Equivariant surgery, isovariant equivalences, and middle dimensional fixed point sets

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ABSTRACT. Complete algebraic obstructions to equivariant surgery are described in some cases satisfying a weakening of a standard dimensional inequality called the Gap Hypothesis. In such cases one has an extra obstruction beyond those which exist when the standard inequality holds. Equivariant homotopy equivalences can be deformed to satisfy a stronger condition called isovariance if the Gap Hypothesis holds, and similar results are obtained for examples satisfying the types of conditions assumed in this paper.

During the nineteen seventies T. Petrie and others observed that many basic techniques of geometric topology – most notably those from the theory of surgery on manifolds – can be applied very effectively to manifolds with group actions if one assumes a condition called the Gap Hypothesis (*cf.* [DoS2, p. 20]). If the group G in question is cyclic of prime order, then the condition is $\dim X > 2\dim X^G + 1$, where as usual X^G denotes the fixed point set. Adjacent cases satisfying the conditions $\dim X = 2\dim X^G + \varepsilon$ (where $\varepsilon = 0, 1$) have been studied successfully in several contexts (*cf.* [Sc4, §II.4]). In particular if G is the cyclic group $\mathbb{Z}_2$ then results of the first author [D1] show that some results involving the Gap Hypothesis extend to cases where $\dim X = 2\dim X^G$ with somewhat different proofs but other results do not extend at all. The purpose of this paper is to consider actions of $G = \mathbb{Z}_p$ where p is an odd prime and $\dim X = 2\dim X^G$. In many respects the conclusions are similar to those in cases where $\dim X \geq 2\dim X^G + 2$, but we shall also show that the self intersection of the fixed point set defines a new equivariant surgery invariant that has no counterpart in cases where the Gap Hypothesis holds.

Summary of results

Equivariant surgery proceeds by a sequence of steps, assuming that a map is an equivalence of the desired type over a distinguished subset and analyzing the possibility of converting the map to an equivalence of the desired type over a larger distinguished subset. In particular, for actions of $\mathbb{Z}_p$ the first step is to convert the map into an equivalence over the fixed point set and the second step is to assume the map is already an equivalence there and measure the obstruction to completing the next step. From this viewpoint our main result can be stated as follows:

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Theorem I. *Let $G = \mathbb{Z}_p$ where p is an odd prime, let $f : M \to X$ be a degree 1 equivariant normal map of smooth 1-connected G-manifolds of dimension $4n \geq 8$ such that M^G and X^G are closed $2n$-manifolds, and assume that the associated map f^G of fixed point sets is a homotopy equivalence. Then there is a well defined class $\sigma(f)$ in the homotopy Wall group $L^h_{4n}(G)$ such that f is G-normally cobordant to a G-homotopy equivalence rel fixed point sets if and only if $\sigma(f) = 0$ and the self intersection numbers of M^G and X^G are equal.*

The statement of the theorem suggests that the vanishing of $\sigma(f)$ does not imply that the self intersection numbers of X^G and M^G are equal; specific examples with $\sigma(f) = 0$ but unequal self intersection numbers are constructed in Section 4. Theorem I has immediate applications to equivariant codimension one splitting theorems and isovariant homotopy equivalences that are parallel to results of [Sc3] in the case $G = \mathbb{Z}_2$. The full statements and proofs will be given in Section 2 (*e.g.*, see Theorems II and 2.1–2.2).

Since the main result deals with equivariant surgery under a condition that is a slight weakening of the Gap Hypothesis, the most direct approach to proving Theorem I is to begin with the ideas that succeed if the Gap Hypothesis holds and to modify the argument as necessary. The invariant $\sigma(f)$ is the obstruction one would obtain if the Gap Hypothesis were true, and in our case it can be defined using results on periodicity in equivariant surgery theory from [DoS2, Ch. III]. On the other hand, the need to consider intersection (and self intersection) numbers for fixed point sets is the basic technical complication that arises. The proof of Theorem I and some straightforward extensions of the latter are given in Section 1.

In [BkM] A. Bak and M. Morimoto announce a general approach to equivariant surgery with middle dimensional fixed point sets that involves Grothendieck-Witt groups of forms with extra structure called positioning data. Their work concentrates on the algebraic induction properties of such groups and their applications to one fixed point actions on spheres. The results of our paper deal with two somewhat different aspects of the topic; namely, explicit descriptions of the equivariant surgery obstructions themselves as in [D1–2] and applications to questions involving codimension one splittings and isovariance obstructions along the lines of [Sc3]. If the **Half Dimension Gap Hypothesis** is the statement

(‡) *for each pair of isotropy subgroups $H \supsetneqq K$ and each pair of components $B \subset M^H$, $C \subset M^K$ such that $B \subsetneqq C$ we have $\dim B \leq \frac{1}{2}(\dim C)$, and furthermore the standard Gap Hypothesis holds for all but at most one component of M^K*

then it seems likely that one can combine the results announced in [BkM] with the methods of this paper to obtain a conclusion of the following sort: *For odd order groups G and smooth even dimensional G-manifolds of dimension ≥ 6 satisfying the Half Dimensional Gap Hypothesis, the equivariant surgery obstructions for the problem* ($f : M \to X$, bundle data) *are completely determined by*

(1) *a stable equivariant surgery obstruction valued in the equivariant surgery obstruction groups of* [DP], [DR] *or* [LM], *which is given by applying the periodic stabilization construction of* [DoS2, §III.5],

(2) *for each relevant pair of subgroups $H \subset K$, the difference between the self intersection numbers of X^K in X^H and M^K in M^H.*

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1. Proof of Theorem I and generalizations

There are two approaches to constructing the obstruction class $\sigma(f)$ in Theorem I. It can be defined using the results of [DoS2, Ch. III] on periodicity in equivariant surgery, and it can also be described in terms of a highly connected equivariant normal map as in, say, [DR] or [LM]. The second approach is more direct, but the first yields the quickest proof that the obstruction is well defined. Both approaches will be used in this section.

