and furthermore we have

$$d(G,H) \;=\; \frac{(1+k)}{2}\cdot d(A,B) = \frac{(1+k)}{2}\cdot y \;=\; \frac{(y+ky)}{2} \;=\; \frac{(x+y)}{2}$$

as stated in the exercise.

To prove the remaining parts of the exercises, it suffices to show that the midpoints of $[AC]$ and $[BD]$ lie on the line GH. Let S and T denote these respective midpoints. Then we know that GS is parallel to AB since it joins the midpoints of two sides of ΔABD, and by Playfair's Postulate it follows that GS must be the same as GH, which is also a line through G which is parallel to AB. Similarly, the lines HG and HT are parallel to AB, and therefore $GH = HG = HT$. Thus we have shown that both S and T lie on GH. Note that $S \neq T$ unless the quadrilateral is a parallelogram (a standard result in plane geometry states that if the diagonals bisect each other, then the quadrilateral is a parallelogram).■

13. First of all, we have $A * E * B$ because $E \in (AB$ and $d(A,E) = x < y = d(A,B)$. Choose $k > 0$ so that $C - D = k(B - A)$ and hence $x = ky$. If $m > 0$ is chosen such that $E - A = m(B - A)$, then we have

$$m \cdot d(A,B) \;=\; d(A,E) \;=\; d(C,D) \;=\; k \cdot d(A,B)$$

so that $m = k$. Therefore we also have $E - A = k(B - A) = C - D$. Since $A,\ C,\ D$ are noncollinear, this means that $A,\ E,\ C,\ D$ (in that order) form the vertices of a parallelogram. In particular, we know that AD is parallel to CE.■

14. By the preceding exercise we know that $A,\ E,\ C,\ D$ (in that order) form the vertices of a parallelogram. Since consecutive angles of a parallelogram are supplementary, it follows that $|\angle DAB| + |\angle AEC| = 180$. However, by the preceding exercise we also know that $A * E * C$ and thus by the Supplement Postulate we also know that $|\angle AEC| + |\angle CEB| = 180$. Combining these, we see that $|\angle DAB| = |\angle CEB|$ in all cases. Furthermore, since the opposite sides of a parallelogram have equal length, we also know that $d(A,D) = d(E,C)$. We shall use this fact repeatedly in proving the equivalence of the three conditions in the exercise.

Proof that (1) $\Longrightarrow$ (2). In this case we are given $d(A,D) = d(B,C)$. By the discussion above we have $d(B,C) = d(A,D) = d(E,C)$. Therefore the Isosceles Triangle Theorem implies that $|\angle CEB| = |\angle CBE|$, and since $\angle CBE = \angle CBA$ it follows that $|\angle CEB| = |\angle CBA|$. On the other hand, by the general discussion we have $|\angle DAB| = |\angle CEB|$, and therefore it follows that $|\angle DAB| = |\angle CBA|$ as required.■

Proof that (2) $\Longrightarrow$ (3). Since consecutive angles in a parallelogram are supplementary, it follows that $|\angle ADC| = 180 - |\angle DAB|$. Since $|\angle DAB| = |\angle CBA|$, it will suffice to show that $|\angle BCD| = 180 - |\angle CBA|$. There are several ways to do this, but the fastest might be to switches the roles of A and C with those of B and D in the preceding discussion; this can be done because the hypothesis does not change if one switches symbols in this fashion. — An alternative approach (which we shall only sketch) is to check that E lies in the interior of $\angle BCD$ and that $|\angle ECD| = |\angle DAB|$ (opposite angles of a parallelogram have equal measure), so that $|\angle BCD| = |\angle ECD| + |\angle ECB| = |\angle ECD| + |\angle DAB| = |\angle ECD| + |\angle CEB|$. Since $|\angle ECD| + |\angle CEB| + |\angle ABC| = 180$ it follows that $|\angle BCD| = 180 - |\angle CBA|$ as required.■

Proof that (3) $\Longrightarrow$ (1). One way of doing this is to show that (3) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1). Each of these can be done by reversing the steps in the preceding parts of the exercise.■

15. Let X and Y be the midpoints of $[AB]$ and $[CD]$ respectively. By construction we then have $d(A,X) = d(X,B)$ and $d(C,Y) - d(Y,C)$

