COMMENTS AND CHANGES FOR

EXERCISES IN SET THEORY

This document has several purposes. The first is to give explicit formulations of some exercises for Unit V that were only described indirectly by references to Halmos or to results and exercises from previous units. The second is to give further hints for solving some of these exercises, and the third is to provide a logical sequence for the exercises which will make it easier to work them in an orderly fashion. Finally, a few additional exercises will also be inserted to clarify the logical sequence still further.

V. Infinite constructions on sets

V.1 : Large set-theoretic constructions

Exercises to work

1. Here is a formal statement of the problem:

Suppose that we have nonempty indexed families of sets $\{A_j \mid j \in J\}$ and $\{C_j \mid j \in J\}$ such that $A_j \subset C_j$ for all $j \in J$. Prove the following inclusion relationships:

$$\bigcup_{j\in J} A_j \subset \bigcup_{j\in J} C_j \qquad\qquad \bigcap_{j\in J} A_j \subset \bigcap_{j\in J} C_j$$

2. Here is a formal statement of the problem:

Suppose that S is a set and we have a nonempty indexed families of subsets of S of the form $\{A_j \mid j \in J\}$. Prove the following identities:

$$S - \bigcup_{j\in J} A_j = \bigcap_{j\in J} S - A_j$$

$$S - \bigcap_{j\in J} A_j = \bigcup_{j\in J} S - A_j$$

3. Here is a formal statement of the problem:

(a) Given that $\{A_i\}_{i\in X}$ and $\{B_j\}_{j\in Y}$ are nonempty indexed families of sets, prove the following identities:

$$\left(\bigcup_i A_i\right) \cap \left(\bigcup_j B_j\right) = \bigcup_{i,j} (A_i \cap B_j)$$

$$\left(\bigcap_i A_i\right) \cup \left(\bigcap_j B_j\right) = \bigcap_{i,j} (A_i \cup B_j)$$

(b) Suppose that $\{I_j\}$ is a family of sets indexed by J, and write $K = \cup_j I_j$. Let A_k be a family of sets indexed by K. Prove the following identities:

$$\bigcup_{k \in K} A_k = \bigcup_{j \in J} \left(\bigcup_{i \in I_j} A_i\right)$$

$$\bigcap_{k \in K} A_k = \bigcap_{j \in J} \left(\bigcap_{i \in I_j} A_i\right)$$

4. Here is a formal statement of the problem:

(a) Let $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ be indexed families of sets. Prove that

$$\left(\bigcup_i A_i\right) \times \left(\bigcup_j B_j\right) = \bigcup_{i,j} (A_i \times B_j)$$

and that a similar formula holds for intersections provided that the indexing sets are nonempty.

(b) Let $\{X_i\}_{i \in I}$ be an indexed family of sets. Prove that $\cap_i X_i \subset X_j \subset \cup_i X_i$ for all $j \in I$. Furthermore, if U and V are sets such that $U \subset X_j \subset V$ for all j, prove that that $U \subset \cap_i X_i$ and $\cup_i X_i \subset V$.

V.2 : Cardinal arithmetic

NOTATIONAL CONVENTION. If X and Y are sets then $\mathbf{F}(X,Y)$ will denote the set of all functions from X to Y.

Exercises to work

3. Change the problem to read as follows:

(i) Let $\alpha \neq 0$ be a cardinal number. Prove that $\alpha \cdot 0 = 1$, $\alpha^1 = \alpha$, and $1^\alpha = 1$.

(ii) Suppose that α, β and γ are cardinal numbers such that $\alpha \leq \beta$. Prove that $\alpha^\gamma \leq \beta^\gamma$.

(iii) Suppose that $\alpha = \aleph_0$ or $2^{\aleph_0}$. Prove that $\alpha^\alpha = 2^\alpha$. Suppose that $\alpha = \aleph_0$ and $\beta = 2^\alpha$. Prove that $\beta^\alpha = 2^\alpha$. [*Hint:* Use the laws of exponents for transfinite cardinal numbers and recall that $\alpha \cdot \alpha = \alpha$ and $\beta \cdot \beta = \beta$.

4. Change the problem and hint(s) to read as follows:

(i) Let X be a set, and let $\Sigma(X)$ denote the set of bijections from X to itself. Suppose that $\varphi : X \to Y$ is a bijection of sets. Prove that there is a bijection $\varphi_* : \Sigma(X) \to \Sigma(Y)$ such that $\varphi_*(h) = \varphi \circ h \circ \varphi^{-1}$ for all $h \in \Sigma(X)$.

(ii) Suppose that $|X| = \alpha$, where $\alpha = \aleph_0$ or $2^{\aleph_0}$. Prove that $|\Sigma(X)| = \alpha^\alpha = 2^\alpha$. [*Hint:* Assume that X is either $\mathbf{N}$ or $\mathbf{R}$. Since $\Sigma(X) \subset \mathbf{F}(X, X)$ by definition, we have $|\Sigma(X)| \leq \alpha^\alpha$. Define a map from $\mathbf{F}(X, X)$ to $\Sigma(X \times X)$ sending $f : X \to X$ to the permutation which interchanges $(x, 0)$ and $(x, f(x))$ for all x and sends the remaining elements to themselves. Explain why different functions determine different elements of $\Sigma(X \times X)$. Use this, the identity $\alpha \cdot \alpha = \alpha$, and the first part of this exercise to prove that $|\Sigma(X)| \geq \alpha^\alpha$, and then apply the Schröder-Bernstein Theorem to show $|\Sigma(X)| = \alpha^\alpha$. Use the previous exercise to complete the proof.]