Suppose that ($f : M \to X$, bundle data) is an equivariant normal map $f : M \to X$ satisfying the conditions of Theorem I. One can perform surgery below the middle dimension away from the fixed point set as usual (*e.g.*, as in [DR]) to make the map $2n$-connected, and the standard arguments as in [WL] imply that the surgery kernel

$$K_{2n}(M) = \text{Kernel}\, f_* \subset H_{2n}(M; \mathbb{Z})$$

is a stably free $\mathbb{Z}[G]$-module; again as in the standard cases, one can add trivial handles to ensure that the surgery kernel is in fact a free $\mathbb{Z}[G]$-module. The intersection pairing then defines a nondegenerate bilinear form λ on the free $\mathbb{Z}[G]$-module $K_{2n}(M)$, and passage to the associated nonequivariant surgery problem shows that the underlying form over the integers is even. Since G has odd order, it follows that there is a unique way to define a self intersection form μ so that $\mathbf{A} = (K_{2n}(M), \lambda, \mu)$ defines a symmetric Hermitian form in the sense of [WL].

We claim that the class of this form in the Wall group $L^h_{2n}(G)$ depends only on the relative normal bordism class of the original normal map (in this context relative means that the fixed point set remains unchanged). One way to see this is to use the periodic stabilization machinery of [DoS2]. Following [DoS2, p. 81], if G is a finite group and N is a smooth G-manifold, then $N{\uparrow}G$ will denote the product of $|G| = \text{order}(G)$ copies of N with the group action given by permuting coordinates. For our purposes the important example is $Y = \mathbb{C}\mathbf{P}^2{\uparrow}G$ where $G = \mathbb{Z}_p$ and p is an odd prime. Then $f \times \text{id}_Y$ is a G-surgery problem for which the Gap Hypothesis holds, and in this case one has a well defined surgery obstruction $\sigma(f \times \text{id}_Y) \in L^h_{4k+4|G|}(G)$. Furthermore, the methods of [Yo] show that under the usual fourfold periodicity of Wall groups this class corresponds to the element in $L^h_{4k}(G)$ representing $\mathbf{A}$. Since $\sigma(f \times \text{id}_Y)$ is a relative normal bordism invariant of f and it corresponds to the class of $\mathbf{A}$, it follows that the class of the latter is a relative normal bordism invariant as claimed. Therefore we shall denote the class of $\mathbf{A}$ by $\sigma(f)$.

To see that the self intersection number of the fixed point set of the domain is a normal bordism invariant, note that this number is given by $\langle \chi(\nu), [M^G] \rangle$ where $\chi(\nu)$ is the Euler class of the equivariant normal bundle for the embedding $M^G \subset M$, $[M^G]$ is the orientation class, and $\langle -, - \rangle$ denotes the Kronecker index. Invariance then follows by the usual proof of bordism invariance for characteristic numbers.

The proof of Theorem I requires the following algebraic input:

Lemma 1.1. *Let M and X be closed oriented G-manifolds such that both M^G and X^G are closed oriented manifolds, and let $f : M \to X$ be an equivariant degree 1 map such that the induced map $f^G : M^G \to X^G$ has degree $q \neq 0$. If $\alpha \in H^*(M;\mathbb{Z})$ and $\beta \in H^*(X;\mathbb{Z})$ are Poincaré dual to the images of the orientation classes $[M^G]$ and $[X^G]$ in $H_*(M,\mathbb{Z})$ and $H_*(X;\mathbb{Z})$ respectively, then $\delta = q\alpha - f^*\beta$ is Poincaré dual to a class in the kernel of the map $f_* : H_*(M;\mathbb{Z}) \to H_*(X;\mathbb{Z})$. Furthermore, if $\dim X = 4k$ and $\dim X^G = 2k$ then the difference between q^2 times the self intersection number of M^G and and the self intersection number of X^G is equal to $\langle \delta^2, [M] \rangle$.*

Proof. The conclusion of the first sentence is a formal consequence of the relations $\alpha \cap [M] = i_*[M^G]$, $\beta \cap [X] = i_*[X^G]$, $f_*[M] = [X]$, $f^G_*[M^G] = q[X^G]$, and the standard identities relating cap and cup products.

To prove the assertion in the second sentence, note first that the self intersection numbers are given by $\langle \alpha^2, [M] \rangle$ and $\langle \beta^2, [X] \rangle$. Since f has degree one the latter is equal to $\langle f^*\beta^2, [M] \rangle$. On the other hand. since δ is Poincaré dual to an element in the kernel of f_*, the identities relating cup and cap products imply that $\delta \cdot f^*\xi = 0$ for all $\xi \in H^{2k}(X;\mathbb{Z})$, so that $q^2\alpha^2 = f^*\beta^2 + \delta^2$. The conclusion of the second sentence follows immediately from this and the formulas in the first sentence of this paragraph.■

Proof of Theorem I. We begin by verifying that the conditions on $\sigma(f)$ and the self intersection numbers are necessary. Let $Y = \mathbb{C}\mathbf{P}^2 \uparrow G$ as above. If f is normally cobordant to an equivariant homotopy equivalence then so is $f \times \mathrm{id}_Y$, and therefore $\sigma(f \times \mathrm{id}_Y) = 0$ (*cf.* [DoS2, §I.4]). Periodicity then implies that $\sigma(f)$ must also be zero. To prove the statement regarding self intersection numbers, suppose first that f is a G-homotopy equivalence, and let α, β, and δ be given as in Lemma 1.1; by assumption $q = \pm 1$ in the cases under consideration. Since f is a homotopy equivalence the kernel of the homology map f_* is trivial, and therefore δ must be trivial. The second sentence of the lemma then implies that the self intersection numbers of M^G and X^G must be equal. Since the self intersection number is invariant under equivariant normal bordisms, it follows that the difference also vanishes if f is normally cobordant to an equivariant homotopy equivalence.

