EXERCISES FOR MATHEMATICS 138A

WINTER 2004

The references denote sections of the text for the course:

> M. P. do Carmo, *Differential geometry of Curves and Surfaces*, Prentice-Hall, Saddle River NJ, 1976, ISBN 0–132–12589–7.

I. Classical Differential Geometry of Curves

I.1 : Cross products

(do Carmo, § 1–4)

Additional exercise

1. Verify that the cross product of vectors in $\mathbf{R}^3$ satisfies the *Jacobi identity*:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \ + \ \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) \ + \ \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \ = \ \mathbf{0} \ .$$

I.2 : Parametrized curves

(do Carmo, § 1–2)

do Carmo, § 1–2, p. 5: 1, 2
do Carmo, § 1–4, p. 15: 8

> [*Hints for p. 15#8:* Strictly speaking there are two cases depending upon whether the lines in question intersect. Suppose that they do not intersect. In this case the shortest distance between the lies is givey by a common perpendicular. Assume that the parametrizations are chosen so that $\mathbf{u}$ and $\mathbf{v}$ lie on this common perpendicular. You may assume the existence of a common perpendicular when working the problem.]

Additional exercises

1. Prove that a regular smooth curve lies on a straight line if and only if there is a point that lies on all its tangent lines.

I.3 : Arc length and reparametrization

(do Carmo, § 1–3)

do Carmo, § 1–3, pp. 8–11: 7

Additional exercises

1. (a) Given $a > 0$, consider the set of all continuously differentiable real valued functions f on $[0,1]$ such that $f(0) = 0$ and $f(1) = a$. Define $L(f)$ by the formula $L(f) = \int_0^a |f'(t)|\, dt$. Show that the minimum value of $L(f)$ is a, and if equality holds then f' is everywhere nonnegative. [*Hints:* Since $f' \leq |f'|$ a similar inequality holds for their definite integrals. This inequality of integrals is strict if and only if $f'(t) < |f'(t)|$ for some t, which happens if and only if $f'(t) < 0$ for that choice of t.]

(b) Let ρ, θ and ϕ denote the usual spherical coordinates, and suppose we have a curve on the sphere of radius 1 about the origin with parametric equations of the form

$$\mathbf{x}(t) \quad = \quad (\cos\theta(t)\, \cos\sin(t),\ \sin\theta(t)\, \sin\phi(t),\ \cos\phi(t)\,)$$

for continuously differentiable functions $\theta(t)$ and $\phi(t)$. Prove that the length of this curve is given by the formula

$$\int_a^b \sqrt{(\theta'(t))^2 \ + \ \sin^2\theta(t)\, (\phi\,'(t))^2}\, dt$$

where the curve is defined on $[a,b]$.

(c) Show that among all regular smooth curves $\mathbf{x}$ that are defined on $[0,1]$, have images on the unit spere, and connect the points $(1,0,0)$ and $(\cos a, \sin a, 0)$ for some $a < \pi$, the curve of shortest length is given by the great circle arc joining the endpoints, and that any other curve with this length is a weak reparametrization of the great circle arc (*i.e.*, if α is the standard great circle arc, then any other curve β must have the form $\beta(t) = \alpha(\,f(t)\,)$, where f is a function from $[0,1]$ to itself that is continuously differentiable and satisfies $f' \geq 0$. [*Hints:* TO BE INSERTED.]

Note. The final part of the problem is a special case of the well known result that the shortest curve on a sphere joining two points is given by the smaller of the arcs on the great circle through the points; in fact, one can use this special case to prove the general statement. [A file containing a detailed proof will probably be inserted into the course directory eventually.]

I.4 : Curvature and torsion

(do Carmo, §§ 1–5, 1–6)

do Carmo, § 1–5, pp. 22-26: 11, 12*cd*
do Carmo, § 1–6, pp. 29-30: 3

Additional exercises

1. Consider the problem of designing a set of railroad tracks that contains a pair of parallel tracks along with a third going from the first to the second smoothly. Mathematically, the parallel tracks themselves may be viewed as corresponding to the parallel lines $y = 0$ and $y = 1$ in the coordinate plane, and the track going from one to the other may be viewed as a regular smooth curve that is the graph of a twice differentiable function f such that $f(x)$ is zero if $t \leq 0$, $f(x) = 1$ if $t \geq 1$, and on $[0,1]$ the function f is given by a polynomial $p(x)$. The existence of a second derivative ensures that the slope of the tangent line would be a continuous function of x, and in addition we want to assume that *the curvature is also a continuous function of* x. Finad a polynomial $p(x)$ of degree 5 such that all the required conditions are fulfilled. [*Hint:* If we are given a graph curve with parametric equatitons $(t,\, y(t))$, then the curvature at parameter value t is given by the formula

$$k(t) \quad = \quad \frac{|y''|}{(1+(y')^2)^{3/2}}$$

and one step in the argument is to use this fact to compute $p''(0)$ and $p''(1)$. In fact, the conditions of the problem uniquely specify the values of p and its first and second derivatives at both 0 and 1. Why does this mean the only values to find are the coefficients of x^3, x^4 and x^5?]

Extra credit. Graph the function f using calculator or computer graphics.

I.5 : Frenet-Serret Formulas

(do Carmo, §§ 1–5, 1–6, 4–Appendix)

Additional exercises

1. Let $\mathbf{x}$ be a regular smooth curve with a continuous third derivative, and let $(\mathbf{T},\, \mathbf{N},\, \mathbf{B}\,)$ be its Frenet trihedron.‘ Prove that there is a vector $\mathbf{W}$ (the *Darboux vector*) such that $\mathbf{T}\,' = \mathbf{W}\times\mathbf{T}$, $\mathbf{N}\,' = \mathbf{W}\times\mathbf{N}$, and $\mathbf{B}\,' = \mathbf{W}\times\mathbf{B}$. What is the length of $\mathbf{W}$?

2. If $\mathbf{x}$ is defined for $t > 0$ by the formula

$$\mathbf{x(t)} \quad = \quad \left(t,\ \frac{1+t}{t},\ \frac{1-t^2}{t}\right)$$

show that $\mathbf{x}$ is planar.

II. Closed Curves as Boundaries

II.1 : Regions, limits and continuity

(do Carmo, 2–Appendix A, 5–Appendix)

Additional exercises

II.2 : Smooth mappings

(do Carmo, 2–Appendix B)

Additional exercises

II.3 : Inverse and Implicit Function Theorems

(do Carmo, 2–Appendix A, 5–Appendix)

Additional exercises

II.4 : Global properties of plane curves

(do Carmo, §1–7)

Additional exercises