Differentiation theorems for multiple integrals

We begin with a result which is closely related to the Fundamental Theorem of Calculus.

THEOREM 1. *Suppose that $f(x)$ is a continuous function defined on an open interval J, and let $x_0 \in J$ be arbitrary. Then there is some $r_1 > 0$ such that $(x_0 - r, x_0 + r)$ is contained in J for $0 < r \leq r_1$, and*

$$\lim_{r \to 0} \frac{1}{2r} \cdot \int_{x_0 - r}^{x_0 + r} f(t)\, dt \quad = \quad f(x_0) \ .$$

Derivation of Theorem 1. Define the antiderivative function to f by

$$g(x) \quad = \quad \int_{x_0}^{x} f(t)\, dt$$

if $x \geq x_0$ and by

$$g(x) \quad = \quad -\int_{x}^{x_0} f(t)\, dt$$

if $x \leq x_0$; both definitions yield $g(x_0) = 0$, so the functions fit together to form a differentiable function on the interval $(x_0 - r_1, x_0 + r_1)$. By the Fundamental Theorem of Calculus we also know that $g'(t) = f(t)$ for all t in the given interval.

Now let $0 < r < r_1$. Then by the Mean Value Theorem there is some Y_r in $(x_0 - r, x_0 + r)$ such that

$$\frac{g(x_0 + r) - g(x_0 - r)}{2r} \quad = \quad f(Y_r)$$

and the identities $g(x_0 + r) \ = \ \int_{x_0}^{x_0 + r} f(t)\, dt$, $g(x_0 - r) \ = \ -\int_{x_0 - r}^{x_0} f(t)\, dt$ allow us to rewrite this as follows:

$$\frac{1}{2r} \cdot \int_{x_0 - r}^{x_0 + r} f(t)\, dt \quad = \quad \frac{1}{2r} \cdot \left(\int_{x_0}^{x_0 + r} f(t)\, dt \ + \ \int_{x_0 - r}^{x_0} f(t)\, dt \right) \quad =$$

$$\frac{g(x_0 + r) - g(x_0 - r)}{2r} \quad = \quad f(Y_r)$$

Since f is continuous and $|Y_r - x_0| < r$, it follows that the limit of the right hand side as $r \to 0$ is equal to $f(x_0)$, and therefore the limit of the left hand side as $r \to 0$ is also equal to $f(x_0)$.

It turns out that Theorem 1 extends to functions of two, three, and even more real variables; we shall only consider the cases with two and three variables since these are the central objects in the first seven chapters of the course text. In order to state these results, we need one piece of notation.

Definition. Let $\mathbf{x}$ be a point in coordinate n-space (where we may restrict to $n = 2, 3$ if we wish), and let $r > 0$. The *closed disk of radius* r *with center* $\mathbf{x}$, written $D_r(\mathbf{x})$, is the set of all $\mathbf{y}$ in coordinate n-space such that $|\mathbf{y} - \mathbf{x}| \leq r$.

We shall state the 2-dimensional and 3-dimensional versions of the Multivariable Differentiation Theorem separately. It is possible to state all the higher dimensional versions of this theorem in a unified fashion, but we shall pass on doing so here (details can be found in graduate level courses on integration theory).

THEOREM 2. *Suppose that $f(\mathbf{x})$ is a continuous function defined on an open region U in the coordinate plane, let $\mathbf{x}_0 \in U$ be arbitrary, and let $A(r) = \pi r^2$. Then there is some $r_1 > 0$ such that $D_r(\mathbf{x}_0)$ is contained in U for $0 < r \leq r_1$, and*

$$\lim_{r \to 0} \frac{1}{A(r)} \cdot \int\int_{D_r(\mathbf{x}_0)} f(\mathbf{u})\, dA \quad = \quad f(x_0)\ .$$

