SOLUTIONS TO EXERCISES FOR

MATHEMATICS 133 — Part 3

Winter 2009

NOTE ON ILLUSTRATIONS. Drawings for several of the solutions in this file are available in the following file:

`http://math.ucr.edu/~res/math133/math133solutions3figures.pdf`

II. Linear algebra and Euclidean geometry

II.3 : Measurement axioms

1. First of all, f is 1–1. Define k_X so that $X = A + k_X(B - A)$. Then $f(X) = f(Y)$ implies $k_X d(A,B) = k_Y d(A,B)$, and since $d(A,B)$ is positive this means $k_X = k_Y$. Also, if r is an arbitrary real number and $k = r/d(A,B)$, then it follows that f maps $X = A + k(B - A)$ to r. Therefore f is onto. Finally, to verify the statement on distances, note that the distance from X to Y is equal to

$$|X - Y| \;=\; \Big|[A + k_X(B-A)] \;-\; [A + k_Y(B-A)]\Big| \;=\; \Big|[k_X(B-A)] \;-\; [k_Y(B-A)]\Big| \;=$$

$$\big|\,(k_X - k_Y)(B-A)\,\big| \;=\; |\,(k_X - k_Y)\,|\cdot|\,(B-A)\,| \;=\; |\,(k_X - k_Y)\,|\cdot d(A,B) \;=$$

$$|\,(k_X - k_Y)d(A,B)\,| \;=\; |\,f(X) - f(Y)\,|$$

which is the identity to be shown.■

2. Follow the hint, and let $h = g \circ f^{-1}$. By construction, h is a 1–1 onto map from the real numbers to themselves such that $|u - v| = |h(u) - h(v)|$ for all u and v. If $k(t) = h(t) - h(0)$, then elementary algebra shows that we also have $|u - v| = |k(u) - k(v)|$ but also $k(0) = 0$. Therefore we have $|k(t)| = |k(t) - k(0)| = |t - 0| = |t|$ for all t. In particular, this means that for each t we have $k(t) = \varepsilon_t \cdot t$, where $\varepsilon_t = \pm 1$. We claim that ε_t is the same for all $t \neq 0$; for $t = 0$ the value of ε does not matter. But suppose that we had $k(u) = u$ and $k(v) = -v$, where $u, v \neq 0$. Then we could not have $|u - v| = |k(u) - k(v)|$; if u and v have the same sign, then the right hand side is greater than the left, and if they have opposite signs, then the right hand side is less than the left (why?). Therefore $k(t) = \varepsilon \cdot t$ where $\varepsilon = \pm 1$, and hence also $h(t) = k(t) + h(0) = \pm t + h(0)$, which is the form required in the exercise.■

3. Write things out using barycentric coordinates. We have $X - A = (2, -6)$, while $B - A = (-5, -9)$ and $C - A = (4, -9)$. Thus we need to solve

$$(2, -6) \;=\; y(-5, -9) \;+\; z(4, -9) \;=\; (-5y + 4z, -9y - 9z)$$

and if we do so we obtain $z = \frac{48}{81} = \frac{16}{27}$ and $y = \frac{6}{81} = \frac{2}{27}$; using the barycentric coordinate equation $x + y + z = 1$ we also obtain $x = \frac{1}{3}$. Thus all three barycentric coordinates are positive and the point lie in the interior.

To work the second part, we have $X - A = (10, k-10)$, and we need to consider the system

$$(10, k-10) \quad = \quad y(-5,-9) \ + \ z(4,-9) \quad = \quad (-5y+4z, -9y-9z)$$

and determine those values of k for which $z > 0$ and $x = 1 - y - z > 0$. Solving for the barycentric coordinates, we find

$$z \ = \ \frac{140-5k}{81}\ , \qquad y \ = \ \frac{-50-4k}{81}\ , \qquad x \ = \ \frac{9k-9}{81}\ .$$

The point will lie in the interior if and only if x and z are positive, which is the same as saying that the numerators $140 - 5k$ and $9k - 9$ should both be positive. This happens if and only if $1 < k < 28$.■

4. In this case we need to work the first part of the problem when X is either $(30, 200)$ or $(75, 135)$, so that $X - A$ is either $(23, 190)$ or $(68, 125)$. The barycentric coordinate z is negative in the first case, and in the second case we have $z > 0 > x$, and therefore neither point lies in the interior of the angle.■

5. We now have $X - A = (-1, 0)$, while $B - A = (3, -6)$ and $C - A = (-1, 20)$. Thus we need to solve

$$(-1, 0) \quad = \quad y(3,-6) \ + \ z(-1,-20) \quad = \quad (3y - z, -6y - 20z)$$

and if we do so we obtain $z = \frac{1}{11}$, $y = -\frac{10}{33}$, and $x = \frac{40}{33}$, so that the point X lies in the interior of the angle.