5. In this exercise, the Well-Ordering Principle, the Axiom of Choice, and Zorn's lemma may be assumed. Furthermore, the following hint should be added:

[*Hint:* First show that for each positive integer n the set $A[n]$ of all subsets with n elements has the same cardinality as $\mathbf{R}$; for example, one can construct a 1–1 mapping from $A[n]$ to $\mathbf{R}^n$ by putting the elements in the standard order $a_1 < \cdots < a_n$ and sending the subset to the ordered n-tuple $(a_1, \cdots, a_n)$. Conversely, one can define an injection from $\mathbf{R}$ to $A[n]$ sending r to $\{r, r+1, \cdots, r+n\}$ and then apply the Schröder-Bernstein Theorem. Using this, show it suffices to prove that the set $A[\omega]$ of all countably infinite subsets of $\mathbf{R}$ has the same cardinality as the latter. Why is there an injection from $\mathbf{R}$ to $A[\omega]$? Given a countably infinite subset B, pick a bijection f_B from $\mathbf{N}$ to B, and let $j_B : \mathbf{N} \to \mathbf{R}$ be the composite of f_B with the inclusion $B \subset A$. Why does the map sending B to j_B define an injection from $A[\omega]$ to the set of functions $\mathbf{R}^{\mathbf{N}}$? Use this, the Schröder-Bernstein Theorem, and Exercise 3(iii) to compute $|A[\omega]|$.]

V.3 : Well-ordered sets

Exercises to work

1. In this exercise, the Well-Ordering Principle, the Axiom of Choice, and Zorn's lemma may be assumed. Furthermore, the following hint should be added:

[*Hint:* Let $\sigma(A) = A \cup \{A\}$ with the original partial ordering on A and such that $A \in \sigma(A)$ is strictly larger than every element of A; note that $\sigma(A)$ consists of A plus one additional element because $A \notin A$. Take a well-ordered set X in 1–1 correspondence with $\mathbf{P}(A)$, and use transfinite recursion to define a nondecreasing map from X to $\sigma(A)$ such that $f(x) \in A$ whenever there is some $b \in A$ such that $b > f(y)$ for all $y < x$ and $f(x) = A$ otherwise. Part of the proof is showing that it is possible to construct such a map. If W is the set of all $y < A$ such that $y = f(x)$ for some x, why is W cofinal in A?]

V.4 : The Axiom of Choice

Exercises to work

7. The first part of this problem has been inserted into the revised Exercise V.2.3(ii). Here is a formal statement of the second part of the original problem:

Suppose that α and β are finite but greater than 1 and γ is infinite. Prove that $\alpha^\gamma = \beta^\gamma$.

10. The following lengthy hint should be added:

[*Hint:* There are several steps in the argument. First, use Exercise V.2.4 to prove the result when $|A| \leq 2^{\aleph_0}$. From this point on assume that $|A| \geq 2^{\aleph_0}$. Next, prove that the set of countably infinite subsets is in 1–1 correspondence with the set $A^{\mathbf{N}}$ of functions from $\mathbf{N}$ to A as follows: Given a countably infinite subset E and a specific 1–1 correspondence from $\mathbf{N}$ onto E one obtains a map from $\mathbf{N}$ to A. Why is this 1–1? By the Schröder-Bernstein Theorem it is enough to define a map from $A^{\mathbf{N}}$ to countably infinite subsets of A. There is a 1–1 map from such functions to countable subsets of $\mathbf{N} \times A$ given by taking the graphs of functions. How can one use this to define the desired map to countable subsets of A? A comparison of $|A|$ and $|\mathbf{N} \times A|$ will probably be helpful here. Finally, use a modified version of the Zorn's Lemma argument proving $\alpha \cdot \alpha = \alpha$ for infinite cardinals α to prove that $\alpha^\omega = \alpha$. Specifically, consider the collection of all pairs (B, φ) consisting of $B \subset A$ satisfying $|B| \geq 2^{\aleph_0}$ and a bijection $\varphi : B^{\mathbf{N}} \to B$, with a partial ordering such that $(B, \varphi) \leq (D, \psi)$ if and only if $B \subset D$ and $\psi = \varphi$ on B. Use the previous exercise to show this set is nonempty. Verify that Zorn's Lemma applies and hence there is a maximal pair, say (B, φ). If $|B| = |A|$ we are done, so suppose instead that $|B| < |A|$. In this case show there is some $C \subset A$ such that $C \subset A - B$ and $|C| = |B|$. Explain why $(B \cup C)^{\mathbf{N}}$ can be written as a union of pairwise disjoint subsets S_Y, where Y runs over all subsets of $\mathbf{N}$ such that $\mathbf{a} \in S_Y$ if and only if $a_k \in B$ for $k \in Y$ and $a_k \in C$ otherwise. Why is each such set in 1–1 correspondence with $B^{\mathbf{N}}$ and $C^{\mathbf{N}}$? If $\mathbf{P}_1(Y)$ denotes the proper subsets of $\mathbf{N}$, explain why there is a bijection from $\mathbf{P}_1(Y) \times C$ to C. Show there is a bijection from $(B \cup C)^{\mathbf{N}}$ to $B \cup C$ sending $S_{\mathbf{N}} = B^{\mathbf{N}}$ to B by the maximal map and sending the other sets S_Y to C by the composites of $S_Y \to \{Y\} \times C$ and $\mathbf{P}_1(\mathbf{N}) \times C \to C$. Why is this a contradiction, and what is the source of the contradiction?]

12. We need to assume that the family $\mathbf{F}$ of finite character is nonempty (otherwise the conclusion is vacuously false!).

V.5 : Simplified axioms for the basic number systems

Note on Exercises V.5.4–V.5.6. The final three exercises for this section deal with questions considered only in the Appendix to this section. None of this material will be covered on course quizzes and examinations, and therefore written solutions to them will not be posted.