Conversely, suppose that $\sigma(f) = 0$ and the self intersection numbers of X^G and Y^G are equal. Without loss of generality we may also assume that f is $2k$-connected and that $K_{2n}(M)$ is a free $\mathbb{Z}[G]$-module. The vanishing of $\sigma(f)$ implies that we may add a finite number of trivial $2n$-handles to $M - M^G$ to obtain a new equivariant normal map $f' : M' \to X$ such that $K_{2n}(M')$ has a free subkernel, so we may as well assume that f already has this property. Choose a free basis $\{e_1, f_1, \cdots, e_r, f_r\}$ such that $\lambda(e_i, e_j) = \lambda(f_i, f_j) = 0$ for all i, j and $\lambda(e_i, f_j) = 0$ if $i \neq j$ and 1 if $i = j$. Let $\Delta \in K_{2n}(M)$ be Poincaré dual to δ (defined as in the lemma). By construction δ is invariant under the action of G on cohomology, and therefore

Δ is invariant under the action of G on homology. Therefore we may expand Δ as a sum $\sum_j a_j T e_j + b_j T f_j$ where a_j and b_j are integers and T denotes the sum of all group elements. The assumption on intersection numbers implies that the self intersection of Δ is trivial, and this translates into the equation

$$2|G| \sum_j a_j b_j = 0$$

which in turn implies $\sum_j a_j b_j = 0$.

We claim that Δ in fact lies in a subkernel $\widetilde{S}$. This is an immediate consequence of the following observation, the proof of which is a fairly straightforward exercise:

($\star$) *Let φ be the standard hyperbolic form on the free $\mathbb{Z}$-module A_{2r} with free basis $\{e_1, f_1, \cdots, e_r, f_r\}$, and let $x \in A_{2r}$ satisfy $\varphi(x,x) = 0$. Then there is a subkernel $S \subset A_{2r}$ such that $x \in S$.*

The claim follows from ($\star$) by taking $x = \sum_j a_j e_j + b_j f_j$, then tensoring φ with $\mathbb{Z}[G]$, and finally setting $\widetilde{S}$ equal to $S \otimes \mathbb{Z}[G]$.

Let $\{y_1, \cdots, y_r\}$ be a symplectic basis for $\widetilde{S}$ (*i.e.*, $\lambda(y_a, y_b) = 0$ for all a, b), and for each j let γ_j be a cohomology class that is Poincaré dual to y_j. Then all the self intersection numbers $y_j \cdot \Delta$ must vanish because $\Delta \in \widetilde{S}$. Since these intersection numbers are given by the integers $\langle \gamma_j \cdot \delta, [M] \rangle$ and each class γ_j satisfies $\gamma_j \cdot f^*\xi = 0$ for all $\xi \in H^{2n}(X;\mathbb{Z})$, the formula $\alpha = f^*\beta + \delta$ in the lemma implies the intersection number formula

$$y_j \cdot i_*[M^G] = \langle \gamma_j \cdot \alpha, [M] \rangle = \langle \gamma_j \cdot (f^*\beta + \delta), [M] \rangle =$$

$$\langle \gamma_j \cdot (f^*\beta), [M] \rangle + \langle \gamma_j \cdot \delta, [M] \rangle = 0 + 0 = 0.$$

The latter in turn implies that one can represent the classes y_j by smoothly embedded spheres V_j such that the following hold:

(a) Each V_j is disjoint from M^G.
(b) The normal bundle of each V_j is trivial and has a distinguished framing that is suitable for performing surgery.
(c) For each j and $g \in G - \{1\}$ we have $V_j \cap g \cdot V_j = \varnothing$.
(d) If $j \neq k$ and $g \in G$ we have $V_j \cap g \cdot V_k = \varnothing$.

As in [D1] one can now perform equivariant surgery on these embedded spheres (and their translates) to obtain an equivariant homotopy equivalence.■

Surgery up to pseudoequivalence. Recall from [DP] or [DoS2] that an equivariant pseudoequivalence is an equivariant map that is a nonequivariant homotopy equivalence (but not necessarily an equivariant homotopy equivalence). In analogy with [DoS1] there is a version of Theorem I for equivariant surgery up to pseudoequivalence.

Theorem 1.2. *Let $G = \mathbb{Z}_p$ where p is an odd prime, let ($f : M \to X$,* bundle data) *be a degree* 1 *equivariant normal map of smooth* 1*-connected G-manifolds of dimension $4n \geq 8$ such that M^G and X^G are closed $2n$-manifolds, and assume that the associated map f^G of fixed point sets has degree q prime to p and induces isomorphisms on mod p homology. Then there is a well defined class $\sigma(f)$ in the homotopy Wall group $L^h_{4s}(G)$ such that f is G-normally cobordant to a G-pseudoequivalence rel fixed point sets if and only if $\sigma(f) = 0$ and the self intersection number of M^G is q^2 times the self intersection number of X^G.*

Note. As in [DoS1], for an arbitrary degree 1 equivariant normal map the obstruction to completing equivariant surgery on the fixed point set lies in a Cappell-Shaneson homology surgery obstruction group $\Gamma^h_n(\mathbb{Z}[\pi_1(X)] \to \mathbb{Z}_p)$ as defined in [CS].

Sketch of proof of Theorem 1.2. We shall only indicate the points that cannot be obtained from the proof of Theorem I by minor changes in wording. The first potential difficulty involves the $\mathbb{Z}[\mathbb{Z}_p]$ module structure on the surgery kernel $K_{2n}(M)$; the general theory of [DP] shows this module is projective and determines a class in $\widetilde{K_0}(\mathbb{Z}[\mathbb{Z}_p])$ that lies in the defect subgroup $D(\mathbb{Z}_p) = B_0(\mathbb{Z}_p)$; since $D(G)$ vanishes if $G = \mathbb{Z}_p$, it follows that the surgery kernel is stably free, and as usual we may add trivial handles to make it a free module. One can again prove the image of the surgery kernel (with its extra algebraic structure) is a well defined element of the Wall group by a periodic stabilization argument, but here one needs to combine the periodicity results in [DoS2, Ch. V] with the general machinery of [DoS2, §III.5]. In the paragraph following the proof of Lemma 1.1 the degree q if f^G is not necessarily equal to 1; however, in this case one still has $\delta = 0$, and the second sentence of Lemma 1.1 now implies that q^2 times the self intersection number of M^G is equal to the self intersection number of X. The rest of the argument proceeds exactly as in the proof of Theorem I.■

Disconnected fixed point sets. The preceding results can also be extended as follows:

Complement 1.3. *Both Theorem* I *and Theorem* 1.2 *extend to situations where X^G is not connected but X satisfies the Half Dimension Gap Hypothesis.*

This is true because the embedded spheres that are needed for equivariant surgery can always be chosen to be disjoint from the lower dimensional components of the fixed point set.■

Analogs for equivariant simple homotopy equivalences. There is also a version of Theorem I in this case if one assumes that the normal representations on the components of M^G and X^G are equivalent.