To work the second part, we have $X - A = (21, k-8)$, and we need to consider the system

$$(21, k-8) \quad = \quad (3y - z, -6y - 20z)$$

and determine those values of k for which $z > 0$ and $x = 1 - y - z > 0$. Solving for the barycentric coordinates, we find

$$z \ = \ \frac{3k+98}{-66}\ , \qquad y \ = \ \frac{k-428}{-66}\ , \qquad x \ = \ \frac{4k-264}{66}\ .$$

The point will lie in the interior if and only if x and z are positive, which is the same as saying that the numerators for x and z should be positive and negative respectively. The inequality $3k+98 < 0$ implies $k < 0$, while the inequality $4k - 164 > 0$ implies $k > 0$. Thus there are no values of k which satisfy both inequalities, and no points on the given line lie in the interior of the given angle. — It might be useful to plot points and sketch the angle to confirm this conclusion; in fact, if a point (u, v) lies in the interior of the given angle then we must have $u < -1$.■

6. Suppose that $X \in (AC)$, so that $A * X * C$ is true. By theorems on plane separation, this implies that A and X lie on the same side of BC, and C and X lie on the same side of AB. But these are the two criteria for a point to lie in the interior of $\angle ABC$, and therefore we know that X lies in the interior of this angle.■

7. It looks as if the open ray $(DE$ meets the triangle in exactly one point. See the illustration in the file of figures.■

8. The segment (AC) should be corrected to (AX). With this correction, proceed as follows: If Y lies on (AX), then $A * Y * X$ is true, so that X and Y lie on the same sides of AB and AC. However, $B * X * C$ implies that X and B lie on the same side of AC and X and C lie on the same side of AB. Therefore we also know that Y and B lie on the same side of AC and Y and C lie on the same side of AB. All that remains is to show that Y and A lie on the same side of BC. But this follows because $A * Y * X$ and $X \in BC$.■

9. By the Protractor Postulate there is a point E on the side of BC opposite A such that $|\angle EBC| = |\angle ABC|$, and by a consequence of the Ruler Postulate there is a point $D \in (AE$ such that $d(D,B) = d(A,B)$. By construction the distance equation holds, and the angle measurement equation holds because $[AD = [AE$. Finally D is on the side of BC opposite A because $D \in (BE$, while E and A lie on opposite sides and all points of $(BE$ lie on a single side of BC.■

10. The interior of the triangle is contained in the interior of $\angle BAC$, so by the Crossbar Theorem we know that $(AD$ meets (BC) in some point E. It will suffice to show that we have the order relationship $A*D*E$ (take $A = X$ and $E = Y$). But this follows because A and D are on the same side of BC, which implies either $A*D*E$ or $D*A*E$. The second of these is incompatible with the known condition $E \in (AD$, so therefore the first must be true and we have shown what we needed.■

II.4 : Congruence, superposition and isometries

1. Follow the hint. By **SSS** we have $\Delta BDC \cong \Delta BDE$, so that $|\angle EDB| = |\angle CDB|$. Since D lies on the segment (CE) it is in the interior of $\angle EBA = \angle ABC$, and therefore by additivity we have

$$|\angle ABC| \;=\; |\angle EDB| + |\angle CDB| \;=\; 2\cdot|\angle DBC| \;=\; 2\cdot|\angle DBA|\ .$$

This shows that the ray $(BD$ bisects $\angle ABC$ and proves existence.

To prove uniqueness, suppose that $(BG$ is an arbitrary bisector ray. Since $(BG$ lies in the interior of $\angle BAC$, it follows that it lies on the same side of AB as C. By hypothesis its measure is $\frac{1}{2}|\angle ABC|$, which is the same as $|\angle ABD|$. Therefore the uniqueness part of the Protractor Postulate implies that $[BG = [BD$.■

2. Take a 3–4–5 right triangle with a right angle at B and $d(A,B) = 3$. Then $\Delta ABC \cong \Delta BCA$ is false because $3 = d(A,B) \neq 4 = d(B,C)$.■

3. Suppose that $\Delta ABC \cong \Delta DEF$. Then $d(A,B) = d(D,E)$, $d(A,C) = d(D,F)$, and $d(B,C) = d(E,F)$. If we rewrite these as $d(A,C) = d(D,F)$, $d(A,B) = d(D,E)$ and $d(C,B) = d(B,C) = d(E,F) = d(F,E)$, then we may apply **SSS** to conclude that $\Delta ACB \cong \Delta DFE$.