By the preceding exercise we know that $d(A,D) = d(B,C)$ and also that $|\angle DAX| = |\angle CBX|$ as well as $|\angle ADY| = |\angle BCY|$. It follows that $\Delta DAX \cong \Delta CBX$ and $\Delta ADY \cong \Delta BCY$. These congruences imply $d(X,D) = d(X,C)$ and $d(Y,A) = d(Y,B)$. The first of these implies that XY is the perpendicular bisector of $[CD]$, and the second implies that XY is the perpendicular bisector of $[AB]$. Therefore the line XY is perpendicular to both AB and CD.■

16. We need to find s and t such that $0 < s,t < 1$ and $sC + (1-s)A = tD + (1-t)B$. If we substitute the coordinate expressions for the four points A, B, C, D we obtain the following equations for the coordinates:

$$\begin{aligned} \tfrac{1}{2}sy \ + \ (1-s)(-\tfrac{1}{2}x) \quad &= \quad -\tfrac{1}{2}ty \ + \ (1-t)(\tfrac{1}{2}x) \\ sh \quad &= \quad th \end{aligned}$$

The second equation implies $s = t$, and if we substitute this into the first equation we obtain

$$\tfrac{1}{2}sy \ + \ (1-s)(-\tfrac{1}{2}x) \quad = \quad -\tfrac{1}{2}sy \ + \ (1-s)(\tfrac{1}{2}x)$$

which means that the right hand side is the negative of the left hand side and hence both are equal to zero. Therefore the equations imply $sy = (1-s)x$ and $k = sh$.

We can solve this for s to obtain $s = x/(x+y)$, and if we substitute this and $k = sh$ into $k/(h-k)$, it follows that the latter is equal to x/y.■

17. (*a*) Following the hint, one first notes that

$$d(A,B) \quad = \quad d(B,C) \quad = \quad d(C,D) \quad = \quad d(D,A) \quad = \quad \sqrt{p^2+q^2}$$

and then one notes that $A - B = (p, -q) = D - C$. Therefore one has

$$AB = A + \mathbf{R} \cdot (p, -q) , \qquad CD = C + \mathbf{R} \cdot (p, -q)$$

so that the lines AB and CD are either parallel or identical. The latter is impossible because it would imply that A, B, C would be collinear. Since the defining equation for AB is $f(x, y) = qx + py - pq = 0$ and $f(C) = q(-p) + p \cdot 0 - pq = -2pq < 0$, we know that A, B, C cannot be collinear, and hence $AB || CD$.

If we interchange the roles of B and D in the preceding argument, we obtain the analogous conclusion that $AD || BC$. Combining these with the observations in the first paragraph, we conclude that A, B, C, D form the vertices of a parallelogram, and (by the first sentence of that paragraph) this parallelogram is a rhombus.■

(b) As suggested in the hint, let T be the orthogonal linear transformation on $\mathbf{R}^2$ defined by $T(x, y) = (x, -y)$; geometrically and physically, the mapping T corresponds to reflection about the x-axis. By definition the map T sends A and C to themselves, and it interchanges B and D. Since a set of points in $\mathbf{R}^2$ is collinear if and only if its image is collinear, it follows that T must interchange the lines AB and AD, and it must also interchange the lines BC and BD.

Suppose now that $F \in AB$ and $G \in CD$ are such that $FG \perp AB$ and $FG \perp CD$. Since T preserves angle measurements, it follows that the line $T(F)\,T(G)$ is perpendicular to both AD and $CB = BC$. Furthermore, since T is distance-preserving, it follows that

$$d(F, G) = d\,(\,T(F), T(G)\,) \ .$$

By construction, the left hand side is equal to the distance between the parallel lines AB and CD, whild the right hand side of the equation is equal to the distance between the parallel lines AD and BC.■

18. As suggested by the drawing, we have $b + 2c = a$ and $b = c\sqrt{2}$. Therefore we have

$$b = a - 2\,c + a - \frac{2b}{\sqrt{2}} = a - b\sqrt{2}$$

which means that $a = (1 + \sqrt{2})b$; if we solve for b and put the result into simplified radical form, we find that $b = \left(\sqrt{2} - 1\right)\, a$.■