Theorem 1.4. *Let $G = \mathbb{Z}_p$ where p is an odd prime, let $f : M \to X$ be a degree* 1 *equivariant normal map of smooth* 1*-connected G-manifolds of dimension $4n \geq 8$ that satisfy the Half Dimension Gap Hypothesis, and assume that the associated map f^G of fixed point sets is a simple homotopy equivalence such that the normal representations of corresponding* *components of M^G and X^G are linearly equivalent. Then there is a well defined class $\sigma(f)$ in the simple Wall group $L^s_{4n}(G)$ such that f is G-normally cobordant to a G-simple homotopy equivalence rel fixed point sets if and only if $\sigma(f) = 0$ and the self intersection numbers of M^G and X^G are equal.*

Sketch of proof. We shall only discuss two points that require additional thought. The condition on normal representations ensures that the chain level Poincaré duality map is an algebraically simple equivariant chain homotopy equivalence (see [DR, §5c, pp. 41–51a]). Also, the change of subkernel in the proof of Theorem I is induced from a change over the integers, and it follows that the new subkernel is a simple subkernel.■

2. Codimension one splitting

Theorem I of the preceding section yields an analog of [Sc3, Thm. A, p. 184] for actions of $\mathbb{Z}_p$ where p is an odd prime.

Theorem II. *Let p be an odd prime, and let N^{4k} be a closed 1-connected smooth $\mathbb{Z}_p$-manifold such that $k > 2$ and the fixed point set N_0 is a 1-connected closed $2k$-manifold. Assume that N admits an invariant decomposition $A \cup_C B$ where A and B are compact smooth $\mathbb{Z}_p$-manifolds such that $\partial A = \partial B = C$, the fixed point set N_0 is contained in A, and A, B and C are all 1-connected. If N^* is another closed smooth $\mathbb{Z}_p$-manifold and $f : N^* \to N$ is a $\mathbb{Z}_p$-homotopy equivalence, then there is an invariant splitting $N^* = A^* \cup B^*$ and a map f' equivariantly homotopic to f such that f' induces a $\mathbb{Z}_p$-homotopy equivalence of triads from $(N^*; A^*, B^*)$ to $(N; A, B)$.*

Remark. In [Sc3, Thm. A, p. 184] the conclusion states that the original map f induces an equivariant homotopy equivalence of triads; as in the statement of the theorem above, a more correct statement of the conclusion in [Sc3, Thm. A] is that f is equivariantly homotopic to an equivariant homotopy equivalence of triads.

Proof. The argument is similar to the proof of [Sc3, Thm. A, pp. 184-185]. Let N_0^* be the fixed point set of the $\mathbb{Z}_p$ action on N^*. Transversality considerations show that f is equivariantly homotopic to a map g such that g is transverse to C, and from this one obtains an equivariant map of triads

$$(g; g_A, g_B) : (N^*; g^{-1}(A), g^{-1}(B)) \to (N; A, B)$$

where $N_0^* \subset g^{-1}(A)$. As in [Sc3, §1] Wall's $\pi - \pi$ theorem [WL, §4] leads to the existence of a $\mathbb{Z}_p$-normal cobordism of an equivariant normal map of triads $(h; h_A, h_B)$ such that h_B is an equivariant homotopy equivalence and the normal cobordism is a product $V \times [0, 1]$ over some neighborhood of the fixed point set.

Consider now the map h_A; this nearly satisfies the conditions of the main theorem, the chief difference being that the domain and codomain have boundaries that are disjoint from the fixed point set and that h_A determines an equivariant homotopy equivalence from one boundary to the other. The argument of the main theorem goes through to show that h_A can be surgered equivariantly to a $\mathbb{Z}_p$-homotopy equivalence, with the boundary left untouched and all surgeries disjoint from the fixed point set, if and only the self intersection numbers of the fixed point sets are the same and a surgery obstruction in $L^h_{4k}(\mathbb{Z}_p)$ vanishes. Since h_B is an equivariant homotopy equivalence and everything in sight is simply connected, it follows that the surgery obstructions for h and h_A are equal. On the other hand, h is $\mathbb{Z}_p$-normally cobordant to the original $\mathbb{Z}_p$ homotopy equivalence f, and therefore this surgery obstruction is zero. Similar considerations imply that the difference between the self intersection numbers is zero. Therefore there is a $\mathbb{Z}_p$-homotopy equivalence $h^\#_A$ that is normally cobordant to h_A with the boundary and fixed point sets left untouched. If $h^\#_B = h_B$ and $h^\# = h^\#_A \cup h^\#_B$ then $h^\#$ defines a $\mathbb{Z}_p$-homotopy equivalence of triads $(h^\#; h^\#_A, h^\#_B)$ that is equivariantly normally cobordant to $(g; g_A, g_B)$ and f (where one has a normal cobordism of triads in the former case).

Let W be the domain of the normal cobordism between f and $h^\#$. We claim that it is possible to perform $\mathbb{Z}_p$-surgery on W, leaving the top and bottom boundary components and a neighborhood of the fixed point set untouched, to obtain a $\mathbb{Z}_p$-homotopy equivalence $W'' \to N \times [0,1]$. By construction W is obtained by adding handles to $N^* \times [0,\varepsilon]$ away from the fixed point set, and if $F : W \to N \times [0,1]$ is the associated degree 1 map then F is given by a map of triads

$$F : \big(W; \mathrm{Tube}((N^*)^G \times [0,1]), \mathcal{U}^\#\big) \to \big(N^* \times [0,1]; \mathrm{Tube}(N^G \times [0,1]), \mathcal{U}\big)$$

