Smoothable submanifolds of a smooth manifold

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The purpose of this note is to describe a result from geometric topology which is well-known to workers in the area but difficult to locate in the literature. Our discussion will be somewhat informal, the goal being mainly to explain how the result can be extracted from the literature.

Question. *Suppose we have a smooth manifold N^n and a topological submanifold $M^m \subset N^n$, where $m < n$. Can one describe necessary and sufficient conditions under which M^m can be approximated (within its homeomorphism type) by a smooth submanifold of N^n?*

The main result on this question is essentially answered by the work of R. Kirby and L. Siebenmann on triangulations and smoothings of topological manifolds [ks76]; a corresponding result for *piecewise smooth* submanifolds, with no dimensional restrictions, is contained in earlier work of R. Lashof and M. Rothenberg [LR], and the proof of the result for topological submanifolds is similar to the argument in [LR]. A discussion of cases not covered by the main result appears in Section 5.

To simplify the discussion we shall assume that the boundaries of M and N are empty; some remarks about the bounded case appear at the end of this article.

1. The main result

If N^n is a smooth manifold and M^m is a smooth submanifold (both without boundary), then by the (smooth) **Tubular Neighborhood Theorem** (see [Bredon], [Hirsch] or [Lang]), then M^m has a *tubular neighborhood* given by a vector bundle (*i.e.*, a **vector bundle neighborhood**). Specifically, there is an open neighborhood U of M^m in N^n and an $(n-m)$-dimensional vector bundle ξ over M with total space $E(\xi)$ such that the pair (U, M) is homeomorphic — in fact, *diffeomorphic* — to the pair $\big(E(\xi)$, zero section$\big)$; the total space $E(\xi)$ has a canonical smooth structure which is determined up to a suitable notion of equivalence by the vector bundle ξ and the smooth structure on M.

The existence of a topological vector bundle neighborhood implies that the embedding of M in N is **locally flat** (see [rushing], p. 33). Since it is possible to construct uncountable families of inequivalent manifold embeddings for almost all choices of m and n (compare [rushing], Chapter 2), it follows immediately that many topological submanifolds cannot be smoothable.

However, the condition of local flatness by itself is not enough to imply that a submanifold is smoothable. For example, if M^m is a compact manifold (without boundary) which does not admit a smooth structure (*e.g.*, the 10-dimensional manifold constructed in [Kervaire]), then one can construct a locally flat embedding of M^n in $\mathbb{R}^n$ for n sufficiently large (see [microbundles] for a strong global version of this result), but the results of [microbundles] show that M^m cannot have a vector bundle neighborhood.

The main result on the question about smoothable submanifolds is essentially a converse to the smooth Tubular Neighborhood Theorem:

Theorem 1. *Let $n, m \geq 5$, let N^n be a smooth n-manifold, and let $M^m \subset N^n$ be a topological m-manifold that is embedded in N^n. Then there is a smooth structure on M^m such that the inclusion of M in N is isotopic to a smooth embedding if and only if M has a topological vector bundle neighborhood.*

We have already noted that the proof of this result is formally parallel to the earlier result of Lashof and Rothenberg on smoothing piecewise smooth submanifolds [LR]; the main difference is that the latter depends crucially on the Cairns-Hirsch smoothability theorem for piecewise linear manifolds [Hirsch-Mazur], and in the proof of Theorem 1 we shall substitute the Product Structure Theorem of Kirby and Siebenmann ([ks76], FILL IN) for the Cairns-Hirsch Theorem.

Although local flatness does not in general imply the existence of a vector bundle neighborhood, such neighborhoods always exist for locally flat submanifolds if $n - m \leq 2$ (*e.g.*, see [ks73]), and thus we have the following conclusion:

Theorem 2. *Let $k = 1$ or 2, let $m \geq 5$, let N^{m+k} be a smooth $(n+k)$-manifold, and let M^m be a locally flat m-dimensional submanifold of N^{m+k}. Then there is a smooth structure on M^m such that the inclusion of M in N is isotopic to a smooth embedding.*

In Section 5 we shall give examples to show that Theorem 2 does not extend to cases with $M \leq 4$.

2. Smoothing vector bundles

Although there are many treatments of vector bundles in textbooks andother publications, such accounts usually emphasize one of two basic categories — the smooth and topological categories — with very little (if anything) said about the relationship between smooth and topological vector bundles. Since the proof of Theorem 1 requires an explicitunderstanding of this relationship, we shall describe the necessary facts here. Unfortunately, even though the proofs are not particularly difficult, finding them in the literature can be extremely challenging. Therefore we shall limit ourselves to summarizing the crucial points.

In order to simpify the discussion we shall limit our attention to real vector bundles, but one can also treat complex vector bundles similarly by substituting $\mathbb{C}$ for $\mathbb{R}$ (and "unitary" for "orthogonal") throughout; everything also goes through for quaternionic vector bundles, but at some points one must phrase things more carefully in order to compensate for the noncommutativity of the quaternions.