Similarly, we have $d(B,C) = d(E,F)$, $d(B,A) = d(A,B) = d(D,E) = d(E,D)$ and $d(C,A) = d(A,C) = d(D,F) = d(F,D)$, so we may now apply **SSS** to conclude that $\Delta BCA \cong \Delta EFD$.■

4. By the Isosceles Triangle Theorem and the identities $\angle DAB = \angle CAB$ and $\angle EBA = \angle CBA$ we have

$$|\angle DAB| \;=\; |\angle CAB| \;=\; |\angle CBA| \;=\; |\angle EBA|$$

and the midpoint conditions together with the isosceles triangle assumption imply $d(A,D) = \frac{1}{2}d(A,C) = \frac{1}{2}d(A,C) = d(B,E)$. Since $d(A,B) = d(B,A)$, by **SAS** we have $\Delta DAB \cong \Delta EBA$.■

5. The congruence assumption implies $d(A,B) = d(D,E)$ and $|\angle CBA| = |\angle FED|$. Since $\angle GBA = \angle CBA$ (as sets!!) and $\angle HED = \angle FED$, it also follows that $|\angle GBA| = |\angle HED|$. Furthermore, congruence and the bisection hypotheses imply that

$$|\angle GAB| \;=\; \tfrac{1}{2}|\angle CAB| \;=\; \tfrac{1}{2}|\angle FDE| \;=\; |\angle HDE|$$

and therefore we have $\Delta GAB \cong \Delta HDE$.■

6. We now have $d(A,B) = d(D,E)$ and

$$|\angle CBA| \quad = \quad |\angle GBA| \quad = \quad |\angle HED| \quad = \quad |\angle FED| \; .$$

In addition, congruence and angle bisection imply that

$$|\angle BAC| \quad = \quad 2\cdot|\angle GAB| \quad = \quad 2\cdot|\angle HDE| \quad = \quad |\angle EDF| \; .$$

Therefore the **ASA** congruence axiom implies that $\Delta ABC \cong \Delta DEF$.■

7. An affine transformation T has the property

$$T\big(s\mathbf{a} \ + \ (1-s)\mathbf{b}\big) \quad = \quad sT(\mathbf{a}) \ + \ (1-s)T(\mathbf{b})$$

so a point $\mathbf{c}$ is between two points $\mathbf{a}$ and $\mathbf{b}$ of K if and only if its image $T(\mathbf{c})$ is between the two image points $T(\mathbf{a})$ and $(\mathbf{b})$. Therefore $\mathbf{c}$ is between two points of K if and only if its image is between two points of the image of K under the affine transformation. Therefore, if $\mathbf{c}$ is not between two points of K, then its image cannot be between two points in the image of K.■

8. (a) This is just a simple partial derivative calculation involving polynomials.■

(b) We have $T_1(\mathbf{x}) = A_1\mathbf{x} + \mathbf{b}_1$ and $T_2(\mathbf{x}) = A_2\mathbf{x} + \mathbf{b}_2$ where A_1 and A_2 are invertible matrices and $\mathbf{b}_1$ and $\mathbf{b}_2$ are vectors. The composite $T_1 \circ T_2$ sends $\mathbf{x}$ to $A_1A_2\mathbf{x} + A_1\mathbf{b}_2 + \mathbf{b}_1$. Therefore A_1A_2 gives the matrix part of $T_1 \circ T_2$.■

(c) $D(T)$ is the identity if and only if for each i the partial derivatives of the i^{th} coordinate functions are equal to the partial derivatives of the standard function x_i. But the latter holds if and only if the i^{th} coordinate function has the form $x_i + b_i$ for some constant b_i, and this is precisely the condition for T to be a translation.■

(d) This can be done directly, but we shall do it using the ideas described above. We have

$$D\left(S^{-1} \circ T \circ S\right) \quad = \quad D(S^{-1})D(T)D(S)$$

and te relation $S-1 \circ S =$ identity implies $D(S^{-1})D(S) = I$, so that $D(S^{-1}) = D(S)^{-1}$. If we now assume T above is a translation, this gives us

$$D\left(S^{-1} \circ T \circ S\right) \quad = \quad D(S^{-1})D(T)D(S) \quad = \quad D(S^{-1})\, I\, D(S) \quad = \quad I$$

which shows that $S^{-1}TS$ must be a translation.■

9. Follow the hints. By construction every $DS(t)$ is equal to the diagonal matrix whose entries in order are 1 and -1. The product of this matrix with itself is the identity, and therefore D applied to $S(a)S(b)$ is the identity. By the preceding exercise, $S(a)S(b)$ must be a translation. We can find the translation vector fairly directly by evaluating at $(0,0)$, and if we do so we fine that the twofold composite sends $(0,0)$ to $(0, 2a-2b)$.