where $\mathrm{Tube}(\cdots)$ denotes a closed invariant tubular neighborhood and $\mathcal{U}$ and $\mathcal{U}^\#$ denote the closures of the complements of such tubular neighborhoods. Furthermore, F determines an equivariant homotopy equivalence on $\big(\mathrm{Tube}((N^*)^G \times [0,1]), \partial\,\mathrm{Tube}((N^*)^G \times [0,1])\big)$, and consequently the inclusion map induces an isomorphism of surgery kernels from $K_*(\mathcal{U}^\#)$ to $K_*(W)$. Therefore we can perform equivariant surgery on W, away from the boundary and the fixed point set, to obtain a similarly structured $\mathbb{Z}_p$-normal cobordism W_0 that is $(k-1)$-connected; in particular, the new degree 1 map F_0 is also given by a map of triads that is an equivariant homotopy equivalence on a tubular neighborhood of the fixed point set, and there is an isomorphism of surgery kernels as before. Therefore if $\mathcal{U}^*$ is the closure of the complement of $\mathrm{Tube}((N^*)^G \times [0,1]$ in W^* and we take a set of generators for $K_k(W^*) \approx K_k(\mathcal{U}^*)$ over the group ring $\mathbb{Z}[\mathbb{Z}_p]$, then we can represent them by embedded k-spheres in $\mathrm{Int}\, W_0 - \mathrm{Fix}(W_0; \mathbb{Z}_p)$. Exactly as in [Wl, §6] these embeddings determine a pair of subkernels in the standard hyperbolic symmetric Hermitian form over the group ring, and from this pair one obtains an invertible matrix α over the group ring that preserves this form. If SU^h denotes the subgroup of all such matrices, then α is not uniquely defined, but since the embedded k-spheres are disjoint from the fixed point set the analysis of [Wl, §6] goes through to show that surgery can be completed, leaving the fixed point set and boundary untouched, if the equivalence class of α under the projection $SU^h \to L^h_{4k+1}(\mathbb{Z}_p)$ is zero. Since the odd Wall groups of odd order groups are always trivial, it follows that one can perform surgery as indicated to obtain an equivariant homotopy equivalence from a suitable compact manifold W'' to $N \times [0,1]$.

If W'' is given as above then W'' is an equivariant h-cobordism; let τ be the equivariant Whitehead torsion of (W'', N); as in [R] this invariant takes values in the ordinary Whitehead group of G. Let X be an h-cobordism of bounded manifolds such that $\partial_0 X \approx h_B^{-1}(B)$ and the Whitehead torsion of $(X, h_B^{-1}(B)\,)$ is equal to $-\tau$, let $\widetilde{X}$ be the universal covering of X, and form a new equivariant h-cobordism W' by attaching $\widetilde{X}$ to $\partial_1 W''$ along $h_B^{-1}(B)$. Then W' has all the properties of W'' that are described in the preceding paragraphs, and in addition the equivariant Whitehead torsion of (W', N) is trivial. Therefore by [Ro, Thm. 3.4, p. 291] the cobordism W'' is $\mathbb{Z}_p$-diffeomorphic to $N^* \times [0,1]$, and one obtains the equivariant homotopy equivalence of the theorem by combining this diffeomorphism with the normal map from W' to $N \times [0,1]$.■

Complement. *A similar result holds if we allow the fixed point set to be the disjoint union of a* 1*-connected closed k-manifold* N_1 *and a union* N_2 *of lower dimensional submanifolds.*

The proof is similar to the argument given above, the main difference being that one must substitute the equivariant $\pi - \pi$ theorem of [DR] for the result from [Wl, §4] in the first paragraph; in fact, the argument in Wall's book goes through if we take the embedded spheres and disks constructed in [Wl] to be disjoint from the fixed point set, and the Gap Hypothesis implies that one can do this by general position.■

Application to isovariance questions

Theorem 2.1. *Let* $G = \mathbb{Z}_p$ *where* p *is prime, let* $f : M \to N$ *be an equivariant homotopy equivalence, where* M *and* N *are closed smooth* 1*-connected* G*-manifolds of dimension* ≥ 5 *that satisfy the Half Dimension Gap Hypothesis such that any components of dimension* $\frac{1}{2}\dim M = \frac{1}{2}\dim N$ *are* 1*-connected. Then* f *is homotopic to an isovariant homotopy equivalence.*

Proof. By the results of [DuS], it suffices to construct equivariant codimension one splittings $M \approx T^* \cup_\partial S^*$, and $N \approx T_0 \cup_\partial S_0$ such that T^* and T_0 are closed tubular neighborhoods of the respective fixed point sets and f is equivariantly homotopic to a map of triads from $(M; T^*, S^*)$ to $(N; T_0, S_0)$. If $\dim M \geq 2\dim M^G + 1$ this is done in [Br], so the conclusion is true in all such cases and we need only consider the case where $\dim M = 2\dim M^G$ and there is only one component of M^G whose dimension is $\frac{1}{2}\dim M$; note that N could have been substituted for N everywhere in the preceding statement because f is an equivariant homotopy equivalence so that $\dim N = \dim M$ and there is a 1–1 dimension preserving correspondence between the components of M^G and N^G.

If $p = 2$ then the conclusion of the theorem follows from [Sc3, Thm. A] (*cf.* the discussion in [Sc3, pp. 190–191]). On the other hand, if $p > 2$ then the codimension of the fixed point set must be even, and therefore $\dim M$ must be divisible by 4. Therefore the conclusion of the theorem follows in this case by applying Theorem II to a splitting $N \approx T_0 \cup_\partial S_0$ where as before T_0 is a closed tubular neighborhood of N^G.■

In [D2] the first author considered a related question; namely, whether normal bordism classes of adjusted normal maps (*i.e.*, homotopy equivalences on the fixed point set) contain isovariant representatives. When combined with the conclusions of [DuS], the results of [D2] show that such representatives always exist if $\dim X \geq 2\dim X^G + 1$ holds and $\dim X^G \geq 5$, and Theorem II yields the following analog when the Half Dimension Gap Hypothesis holds:

Theorem 2.2. *Let $G = \mathbb{Z}_p$ where p is an odd prime, let M and N be closed smooth G-manifolds of dimension ≥ 5 for which the Half Dimension Gap Hypothesis holds, and let $f : M \to N$ be an equivariant degree 1 normal map that is a homotopy equivalence on the fixed point set. Then f is equivariantly normally cobordant, rel fixed point sets, to an isovariant normal map if and only if the self intersection numbers of M^G in M and N^G in N are equal.*

Remarks. **1.** Examples of W. Browder [Br] show that f need not be equivariantly homotopic to an isovariant map even if f is an equivariant homeomorphism on a tubular neighborhood of the fixed point set (see also [Sc5]).

