CASSON AND GORDON MEET HEEGAARD AND FLOER

DANIEL RUBERMAN

To Terry, with admiration and thanks

The computation of the classical knot concordance groups $\mathcal{C}_{\mathrm{top}}$ (topological) and $\mathcal{C}_{\mathrm{smooth}}$ (smooth) remains, despite some 50 years of activity [5], one of the central problems in low-dimensional topology. The work of Levine [14] gave a complete calculation of the knot concordance group in high dimensions, via an isomorphism $\Phi : \mathcal{C} \to \mathbf{Z}^\infty \oplus \mathbf{Z}_2^\infty \oplus \mathbf{Z}_4^\infty$. (Except for a minor issue in ambient dimension 5, there isn't any difference between $\mathcal{C}_{\mathrm{top}}$ and $\mathcal{C}_{\mathrm{smooth}}$ in high dimensions.) Subsequently, Casson and Gordon [1, 2] showed that while Φ is onto in the classical dimension, it is not injective.

In addition to their intrinsic interest, questions about concordance are intimately related to surgery theory and other questions about 4-manifolds. Indeed, the original paper of Fox and Milnor [6] explains that the problem of representing a 2-dimensional homology class in a 4-manifold by an embedded sphere often reduces to asking if some knot is slice. Conversely, the ability to represent certain homology classes by topologically embedded spheres means that methods of surgery theory will show certain knots to be topologically slice. The most famous instance of this is Freedman's proof that Alexander-polynomial 1 knots are slice; there are more recent results along these lines by Friedl-Teichner [7].

More recently, gauge theory has provided very strong obstructions to a homology class in a 4–manifold being represented by an embedded sphere, or more generally by a surface of low genus. Terry was an early contributor to this study–see [11] and the wonderful survey articles [12, 13]. The most recent 'gauge-theoretic' tool, the Heegaard-Floer homology of Ozsváth and Szabó is no exception: many of the genus bounds proved via Donaldson and Seiberg-Witten theory have new (and in some sense easier) proofs via the new theory. These results can be translated into new obstructions to knots being slice, but the Ozsváth-Szabó theory provides a more direct route to such obstructions. They [21, 20], simultaneously with Rasmussen [22], introduced 'knot Floer homology' groups $HFK(K)$ for a knot $K \subset S^3$, from which they derived a numerical invariant $\tau(K) \in \mathbf{Z}$.

The invariant $\tau(K)$ in fact vanishes on slice knots, and provides a homomorphism $\mathcal{C}_{\mathrm{smooth}} \to \mathbf{Z}$ that is distinct from all classical invariants, including the signature. It has a nice property that is, on the other hand, reminiscent

My talk at Tulane, and this note on which it was based, represent joint work with Eli Grigsby and Sašo Strle.

of signature invariants of knots: the τ invariant can in fact be defined for a null-homologous knot in a rational homology sphere Y; in the case at hand Y will be the 2-fold branched cover of a knot K in S^3, and we will consider $\tau(Y,\tilde{K})$ of the branch set $\tilde{K} \subset Y$. As we will explain below, $\tau(Y,\tilde{K})$ is a function from $H^2(Y) \to \mathbb{Z}$, where $H^2(Y)$ parameterizes the spin^c structures on Y. Our idea was that τ could be considered as analogous to the Casson-Gordon invariant $\tau(K,\chi)$ which is a function of $\chi \in \mathrm{Hom}(H_1(Y),\mathbb{Q}/\mathbb{Z})$. The main theorem says that if K is slice, then $\tau(Y,\tilde{K},s)$ vanishes for appropriately chosen $s \in \mathrm{Spin}^c(Y)$, much as $\tau(K,\chi)$ must vanish for appropriate characters χ.

To give a precise statement, we need to quickly review some notions from the realm of Heegaard-Floer theory. Let Y be a rational homology sphere, and let $s \in \mathrm{Spin}^c(Y)$ be a spin^c structure. A Heegaard decomposition of Y (or equivalently, a Morse function) defines a chain complex $\widehat{\mathrm{CF}}(Y,s)$ whose homology is the Heegaard-Floer group $\widehat{\mathrm{HF}}(Y,s)$. Because we are working on a rational homology sphere, there is a rational-valued 'Maslov grading', ie a function $\tilde{\mathrm{gr}} : \widehat{\mathrm{HF}}(Y,s) \to \mathbb{Q}$. There is a canonical summand $\widehat{\mathrm{HF}}_U(Y,s)$ of $\widehat{\mathrm{HF}}(Y,s)$, and the *correction term* for a spin^c structure s, denoted $d(Y,s)$, is the absolute $\mathbb{Q}$ homological grading, $\tilde{\mathrm{gr}}$, of $\widehat{\mathrm{HF}}_U(Y,s)$.