Applying this to the threefold composite, we obtain

$$S(a)S(b)S(c)(x_1,x_2) \quad = \quad S(a)(x_1, x_2+2b-2c) \quad = \quad (x_1, 2a+2c-2b-x_2)$$

which means that the threefold composite is $S(d)$, where $d = a+c-b$.■

10. The most direct way to do this is to prove that there are nonzero vectors $\mathbf{y}$ and $\mathbf{z}$ such that A sends $\mathbf{y}$ to itself, A sends $\mathbf{z}$ to $-\mathbf{z}$, and the vectors $\mathbf{y}$ and $\mathbf{z}$ are perpendicular. We can then get the desired orthonormal vectors by letting $\mathbf{u}$ and $\mathbf{v}$ be $\mathbf{y}$ and $\mathbf{z}$ multiplied by the reciprocals of their repsective lengths.

We can find nonzero vectors $\mathbf{y}$ and $\mathbf{z}$ as above if and only if the equations $(A+I)\mathbf{x} = \mathbf{0}$ and if and only if the equations $(A-I)\mathbf{x} = \mathbf{0}$ have nontrivial solutions, which is the same as showing that the determinants of $A \pm I$ are equal to zero. Direct computation shows that

$$0 \quad = \quad \det(A - kI) \quad = \quad k^2 - \cos^2\theta - \sin^2\theta \quad = \quad k^2 - 1$$

and hence the determinant is zero if $k = \pm 1$. This yields the vectors $\mathbf{y}$ and $\mathbf{z}$.

It is possible to solve directly for these vectors and show they are perpendicular by direct computation, but we shall give a conceptual proof which does not require finding the vectors explicitly. Since A is orthogonal we have

$$\langle \mathbf{y}, \mathbf{z} \rangle \quad = \quad \langle A\mathbf{y}, A\mathbf{z} \rangle \quad = \quad \langle \mathbf{y}, -\mathbf{z} \rangle \quad = \quad -\langle \mathbf{y}, \mathbf{z} \rangle \ .$$

The right hand side is the negative of the left hand side, and this can only happen if both sides are zero. Therefore the two vectors we want are perpendicular to each other.■

11. Follow the hint. To show that $A - I$ is invertible, compute its determinant; this turns out to be $2 - 2\cos\theta$, which is zero if and only if θ is an integral multiple of 2π. We have excluded these choices for θ, and hence the matrix will always be invertible in our situation.

Applying this to the question in the exercise, we need to show there is a unique $\mathbf{z}$ such that $T(\mathbf{z}) = A\mathbf{z} + \mathbf{b}$, or equivalently there is a unique solution to the equation $(A - I)\mathbf{z} = -\mathbf{b}$. Since $A - I$ is invertible, there is indeed a unique solution to this equation, and this suffices to prove the exercise.■

II.5 : Euclidean parallelism

1. Follow the hint, and write $L = \mathbf{x} + V$ and $M = \mathbf{y} + W$. If L and M were parallel then we would have $V = W$, so since they are not we know that $V \neq W$. This implies that V and W are both **proper vector subspaces** of $U = V + W$. Now V and W each have one element bases given by the nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ respectively, so $V + W$ is spanned by $\{\mathbf{v}, \mathbf{w}\}$, which means that $\dim U \leq 2$; since U properly contains V and W, it follows that its dimension is exactly 2.