2. It seems highly unlikely that a similar conclusion holds under a condition weaker than the Half Dimension Gap Hypothesis. We hope to establish this in a subsequent paper on unstable versions of the equivariant function spaces in [BeS].

Sketch of proof of Theorem 2.2 To see that the condition on self intersection numbers is necessary, observe that if h is an isovariant degree 1 normal map that induces a homotopy equivalence of fixed point sets, then h induces a homotopy equivalence of the boundaries of their tubular neighborhoods, and this is enough to imply that the self intersection numbers of the fixed point sets (which are given by the Euler classes of the associated sphere bundles) are equal.

Suppose now that the condition on self intersection numbers is satisfied. The argument combines ideas from [Sc3] with the arguments in [D2] for cases where $\dim M > 2\dim M^G$. Split $N = T_0 \cup_\partial S_0$ as before, and write $T_0 = T_1 \coprod T_1'$ where T_1 is a closed tubular neighborhood of the unique component of N^G with maximal dimension and T_1' is the union of the tubular neighborhoods of the other components. Since it is possible to deform f equivariantly to a map that is transverse to ∂T_1 we may assume without loss of generality that f already has this property. If $S_1 = S_0 \cup T_1'$ then it follows that f induces a map of triads into $(N; T_1, S_1)$.

By construction $(S_1, \partial S_1)$ is at least 2-connected. Since G acts freely on the boundary and the Gap Hypothesis holds for S_1, the arguments of [D2] show that f is equivariantly normally cobordant as a map of triads to an equivariant normal map f_1 such that $f_1 | f_1^{-1}(S_1)$ is an isovariant homotopy equivalence of manifolds with boundary. Let h_1 denote the restriction of f_1 to $f_1^{-1}(T_1)$; as in the proof of Theorem II one can convert h_1 into an equivariant homotopy equivalence rel $f_1^{-1}(\partial S_1)$ if its Wall group obstruction σ_1 is zero (because we already know that the self intersection numbers of the fixed point sets are the same). However, we have no *a priori* information on σ_1, so we need to find a normal cobordism to a new equivariant normal map for which the Wall group obstruction is zero. To do this, construct the nonequivariant normal cobordism X_1^* mapping into $\partial S_1/G$ such that $\partial_0 X_1^* = \partial S_1/G$ maps by the identity, $\partial_1 X_1^*$ maps by a homotopy equivalence, $\pi_1(X_1^*) \approx G$, and the relative surgery obstruction for the normal cobordism is σ_1. Denote the universal covering of X_1^* by X_1. The standard normal cobordism extension construction now yields a new normal cobordism to a map of triads f_2 such that $f_2^{-1}(S_1) = f_1^{-1}(S_1) \cup -X_1$, $f_2^{-1}(T_1) = f_1^{-1}(T_1) \cup X_1$, and f_2 induces an equivariant homotopy equivalence on $f_2^{-1}(\partial T_1)$. If $h_2 = f_2|f_2^{-1}(T_1)$, then h_2 is again an equivariant homotopy equivalence on the boundary (where G acts freely on the latter) and the self intersection numbers of the fixed point sets are again equal, but the Wall group obstruction has changed by $-\sigma_1$ and therefore must be zero. Therefore h_2 is equivariantly normally cobordant rel the boundary to and equivariant homotopy equivalence, say h_3. If we set f_3 equal to $h_3 \cup_\partial f_2|f_2^{-1}(S_1)$ then by construction and the results of [DuS] we know that f_3 is an isovariant normal map that is equivariantly normally cobordant to f.■

The situation for $G = \mathbb{Z}_2$ is considerably less straightforward. In particular, let M and N be closed simply connected smooth $\mathbb{Z}_2$-manifolds of dimension $4k+2 \geq 6$ with connected $(2k+1)$-dimensional fixed point sets, and let $f : M \to N$ be an equivariant degree 1 normal map that is a homotopy equivalence on the fixed point set. In this case the equivariant surgery obstruction is given by the (nonequivariant) Kervaire invariant and the mod 2 rank of the surgery kernel [D1, DoS1], and we claim that these invariants vanish if f is normally cobordant to an isovariant normal map. To see this, suppose that f is isovariant, consider an associated map of triads

$$\big(M; \mathrm{Tube}(M^G), M - \mathrm{Open\ Tube}(M^G)\big) \to \big(N; \mathrm{Tube}(N^G), N - \mathrm{Open\ Tube}(N^G)\big)$$

and take f_1 to be the induced map on $M - \mathrm{Open\ Tube}(M^G)$. Then the obstruction to converting f_1 into an equivariant homotopy equivalence rel the boundary is given by a class in $L_2^h(\mathbb{Z}_2, \text{nontrivial map}) \approx \mathbb{Z}_2$, and by [D1, Example 4.13, p. 181] the Kervaire invariant and mod 2 rank are trivial for the original map f. Conversely, if these invariants vanish then by [Sc3] one can perform surgery to obtain an isovariant normal map. In many cases (*e.g.*, see [Sc3, Prop 2.4 and Thm. 2.5, pp. 189-190]) the mod 2 rank is related to the value of the self intersection form μ on the analog of the homology class Δ considered in the proof of Theorem I, so there is some parallel with Theorem 2.2. However, the results of [DMS] and [Stz] yield many examples for which the target manifold is $\mathbb{C}\mathbf{P}^{2k+1}$ with the complex conjugation involution and the mod 2 rank invariant is trivial but the Kervaire invariant is nontrivial. In contrast if $\dim M \equiv 0 \mod 4$ (hence the involution preserves orientation), then the conclusion of the theorem is true but the proof is somewhat more complicated.

3. Some isovariant deformation theorems

insert material here

4. An example

In this section shall construct an example to shows that the hypothesis on the self intersection numbers in Theorem I is not redundant. More precisely, the example will be an equivariant normal map (f, bundle data) satisfying the general conditions of the theorem and $\sigma(f) = 0$ but not satisfying the condition on self intersection numbers for the fixed point set. This example is probably one of the simplest that can be constructed; the same techniques and more methodical computations yield many other examples with these properties.