The d-invariant has the important property that $d(Y,s) = 0$ whenever $Y = \partial W$ where W is a rational homology ball and the spin^c structure s extends over W. Because the 2-fold cover of S^3 branched along a slice knot bounds a rational homology ball (the branched cover of the 4-ball over the slicing disk) the d-invariant gives a new obstruction to a knot being slice. This has been investigated by Manolescu-Owens [17] and Jabuka-Naik [10]). One point about the use of the d-invariant that is common with the original Casson-Gordon invariants is that one does not know *a priori* which spin^c structures on Y extend over W, so that in applying this obstruction one may have to do a great deal of computation.

Our idea was to strengthen the application of Heegaard-Floer homology by using observation that (with notation as in the last paragraph) not only does $Y = \partial W$, but the preimage of K in Y is slice in W. So we can use another concordance invariant: the τ invariant, which arises from the knot homology theory $HFK(K)$. Very briefly, a null-homologous knot K gives rise to a $\mathbb{Z}$-grading on the Heegaard-Floer chain complex $\widehat{\mathrm{CF}}(Y,s)$. The minimal grading of an element that projects non-trivially to the aforementioned group $\widehat{\mathrm{HF}}_U(Y,s)$ is by definition $\tau(Y,K,s)$. When Y is the 3-sphere, $\tau(K)$ is a concordance invariant, and in fact gives a lower bound for the genus of an oriented surface bounded by K in the 4-ball. Our main technical result states something similar for all of the spin^c structures on Y that extend over W.

Theorem 1. *Let K be a knot in S^3, and Y the 2-fold cover of S^3 branched along K. Denote by $\widetilde{K}$ the preimage of K in Y. If K is slice, then there* *exists a subgroup* $G < H^2(Y;\mathbb{Z})$ *with* $|G|^2 = |H^2(Y;\mathbb{Z})|$ *such that* $d(Y,s) = 0$ *and* $\tau(Y,\widetilde{K},s) = 0$ *for all* $s \in s_0 + G$*, where* s_0 *is the unique spin structure on* Y*.*

An only slightly more elaborate statement holds with 2 replaced by p^r for any prime p.

Having been raised to believe that a theorem is not worth much unless it can be applied to some interesting examples, we went looking for knots for which we could compute the invariants $\tau(Y,\widetilde{K},s)$. There have been important recent advances [18] in the computation of Heegaard-Floer groups, and these have brought the computation of $\tau(Y,\widetilde{K},s)$ within reach for at least one class of knots, the 2-bridge knots. Recall that these are knots $K_{p,q}$ whose double branched cover is the lens space $L(p,q)$. Eli Grigsby [9] showed how to compute the groups $\widehat{\mathrm{HFK}}(L(p,q),\widetilde{K}_{p,q})$ purely combinatorially, and with some extra work one can extract combinatorial calculations of the corresponding τ invariants. The question of which 2-bridge knots are smoothly slice has been definitively answered by Paolo Lisca [16]. However, there remain further questions about this category of knots, in particular the question of showing that a particular 2-bridge knot has *infinite order* in the concordance group.

One issue that arises in the Casson-Gordon invariants, and indeed in all concordance obstructions based on branched covers, is that one does not know *a priori* the restriction map $H^2(W;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ where W and Y are as above the branched covers of the 4-ball and 3-sphere respectively. In the original setting and also in the initial gauge-theoretic extensions [4, 19, 23], this comes out as a lack of information about which cyclic covers of Y extend over W, whereas in the Seiberg-Witten and Heegaard-Floer setting, it is a question about extension of spin^c structures. In principle, to use an obstruction such as that in Theorem 1 to show that some knot isn't slice, one might have to test whether the conclusions fail for *every* $G < H^2(Y;\mathbb{Z})$ with $|G|^2 = |H^2(Y;\mathbb{Z})|$! This clearly gets out of hand when the order of $H^2(Y;\mathbb{Z})$ is large. For example, to use the theorem as stated to determine if the sum of 4 copies of the 2-bridge knot 45/17 is slice would require examination of 9,745,346 such subgroups. As it turns out, there are subgroups on which the d-invariant vanishes, so that invariant alone would not suffice to determine the concordance order of 45/17.