COMPARING CATEGORIES OF VECTOR BUNDLES. The classical examples of vector bundles in differential geometry are tangent bundles and various sorts of tensor bundles over a smooth manifold, and these are smooth vector bundles, at least if the manifold is smooth of class $\mathcal{C}^2$ (*e.g.*, see [Lang]). Onthe other hand, for manypurposes in topology it is more convenient to consider *continuous* and *topological* vector bundles as in M. F. Atiyah's classic set of lecture notes [atiyah]. It is straightforward to check that evey smooth vector bundle has an underlying topological vector bundle, just as smooth manifolds have underlying topological manifolds. For our purposes the following converse relationship is fundamentally important:

Theorem 3. *Let M be a smooth manifold, and let q be a positive integer.*
(i) If ξ is a continuous q-dimensional vector bundle over M, then there is a smooth q-dimensional vector bundle ξ' and a vector bundle isomorphism $\varphi : \xi' \to \xi$; in other words, if $E(\xi')$ and $E(\xi)$ are the total spaces and pand p' are the projections, then there is a homeomorphism $E(\varphi)$:

$E(\xi') \to E(\xi)$ such that $p\varphi = p'$, and for each $x \in M$ the map φ defines a vector bundle isomorphism from the vector space $\xi'_x = p'^{-1}[\{x\}]$ to $\xi_x = p^{-1}[\{x\}]$.
(ii) If ξ and ξ' are smooth q-dimensional vector bundles over M and $\varphi : \xi' \to \xi$ is a continuous vector bundle isomorphism, then φ is isotopic to a smooth vector bundle isomorphism; in other words, there is a homotopy $\Phi : E(\xi') \times [0,1] \to E(\xi)$ such that $\Phi|E \times \{0\}$ is given by φ, for each $\{t\}$ the map $\Phi|E \times \{t\}$ is a vector bundle isomorphism, and $\Phi|E \times \{1\}$ is a diffeomorphism.

VECTOR BUNDLES AND PRINCIPAL BUNDLES. In both the smooth and topological categories, there is a 1–1 correspondence between isomorphism classes of q-dimensional (real) vector bundles over a given base B and principal $\mathbf{GL}(q,\mathbb{R})$-bundles over B. Given a vector bundle $p : E \to B$, the corresponding $\mathbf{GL}(q,\mathbb{R})$-bundle is called the **bundle of q-frames**, and it consists of all ordered q-tuples $(\mathbf{v}_1, \cdots, \mathbf{v}_q)$ where $\mathbf{v}_j \in E$ for all j such that the following hold:

(i) $p(\mathbf{v}_1) = \cdots p(\mathbf{v}_q)$ (call this common point b).
(ii) In the vector space $p^{-1}[\{b\}]$, the vectors $\mathbf{v}_1, \cdots, \mathbf{v}_q$ are linearly independent and hence form a basis for the given vector space.

Conversely, if we start with a principal $\mathbf{GL}(q,\mathbb{R})$-bundle, then the corresponding vector bundle is merely the associated fiber bundle with fiber $\mathbb{R}^q$, where $\mathbf{GL}(q,\mathbb{R})$ acts on $\mathbb{R}^q$ in the usual way by invertible linear transformations. Further details on this correspondence can be found in Section 4.4 of [jiewu] (in particular, see Proposition 4.11 on page 31).

By the preceding discussion, the proof of Theorem 3 reduces to proving a similar result for smooth and topological principal $\mathbf{GL}(n,\mathbb{R})$-bundles over a smooth manifold. We shall analyze this relationship between the two types of bundles using standard results from bundle theory. The classical formulation of bundle theory in the 1949–1950 Séminaire Henri Cartan [hcartan] is particularly useful for our purposes.

The next step in the process is to observe that isomorphism classes of principal $\mathbf{GL}(n,\mathbb{R})$-bundles correspond bijectively to isomorphism classes of principal O_q-bundles, where as usual $O_q \subset \mathbf{GL}(q,\mathbb{R})$ is the orthogonal group. Furthermore, if we are given two principal O_q-bundles and an isomorphism of their extensions to principal $\mathbf{GL}(q,\mathbb{R})$-bundles, then this isomorphism can be deformed to second isomorphism which is an extension of a principal O_q-bundle isomorphism.

By the discussion thus far, Theorem 3 will be true if we can prove the following more general result:

Theorem 4. *Let M be a smooth manifold, and let G be a compact Lie group.*
(i) If ξ is a topological principal G-bundle over M, then there is a smooth principal G-bundle bundle ξ' and a principal bundle isomorphism $\varphi : \xi' \to \xi$; in other words, if $E(\xi')$ and $E(\xi)$ are the total spaces and pand p' are the projections, then there is a homeomorphism $E(\varphi) : E(\xi') \to E(\xi)$ such that $p\varphi = p'$, and for each $x \in M$ the map φ defines a G-equivariant bijection from the fiber $\xi'_x = p'^{-1}[\{x\}]$ to $\xi_x = p^{-1}[\{x\}]$.
(ii) If ξ and ξ' are smooth principal G-bundles bundles over M and $\varphi : \xi' \to \xi$ is a continuous principal bundle isomorphism, then φ is isotopic to a smooth principal bundle isomorphism; in other words, there is a homotopy $\Phi : E(\xi') \times [0,1] \to E(\xi)$ such that $\Phi|E \times \{0\}$ is given by φ, for each $\{t\}$ the map $\Phi|E \times \{t\}$ is a principal bundle isomorphism, and $\Phi|E \times \{1\}$ is a diffeomorphism.