If a pointed space X is a product $X_1 \times \cdots \times X_n$ and A is a subset of $\{1, \cdots, n\}$, then there is an obvious projection map π_A from X to the product P_A of the X_j such that $j \in A$. Since our construction will involve the composites of such maps with the collapse map sending P_A to the smash product of the factors for several choices of A, it will be convenient to denote the codomain of this composite by $X_1(A) \wedge \cdots \wedge X_n(A)$ where $X_j(A)$ is X_j if $j \in A$ and S^0 otherwise; the composite map from $X_1 \times \cdots \times X_n$ to $X_1(A) \wedge \cdots \wedge X_n(A)$ will be denoted by ρ_A.

We now set $G = \mathbb{Z}_3$ and $X = S^4 \times S^4 \times S^{16}$ where G acts trivially on the first two factors and linearly on the third such that the fixed point set is S^4 (hence $X^G = S^4 \times S^4 \times S^4$ and $\dim X = 2 \dim X^G$). We claim that the existence of examples with the desired properties will follow if we can construct a G-vector bundle ω over X with the following properties:

(1) *ω is equivariantly stably fiber homotopically trivial.*
(2) *The (nonequivariant) Hirzebruch L-characteristic number of ω is trivial.*
(3) *If we write $\omega | X^G = \omega_G \oplus \omega^G$ where $\omega^G = (\omega | X^G)^G$, then the (nonequivariant) Hirzebruch L-characteristic number of ω^G is trivial.*
(4) *If $g \in G$ corresponds to the standard generator $e^{(2\pi i/3)} \in \mathbb{C}$ and $L(g, \omega | X^G)$ is the expression in the Atiyah-Singer G-signature formula with $\omega | X^G$ replacing $\tau_X | X^G$ (where τ_X is the equivariant tangent bundle of X), then $L(g, \omega | X^G) = 0$.*
(5) *If we take a complex structure on ω_G determined by the linear G-action on the fibers, then the top dimensional Chern class of ω_G is nonzero.*

Proof of claim. The methods of equivariant transversality theory (compare [P, §§II.1–3]) will yield an equivariant degree one normal map if there is a complex 6-plane bundle ν that is stably equivalent to ω_G as a complex vector bundle. Such a bundle exists for dimensional reasons because the base is 12-dimensional. Condition (3) and the standard results of simply connected surgery then show that we can do surgery to convert the map of fixed points sets into a homotopy equivalence; let $f : M \to X$ be such a map. Since $L^h_{24}(\mathbb{Z}_3)$ is detected by the difference of G-signatures between M and X and the G-signature of X is trivial, conditions (2) and (4) combine with the Hirzebruch Signature Theorem and the Atiyah-Singer G-signature formula to show that $\sigma(f) = 0$. On the other hand, the self intersection number of X^G in X is zero, but if M is the domain of the equivariant degree 1 normal map then the self intersection number of M^G in M is given by evaluating the Euler class of ν on the orientation class of X^G, and this number is nonzero by (5) and the relation $\chi(\nu) = c_6(\nu) = c_6(\omega_G)$.■

We shall use the previously developed notation for vector bundles over a product space to construct vector bundles satisfying conditions $(1) - (5)$ in the list above. If X is the threefold product described above, then all the smash product summands $X_1(A) \wedge X_2(A) \wedge X_3(A)$ are spheres with linear actions; furthermore, since X^G is the product of the fixed point sets of the factors, it follows that the smash product splitting is compatible with passage to the fixed point set. In particular, this means that the restriction map from $\widetilde{KO_G}(X)$ to $\widetilde{KO_G}(X^G)$ splits into a direct sum of restriction maps

$$\widetilde{KO_G}(S(\mathbf{M} \oplus \mathbb{R})) \to \widetilde{KO_G}(S(\mathbf{M}^G \oplus \mathbb{R}))$$

where $\mathbf{M}$ is a G-representation. The methods of [Sc2, Thm. 3.5] yield the following rational information on these restriction maps:

Exactness Property. *If the representation* $\mathbf{M}$ *is nontrivial and even dimensional and* $\mathbf{S}$ *is a standard splitting isomorphism as in* [Se, Remark, pp. 133-134]*, then image of the composite*

$$\begin{tikzcd}
\widetilde{KO_G}(S(\mathbf{M} \oplus \mathbb{R})) \otimes \mathbb{Q} \arrow[r, "i^*"] & \widetilde{KO_G}(S(\mathbf{M}^G \oplus \mathbb{R})) \otimes \mathbb{Q} \arrow[d, "\mathbf{S}\otimes\mathbb{Q}"] \\
 & \widetilde{KO}(S(\mathbf{M}^G \oplus \mathbb{R})) \otimes \mathbb{Q} \oplus \widetilde{K}(S(\mathbf{M}^G \oplus \mathbb{R})) \otimes \mathbb{Q}
\end{tikzcd}$$

consists of all ordered pairs (λ_1, λ_2) *such that the image of* λ_2 *under realification is equal to* $-\lambda_1$*.*$\blacksquare$

We shall be interested in G-vector bundles ω over X that are direct sums

$$\xi \oplus \eta \oplus \alpha \oplus \beta \oplus \gamma = \pi^*_{\{1\}}\xi_0 \oplus \pi^*_{\{1,2\}}\eta_0 \oplus \pi^*_{\{1,2,3\}}\alpha_0 \oplus \pi^*_{\{2,3\}}\beta_0 \oplus \pi^*_{\{3\}}\gamma_0$$

where ξ_0 and η_0 are G-vector bundles with trivial actions that are stably fiber homotopically trivial and have nonzero rational Pontryagin classes. These choices allow relatively easy computations of rational characteristic classes for the following reasons:

(a) The base spaces of the bundles $\xi_0, \cdots$ are suspensions and therefore the rational Chern and Pontryagin classes are additive with respect to direct sums.

(b) If u and v are positive dimensional cohomology classes of X such that $u \in \mathrm{Image}\,\pi^*_A$ and $v \in \mathrm{Image}\,\pi^*_B$ with $A \cap B \neq \varnothing$, then the cup product $u \cdot v$ is trivial.

