APPENDIX D

AUTOMORPHISMS OF THE COMPLEX NUMBERS

This appendix discusses an issue that arose in Section VI.??. It is written at a level somewhat higher than the notes, but not as high as the level of material in Appendix B. The necessary background can be found in the graduate algebra text by T. Hungerford and the introductory granduate level topology text by J. Munkres that are listed in the bibliography.

The Fundamental Theorem of Projective Geometry in Chapter VI (Theorem VI.??) implies that for each positive integer n there is a group homomorphism from the group $\mathsf{Proj}_n(\mathbb{F})$ of geometric symmetries, or *collineations*, of $\mathbb{F}¶^n$ to the automorphism group of $\mathbb{F}$, and this mapping is onto. For some fields such as the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$, the only automorphism is the identity. On the other hand, in the notes we pointed out that the field $\mathbb{C}$ of complex numbers also has a second automorphism given by complex conjugation. This leads immediately to the following:

Quesion. *Are there any automorphisms of $\mathbb{C}$ other than the identity and complex conjutation?*

In fact, there are two ways of answering this question. If we restrict our attention to automorphisms that are continuous as mappings of $\mathbb{C} \cong \mathbb{R}^2$ to itself, then the identity and complex conjugation are the only possibilities. On the other hand, if we consider arbitrary automorphisms of $\mathbb{C}$ with no continuity requirement, then the automorphism group of $\mathbb{C}$ is huge. One illustration of this is the formula for its cardinality given below.

Continuous automorphisms of $\mathbb{C}$

We shall first prove that there are only two continuous automorphisms.

Theorem D.1. *Let $\varphi : \mathbb{C} \to \mathbb{C}$ be an automorphism of the complex numbers that is also continuous as a function of two variables. Then φ is either equal to the identity or to complex conjugation.*

Proof. Every automorphism sends 0 and 1 to themselves, and from this it follows that every automorphism sends the rational numbers $\mathbb{Q} \subset \mathbb{C}$ to itself by the identity.

Let $\mathbb{Q}[\mathbf{i}]$ be the set of all complex numbers of the form $a + b\mathbf{i}$, where $\mathbf{i}^2 = -1$ and $a,\ b \in \mathbb{Q}$. Since $\pm\,\mathbf{i}$ are the only two complex numbers whose squares are equal to -1, it follows that $\varphi(\mathbf{i}) \;=\; \pm\,\mathbf{i}$. Therefore φ agrees with either the identity or conjugation on $\mathbb{Q}[\mathbf{i}]$.

Just as the rationals are dense in $\mathbb{R}$, so also the set $\mathbb{Q}[\mathbf{i}]$ is dense in $\mathbb{C}$. Since two continuous functions from $\mathbb{C}$ to itself are equal if they agree on a dense subset, it follows that φ is equal to the identity or complex conjugation, depending upon whether $\varphi(\mathbf{i}) \;=\; \mathbf{i}$ or $\varphi[\mathbf{i}] \;=\; -\mathbf{i}$.■

Arbitrary automorphisms of $\mathbb{C}$

This discussion will rely heavily on the existence of *transcendence bases* for the $\mathbb{C}$ with respect to its subfield $\mathbb{Q}$. Formal definitions and the basic properties of such subsets are contained in Section VI.1 of Hungerford's book. A brief summary of some important points is given in the Wikipedia reference:[1]

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[1]Previous comments about Wikipedia also apply here.