THEOREM 3. *Suppose that $f(\mathbf{x})$ is a continuous function defined on an open region U in coordinate 3-space, let $\mathbf{x}_0 \in U$ be arbitrary, and let $V(r) = \frac{4}{3}\pi r^3$. Then there is some $r_1 > 0$ such that $D_r(\mathbf{x}_0)$ is contained in U for $0 < r \leq r_1$, and*

$$\lim_{r \to 0} \frac{1}{V(r)} \cdot \int\int\int_{D_r(\mathbf{x}_0)} f(\mathbf{u})\, dV \quad = \quad f(x_0)\ .$$

Both of these results can be established using mean value theorems for multiple integrals like those formulated for double and triple integrals on pages 473 and 445 (respectively) of the course text. However, we shall use a different and more direct (but also more abstract) approach, which uses the classical definition of continuity:

WEIERSTRASS DEFINITION OF CONTINUITY. A real valued function f is continuous at a point $\mathbf{x}$ if and only if for each $\varepsilon > 0$ there is some $\delta > 0$ such that $|f(\mathbf{y}) - f(\mathbf{x})| < \varepsilon$ provided $|\mathbf{y} - \mathbf{x}| < \delta$.

We shall also need the following standard upper estimates for integrals:

Let f be a continuous function defined on the closed region $D_r(\mathbf{x}_0)$ in the coordinate plane or coordinate 3-space. Then there is a positive constant $M > 0$ such that $|f(\mathbf{y})| < M$ for all $\mathbf{y}$ in the given region, and the associated double or triple integral satisfies the respective inequality

$$\left| \int\int_{D_r(\mathbf{x}_0)} f(\mathbf{x})\,dA \right| \;<\; M \cdot A(r)\ , \qquad \left| \int\int\int_{D_r(\mathbf{x}_0)} f(\mathbf{x})\,dV \right| \;<\; M \cdot V(r)$$

where $A(r)$ and $V(r)$ are defined as before.

The 2-dimensional version of this inequality is mentioned in one of the commentaries on Chapter 5 in this directory.

Proof of Theorem 2. This argument is at the level of an introductory real variables course such as Mathematics 151A.

Let $\varepsilon > 0$. We need to find some $\delta > 0$ such that

$$\left| \frac{1}{A(r)} \cdot \int\int_{D_r(\mathbf{x}_0)} f(\mathbf{u})\,dA \;-\; f(x_0) \right| \quad \varepsilon$$

provided $r < \delta$. We can rewrite the expression inside the absolute value sign as

$$\frac{1}{A(r)} \cdot \int\int_{D_r(\mathbf{x}_0)} f(\mathbf{u})\,dA \;-\; \frac{1}{A(r)} \cdot \int\int_{D_r(\mathbf{x}_0)} f(\mathbf{x}_0)\,dA \;=$$

$$\frac{1}{A(r)} \cdot \int\int_{D_r(\mathbf{x}_0)} \big(\, f(\mathbf{u}) - f(\mathbf{x}_0)\,\big)\,dA\ .$$

Since f is continuous and $\varepsilon > 0$, we know that there is some $\delta > 0$ such that $|f(\mathbf{y}) - f(\mathbf{x}_0)| < \varepsilon$ provided $|\mathbf{y} - \mathbf{x}| < \delta$. Therefore if $r < \delta$ then we have

$$\left| \frac{1}{A(r)} \cdot \int\int_{D_r(\mathbf{x}_0)} \big(\, f(\mathbf{u}) - f(\mathbf{x}_0)\,\big)\,dA \right| \;<\; \frac{1}{A(r)} \cdot \varepsilon \cdot A(r) \;\; = \varepsilon$$

and therefore we have the desired limit formula.

Proof of Theorem 3. This argument is a fairly straightforward modification of the previous one.