Now let $P = \mathbf{x} + U$, so that $L \subset P$. We claim that $P \cap M = \emptyset$, and we shall do so by *reductio ad absurdum*. So assume M and P meet at the point $\mathbf{z}$. Then by the Coset Property we know that $M = \mathbf{z} + W$ and $P = \mathbf{z} + U$. Thus we have $\mathbf{x} = \mathbf{z} + b\,\mathbf{v} + c\,\mathbf{w}$ for suitable scalars b, c. Rearranging this, we obtain

$$\mathbf{x} \ - \ b\,\mathbf{v} \quad = \quad \mathbf{y} \ + \ c\,\mathbf{w}$$

and by the first sentence of this proof the latter yields a point on $L \cap M$. But we are given that $L \cap M = \emptyset$, so this is a contradiction. Therefore the line M and the plane P cannot have any points in common.■

2. The lines $S \cap P$ and $S \cap Q$ both lie in the plane S. If the intersection were nonempty and X belonged to that intersection, then we would have

$$X \ \in \ (S \cap P) \ \cap \ (S \cap Q) \quad = \quad (S \cap P \cap Q) \ \subset \ P \cap Q$$

which contradicts the hypothesis that $P \cap Q = \emptyset$.■

3. We shall do this problem using linear algebra. Following the hint, we first show that if S is a plane and $\mathbf{v} \notin S$, then there is a unique plane T such that $\mathbf{v} \in T$ and $S \cap T = \emptyset$.

Existence. Write $S = \mathbf{u} + W$ where W is a 2-dimensional vector subspace of $\mathbb{R}^3$, and let $T = \mathbf{v} + W$. Then $T \neq S$ because $\mathbf{v} \in T$ but $\mathbf{v} \notin S$, and therefore by the Coset Property for translates of the same subspace we know that S and T must be disjoint.

Uniqueness. Suppose we have an arbitrary 2-plane $\mathbf{y} + W'$ for some 2-dimensional vector subspace W'. By the Coset Property we have $\mathbf{y}+W' \ = \ \mathbf{v}+W'$. We claim that $\mathbf{v}+W$ and $\mathbf{v}+W'$ have a point in common if $W \neq W'$. The latter means that $W+W'$ properly contains either W or W'; this proper containment implies that $\dim W + W' \geq 3$, and since $W + W'$ is a vector subspace of $\mathbb{R}^3$ it follows that $W + W' \ = \ \mathbb{R}^3$. This in turn implies that $\dim W \cap W' \ = \ 1$. By the proof of the dimension theorem for subspaces of a vector space, it follows that there are vectors $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ such that the first one defines a basis for $W \cap W'$, the first two define a basis for W, and the first and last define a basis for W', so that all three form a basis for $\mathbb{R}^3$.

We may now write

$$\mathbf{v} \ - \ \mathbf{u} \ = \ t_1\,\mathbf{y}_1 \ + \ t_2\,\mathbf{y}_2 \ + \ t_3\,\mathbf{y}_3$$

for suitable scalars t_i, and we may rewrite this equation as follows:

$$\mathbf{v} \ - \ t_3\,\mathbf{y}_3 \ = \ \mathbf{u} \ + \ t_1\,\mathbf{y}_1 \ + \ t_2\,\mathbf{y}_2$$

The expression on the right hand side of this equation lies in $\mathbf{v} + W' = T$, whild the expression on the left hand side lies in $\mathbf{u} + W = S$, and thus we conclude that $S \cap T \neq \emptyset$. — This completes the proof of the assertion in the second sentence of this solution.

Conclusion of the argument. Suppose that $\mathbf{z} \in P \cap Q$. Then P and Q are two planes through $\mathbf{z}$ such that each is disjoint from S. Since this contradicts that result that we established above, it follows that there cannot be a point which lies on both P and Q.■

4. Again follow the hint; we shall use the notation introduced there.

The vector $Q(\mathbf{z})$ is perpendicular to $\mathbf{e}$ and $\mathbf{f}$ by the argument for deriving the Gram-Schmidt orthonormalization process (see Section I.1 of the notes). Since $\mathbf{v}$ and $\mathbf{w}$ are linear combinations of $\mathbf{e}$ and $\mathbf{f}$, it follows that $Q(\mathbf{z})$ is also perpendicular to $\mathbf{v}$ and $\mathbf{w}$. The Pythagorean principle for inner products now implies that

$$\left|\mathbf{z} - s\mathbf{e} - t\mathbf{f}\right|^2 \ = \ |Q(\mathbf{z})|^2 \ + \ |s - \langle \mathbf{z},\, \mathbf{e}\rangle \mathbf{e}|^2 \ + \ |t - \langle \mathbf{z},\, \mathbf{f}\rangle \mathbf{f}|^2$$

and this expression takes its minimum value for the values of s and t which make the last two summands equal to zero. Now this is precisely the condition under which $\mathbf{z} - s\mathbf{e} - t\mathbf{f}$ is perpendicular to (all linear combinations of) $\mathbf{e}$ and $\mathbf{f}$.