Sketch of proof. Given a compact Lie group G and a positive integer n, one has the usual sort of n-universal principal G-bundle $p_n : E_n \to B_n$ such that if M^m is a manifold whose dimension m is sufficiently small with respect to m, then principal G-bundles over M are classified up to isomorphism by homotopy classes of maps from M into B_n. A closer examination of the construction in [hcartan] yields two stronger conclusions. First, an isomorphism of principal G-bundles over M determines a homotopy of classifying maps. Second, an isotopy of principal G-bundle isomorphisms determines a homotopy of homotopies of classifying maps.

The assertions in the previous paragraph are true for both the topological and smooth categories. In order to check this for the smooth category, it is necessary to find an n-universal principal G-bundle which is smooth. However, thes can be done fairly directly as in Steenrod's book: For each compact Lie group G there is a smooth injective homomorphism $G \to O_p$ for some $p > 0$, and one can then take the n-universal bundle to be the smooth principal bundle

$$G \longrightarrow O_{n+p+1}/O_{n+1} \longrightarrow O_{n+p+1}/O_{n+1} \times G$$

where the first arrow represents the composite

$$G \to O_p \to O_{n+1} \times O_p \to O_{n+1+p} \to O_{n+p+1}/O_{n+1} \ .$$

To prove the theorem, take a topological principal bundle ξ over M, and choose a continuous mapping f from M to the base space of a smooth n-universal bundle B_n, where $n > m$ such that ξ is isomorphic to the pullback of the universal bundle γ via f. Standard results on smooth approximations to continuous functions (*e.g.*, see [munkresamstudy]) show that f is homotopic to a smooth map f_1. By construction the pullback bundle $f_1^*\gamma$ is a smooth bundle, and as noted before the homotopy from f to f_1 defines a topological principal bundle isomorphism from $\xi \cong f^*\gamma$ to $f_1^*\gamma$. This proves the first part. To prove the second part, note that in this case the topological isomorphism of smooth principal bundles from $\xi_1 \cong f_1^*\gamma$ and and $\xi_2 \cong f_2^*\gamma$ (where f_i is smooth) will determine a continuous homotopy $H : M \times [0,1] \to B_n$ from the smooth map f_1 to the smooth map f_2. Constructing the desired isotopy amounts to constructing a relative homotopy $K : M \times [0,1] \times [0,1]$ from H to a smooth homotopy H' such that the homotopy is fixed on $M \times \{0,1\} \times [0,1]$; such a relative homotopy will define a relative isotopy from the original continuous isomorphism of principal bundles to a smooth isomorphism. In this case, the existence of a suitable relative homotopy follows from a relative version of the results on approximating continuous mappings by smooth ones.

This completes the proofs of Theorems 3 and 4. In particular, these results have the following important consequence that we shall need n the next section:

Theorem 5. *Let M be a smooth manifold, let q be a positive integer, and let ξ be a continuous q-dimensional vector bundle over M. Then there is a canonical smooth structure on the total space $E(\xi)$ such that the embedding of the zero section is a smooth embedding.*

3. Proof of the main results

We have noted that Theorem 2 follows from Theorem 1 and the result of Kirby-Siebenmann, so we shall concentrate on the proof of Theorem 1. Furthermore, since we know the "only if" implication is true, it will suffice to restrict our attention to the "if" implication.

Suppose now that we have $n > m \geq 5$, and we are given a smooth manifold N^n with an embedded topological submanifold M^m such that M has a topological vector bundle neighborhood. Let ξ be the vector bundle such that M has an open neighborhood U in N which is homeomorphic to $E(\xi)$ with M corresponding to the zero section. The smooth structure on N determines a smooth structure on U and hence on $E(\xi)$. Let $p : E(\xi) \to M$ be the projection map.

The next step in the proof is to show that M is smoothable, and the argument closely resembles the corresponding part of [LR]. We know that for some positive integer p there is a p-dimensional inverse vector bundle ω^p over M such that the Whitney sum $\xi \oplus \omega$ is a trivial vector bundle and hence $E(\xi \oplus \omega) \cong M \times \mathbb{R}^{q+p}$. For every vector bundle β over M there is a standard vector bundle identity

$$E(p^*\beta) \;\cong\; E(\xi \oplus \beta)$$

and if we apply this to ω we see that $E(p{*}^{\omega})$ is homeomorphic to $M \times \mathbb{R}^{q+p}$.

Since we have a smooth structure on $E(\xi)$, the results of the preceding section imply that

$$E(p^*\beta) \;\cong\; E(\xi \oplus \beta) \;\cong\; M \times \mathbb{R}^{q+p}$$

also has a smooth structure. Since $m \geq 5$, we can apply the Product Structure Theorem in [ks76] (see Theorem 5.1, ?????) to conclude that M is smoothable.