Let $\varepsilon > 0$. We need to find some $\delta > 0$ such that

$$\left| \frac{1}{V(r)} \cdot \int\int\int_{D_r(\mathbf{x}_0)} f(\mathbf{u})\,dA \;-\; f(x_0) \right| \quad \varepsilon$$

provided $r < \delta$. We can rewrite the expression inside the absolute value sign as

$$\frac{1}{V(r)} \cdot \int\int\int_{D_r(\mathbf{x}_0)} f(\mathbf{u})\, dV \;-\; \frac{1}{V(r)} \cdot \int\int\int_{D_r(\mathbf{x}_0)} f(\mathbf{x}_0)\, dV \;=$$

$$\frac{1}{V(r)} \cdot \int\int\int_{D_r(\mathbf{x}_0)} \big(f(\mathbf{u}) - f(\mathbf{x}_0) \big)\, dV \;.$$

Since f is continuous and $\varepsilon > 0$, we know that there is some $\delta > 0$ such that $|f(\mathbf{y}) - f(\mathbf{x}_0)| < \varepsilon$ provided $|\mathbf{y} - \mathbf{x}| < \delta$. Therefore if $r < \delta$ then we have

$$\left| \frac{1}{V(r)} \cdot \int\int\int_{D_r(\mathbf{x}_0)} \big(f(\mathbf{u}) - f(\mathbf{x}_0) \big)\, dV \right| \;<\; \frac{1}{V(r)} \cdot \varepsilon \cdot V(r) \;=\varepsilon$$

and therefore we have the desired limit formula.

A related result for vector fields

In certain contexts (like the derivations of Faraday's Law and Ampère's Law) one needs a version of the Multidifferentiation Theorem for 3-dimensional vector fields instead of scalar valued functions. We shall foormulate the statement of such a result and make a few comments on how it can be recovered from Theorem 2.

Let $\mathbf{F}$ be a vector field defined in a region of 3-dimensional coordinate space (as usual, the coordinate functions are assumed to have continuous partial derivatives). Given a point $\mathbf{p} = (c_1, c_2, c_3)$ in the region, a typical plane through $\mathbf{p}$ has a parametrization of the form

$$\mathbf{X}(u, v) \;=\; (a_1 u + b_1 v + c_1,\; a_2 u + b_2 v + c_2,\; a_3 u + b_3 v + c_3)$$

where the vectors $\mathbf{X}_1 = (a_1, a_2, a_3)$ and $\mathbf{X}_2 = (b_1, b_2, b_3)$ are linearly independent; the latter is equivalent to assuming that their cross product $\mathbf{X}_1 \times \mathbf{X}_2$, which is just the normal vector $\mathbf{N}$ to the plane, is nonzero.

We shall be interested in parametrized pieces of the plane which are obtained by restricting $\mathbf{X}$ to disks of the form $u^2 + v^2 \leq r^2$ for r sufficiently close to zero (specifically, we want r so small that the image of this piece of the plane will lie inside the region on which the vector field $\mathbf{F}$ is defined). The parametrized surface obtained in this manner, with its usual cross product orientation, will be denoted by $S_r(\mathbf{p}; \mathbf{X})$.

If we combine the preceding discussion with Theorem 2 and the usual formulas for surface integrals in terms of ordinary double integrals, we obtain the following result:

THEOREM 4. *Suppose that* $\mathbf{F}(\mathbf{x})$ *is a smooth vector field defined on an open region* U *in coordinate* 3*-space, and let* $\mathbf{p} \in U$ *be arbitrary. Assume further that we are given a* *plane through* $\mathbf{p}$ *parametrized linearly by some function —bf X as above. Then there is some* $r_1 > 0$ *such that* $S_r(\mathbf{p};\mathbf{X})$ *is contained in* U *for* $0 < r \le r_1$, *and*

$$\lim_{r\to 0} \frac{1}{\mathrm{area}(S_r(\mathbf{p};\mathbf{X})} \cdot \int\int_{S_r(\mathbf{p};\mathbf{X})} \mathbf{F}(\mathbf{u}) \cdot d\mathbf{S} \quad = \quad \mathbf{F}(\mathbf{p}) \cdot \mathbf{N}\ .$$