In particular, it follows that conditions (2) – (4) in the preceding list translate into the following relations on the characteristic classes of $\xi, \eta, \cdots$ [*etc.*]:

(2*) $L_6(\alpha) + L_5(\beta)L_1(\xi) + L_4(\gamma)L_2(\eta) = 0.$
(3*) $L_3(\alpha^G) + L_2(\beta^G)L_1(\xi) + L_1(\gamma^G)L_2(\eta) = 0.$
(4*) $\Phi_6(\frac{2\pi}{3})c_6(\alpha_G) + \Phi_4(\frac{2\pi}{3})c_4(\beta_G)L_1(\xi) + \Phi_2(\frac{2\pi}{3})c_2(\gamma_G)L_2(\eta) = 0.$

As in [Sc1, (1.3), p. 499] the generating function for the coefficients $\Phi_k(\theta)$ is the power series

$$1 + iz\csc(iz - \theta).$$

By observation (a) above this defines a system of three homogeneous linear equations in α, β, γ if ξ and η are fixed arbitrarily (with respect to the previously stated conditions on fiber homotopy triviality and rational characteristic classes). In particular, if $\{\alpha, \beta, \gamma\}$ defines a solution and k is an arbitrary integer then the sums $\{k\alpha, k\beta, k\gamma\}$ of k copies of these bundles with themselves also defines a solution. But the results of [BeS, Prop. 11.2] and [Wn] on classifying equivariant fibrations and the finiteness of the homotopy groups for the stable space of self maps BF_G (see [BeS, §5]) imply that the G-vector bundles $k\alpha, k\beta, k\gamma$ are all stably equivariantly fiber homotopically trivial for all multiples of a sufficiently large number k_0, and because of this the construction of a G-vector bundle satisfying (1) – (5) reduces to the construction of examples satisfying (2) – (5). Furthermore, if one has a solution of (3^*) and (4^*) for which (5) holds and the appropriate characteristic classes are not all zero, we claim that one can obtain a solution of the entire system for which the values of the rational characteristic classes in (2^*) – (4^*) and (5) are nonzero multiples of the values in the original solution; if so, then everything reduces to the construction of examples satisfying (3^*) – (4^*) and (5). To establish this claim, suppose we have α, β, γ such that the latter hold. It will suffice to show that we can find α', β', γ' whose restrictions to X^G are (stably equivalent to) k-fold sums of $\alpha|X^G, \beta|X^G, \gamma|X^G$ for some $k > 0$ and (2^*) – (4^*) and (5) all hold. Perhaps the simplest way of constructing these G-vector bundles is to start with a vector bundle δ over S^{24} for which $p_6 \neq 0$ and pull it back to a G-vector bundle over X by the following equivariant composite:

$$X \xrightarrow[\text{proj.}]{\text{orbit}} X/G \xrightarrow{\text{degree 1}} S^{24}$$

It will then follow that

$$k(L_6(\alpha) + L_5(\beta)L_1(\xi) + L_4(\gamma)L_2(\eta)) = mL_6(\delta)$$

for suitable nonzero integers k and m. If we set $\beta' = k\beta$, $\gamma' = k\gamma$ and $\alpha' = k\alpha - m\delta$ then (3^*) – (4^*) and (5) will remain true (the restriction of δ to X^G is stably trivial and the effect of the substitution is simply to multiply the variable characteristic classes by k), and in addition (2^*) will also be true. Therefore everything reduces to showing that (3^*) – (4^*) and (5) hold for suitably chosen α, β, γ.

Modulo decomposables the Hirzebruch classes are given by

$$L_1 \equiv \frac{p_1}{3}, \qquad L_2 \equiv \frac{7p_2}{45}, \qquad L_3 \equiv \frac{62p_3}{315}$$

(*cf.* [Hi, p. 12]). Furthermore, the Exactness Property and standard relations between Chern and Pontryagin classes imply that $p_k(\lambda^G) = (-1)^{k+1}2c_{2k}(\lambda_G)$, where λ is α, β or γ. Finally, direct computation of the generating function shows that

$$\Phi_6(\tfrac{2\pi}{3}) = -\tfrac{403}{540}, \qquad \Phi_4(\tfrac{2\pi}{3}) = \tfrac{7}{9}, \qquad \Phi_2(\tfrac{2\pi}{3}) = -\tfrac{2}{3}$$

so that finding nontrivial solutions to (3^*) and (4^*) reduces to solving the following system of homogeneous linear equations in $c_6(\alpha_G)$, $c_4(\beta_G)$ and $c_2(\gamma_G)$:

($3^\ddagger$) $\quad \tfrac{124}{315}c_6(\alpha_G) - \tfrac{14}{45}c_4(\beta_G)L_1(\xi) + \tfrac{2}{3}c_1(\gamma_G)L_2(\eta) = 0.$

$(4^{\ddagger})\ -\frac{403}{540}c_6(\alpha_G) + \frac{7}{9}c_4(\beta_G)L_1(\xi) - \frac{2}{3}c_2(\gamma_G)L_2(\eta) = 0.$

Since the Chern and Pontryagin characters define isomorphisms between rationalized reduced (real and complex) K-theory and the direct sum of the reduced rational cohomology groups in dimensions divisible by 2 (complex case) or 4 (real case), one can find vector bundles whose characteristic classes satisfy the conditions $(3^*) - (4^*)$ and (5) if and only if one can can find algebraic solutions of the corresponding system $(3^{\ddagger}) - (4^{\ddagger})$ and (5) of equations with $c_6(\alpha_G)$, $c_4(\beta_G)$ and $c_2(\gamma_G)$ replaced by abstract cohomology classes A, B, and C respectively, where $A \neq 0$. The nontriviality of $L_1(\xi)$ and $L_2(\eta)$ implies that one can find solutions of the second system of equations with $A \neq 0$ because the coefficients of B and C in the first equation are not proportional to the coefficients of B and C in the second. Therefore one can find α, β, γ such that $(3^*) - (4^*)$ and (5) hold, and this proves that the difference of the self intersection numbers of the fixed point set can be nonzero even if the standard Wall group obstruction vanishes.■

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