Finally since $\mathbf{e}$ and $\mathbf{f}$ span the same subspace of $\mathbb{R}^3$ as $\mathbf{v}$ and $\mathbf{w}$, it follows that the set of all vectors expressible in the form $\mathbf{z} - s\mathbf{e} - t\mathbf{f}$ is identical to the corresponding set of vectors expressible as $\mathbf{z} - a\mathbf{v} - b\mathbf{w}$, and therefore the minimum values for the (squares of the) lengths of vectors of the two types must also be equal. Combining this with the previous observations, we obtain the conclusion stated in the exercise.■

5. Write $\mathbf{x} = p\mathbf{a}$ and $\mathbf{y} = \mathbf{b} + q(\mathbf{c} - \mathbf{b})$, where p and q are scalars. Then we have

$$\mathbf{y} \ - \ \mathbf{x} \ = \ \mathbf{b} \ + \ q(\mathbf{c} - \mathbf{b}) \ + \ p\mathbf{a}$$

and since the original lines are skew lines we know that **a** and **c** − **b** are linearly independent by (the solution to) Exercise 1. If we now apply Exercise 2, we see that the length of the displayed vector is minimized when the vector in question is perpendicular to **a** and **c** − **b**, which is the same as saying that the line **xy** is perpendicular to **0a** and **bc**.■

6. It will suffice to show that $B - A$ and $D - C$ are nonzero multiples of each other (so that the lines AB and CD are either equal or parallel, and we know they are not equal), and similarly that $D - A$ and $B - C$ are nonzero multiples of each other.

By our hypotheses we have $A = \frac{1}{2}(W+X)$, $B = \frac{1}{2}(X+Y)$, $C = \frac{1}{2}(Y+Z)$, and $D = \frac{1}{2}(Z+W)$, so that

$$B - A = \tfrac{1}{2}(Y - W), \qquad D - C = \tfrac{1}{2}(W - Y)$$

and this shows AB is parallel to CD. Similarly, we have

$$D - A = \tfrac{1}{2}(Z - X), \qquad C - B = \tfrac{1}{2}(Z - X)$$

and this shows AD is parallel to BC.

7. Let M be the unique line through D which is equal or parallel to AC. Then $B \notin M$, and since AC has points in common with AB and BC it follows that M also has points X and Y in common with these two lines.

There are three cases, depending upon whether

(*i*) $D \in AC$,

(*ii*) D and B lie on the same side of AC,

(*iii*) D and B lie on opposite sides of AC.

In the first case it suffices to show that $D \in (AC)$, and in the other cases it we need to show that $X \in (BA$ and $Y \in (BC$. Note that $X \neq A$ and $Y \neq C$ in the last two cases because M must be parallel to AC.

In the first case, we have $D \in AC$ and D cannot be A or C. Now $A * C * D$ would imply A and D lie on opposite sides of BC, and $D * A * C$ would imply C and D lie on opposite sides of AB, so the condition $D \in \text{Int}\,\angle ABC$ forces the conclusion $A * D * C$, so that $D \in (AC)$.

In the second case, by the Crossbar Theorem we know that $(BD$ and (AC) have a point E in common. We claim that $B * D * E$ holds. Since $E \in (BD$, the other possibilities are $D = E$ or $B * E * D$. The first of these is impossible because we have a pair of parallel lines such that one contains D and the other contains E, and the second contradicts our assumption that B and D lie on the same side of AC. Therefore the line M contains a point between A and E. If we apply this and Pasch's Theorem to ΔABE and ΔCBE, we find that M must contain points on (AB) and (BC) because the line $AC = AE = CE$ is parallel to M. These points on M must be X and Y respectively, so we know that $X \in (BA) \subset (BA$ and $Y \in (BC) \subset (BC$.

The third case is similar, and once again we have the point E, but this time we claim that $B * E * D$ holds. As before, we cannot have $D = E$, but now $B * D * E$ would imply that D and B were on the same side of AC, so we are forced to conclude that $B * E * D$. In this case we know that the line AC contains the point E betwween B and D. If we apply this and Pasch's Theorem to ΔXBD and ΔYBD, we find that AC must contain points on (BX) and (BY) because the line $M = DX = DY$ is parallel to AC. These points on AC must be A and C respectively, so we know that $A \in (BX)$ and $C \in (BY)$. Therefore we have $B * A * X$ and $B * C * Y$, which means that $X \in (BA$ and $Y \in (BC$.■