To get around this problem, we developed a way to package the functions $d(\cdot) : \mathrm{Spin}^c(Y) \to \mathbb{Q}$ and $\tau(Y,\widetilde{K},\cdot) : \mathrm{Spin}^c(Y) \to \mathbb{Z}$ that does not require examination of subgroups. It is easiest to explain in the case that $H^2(Y;\mathbb{Z})$ is cyclic, which we now assume. We define two invariants $\mathcal{T}_p$ (resp. $\mathcal{D}_p$) to be the absolute value of

$$\sum_{\{\mathfrak{s}\in\mathrm{Spin}^c(Y) | \mathfrak{s} \text{ has order p}\}} \tau_{\mathfrak{s}}(\widetilde{K}) \qquad (\text{resp. } d_{\mathfrak{s}}(Y)).$$

We then showed

Theorem 2. *Let $K \subset S^3$ be a knot and $p \in \mathbb{Z}_+$ prime or 1. If there exists a positive $n \in \mathbb{Z}$ such that $\#_n K$ is smoothly slice, then $\mathcal{T}_p(K) = \mathcal{D}_p(K) = 0$.*

To finish the discussion of the example for K the 2-bridge knot 45/17, the invariants $\mathcal{D}_3(K)$ and $\mathcal{D}_5(K)$ both vanish, but $\mathcal{T}_3(K) = \mathcal{T}_5(K) = -1$. Hence we conclude that this knot has infinite order in the smooth concordance group. It would of course be of great interest to know if it has finite order in the topological concordance group.

Postscript: There have been some interesting recent developments in applying gauge theory to knot concordance since we posted the preprint on which this talk was based. Lisca [15] showed how to apply Donaldson's diagonalization theorem to deduce that every 2-bridge knot, other than those already known to be ribbon [1, 2, 3] has infinite concordance order. Combining this essentially algebraic technique with the d-invariant, Greene and Jabuka [8] extended this result to pretzel knots all of whose twisting numbers are odd. It seems likely that much more can be said with these techniques.

References

[1] A. Casson and C. Gordon, *On slice knots in dimension three*, Proc. Symp. Pure Math., **32** (1978).
[2] ———, *Cobordism of classical knots*, in "À la recherche de la Topologie Perdue", A. Marin and L. Guillou, eds., Progress in Mathematics, Birkhauser, Boston, 1986.
[3] A. Casson and J. Harer, *Some homology lens spaces which bound rational homology balls*, Pacific J. Math., **96** (1981), 23–36.
[4] R. Fintushel and R. Stern, *Rational homology cobordisms of spherical space forms*, Topology, **26** (1987), 385393.
[5] R. H. Fox and J. W. Milnor, *Singularities of 2-spheres in 4-space and equivalence of knots*, Bull. A.M.S., **63** (1957), 406.
[6] ———, *Singularities of 2-spheres in 4-space and cobordism of knots*, Osaka J. Math., **3** (1966), 257–267.
[7] S. Friedl and P. Teichner, *New topologically slice knots*, Geom. Topol., **9** (2005), 2129–2158 (electronic).
[8] J. Greene and S. Jabuka, *The slice-ribbon conjecture for 3-stranded knots*. arXiv:0706.3398v2 [math.GT].
[9] J. E. Grigsby, *Combinatorial description of knot floer homology of cyclic branched covers*. http://www.arxiv.org/abs/math.GT/0610238, 2006.
[10] S. Jabuka and S. Naik, *Order in the concordance group and Heegaard Floer homology*. http://www.arxiv.org/abs/math.GT/0611023, 2006.
[11] T. Lawson, *Representing homology classes of almost definite 4-manifolds*, Michigan Math. J., **34** (1987), 85–91.
[12] ———, *Smooth embeddings of 2-spheres in 4-manifolds*, Exposition. Math., **10** (1992), 289–309.
[13] ———, *The minimal genus problem*, Exposition. Math., **15** (1997), 385–431.
[14] J. Levine, *Invariants of knot cobordism*, Inventiones Math., **8** (1969), 98–110.
[15] P. Lisca, *Sums of lens spaces bounding rational balls*. arXiv:0705.1950v1 [math.GT].
[16] ———, *Lens spaces, rational balls and the ribbon conjecture*, Geom. Topol., **11** (2007), 429–472 (electronic).
[17] C. Manolescu and B. Owens, *A concordance invariant from the Floer homology of double branched covers*. http://www.arxiv.org/abs/math.GT/0508065, 2005.

[18] C. Manolescu, P. Ozsváth, and S. Sarkar, *A combinatorial description of knot Floer homology.* math.GT/0607691, 2006.

[19] G. Matić, *SO(3)–connections and rational homology cobordisms*, J. Diff. Geo., **28** (1988), 277–307.

[20] P. Ozsváth and Z. Szabó, *Holomorphic disks and genus bounds*, Geom. Topol., **8** (2004), 311–334 (electronic).

[21] ———, *Holomorphic disks and knot invariants*, Adv. Math., **186** (2004), 58–116.

[22] J. A. Rasmussen, *Floer homology and knot complements.* PhD thesis, Harvard University, 2003.

[23] D. Ruberman, *Rational homology cobordisms of rational space forms*, Topology, **27** (1988), 401–414.