SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 2

Fall 2008

II. Metric and topological spaces

II.2 : Closed sets and limit points

Problems from Munkres, § 17, *pp.* 101 – 102

2. Show that if A is closed in Y and Y is closed in X then A is closed in X.

SOLUTION.

Since A is closed in Y we can write $A = F \cap Y$ where F is closed in X. Since an intersection fo closed set is closed and Y is closed in X , it follows that $F \cap Y = A$ is also closed in x.■

8. Let A, $B \subset X$, and determine whether the following inclusions hold; if equality fails, determine whether containment one way or the other holds.

(a) $\overline{A \cap B} = \overline{A} \cap \overline{B}$

SOLUTION.

Since $C \subset Y$ implies $\overline{C} \subset \overline{Y}$ it follows that $\overline{A \cap B} \subset \overline{A}$ and $\overline{A \cap B} \subset \overline{B}$, which yields the inclusion

$$\overline{A \cap B} = \overline{A} \cap \overline{B}\ .$$

To see that the inclusion may be proper, take A and B to be the open intervals $(0, 1)$ and $(1, 2)$ in the real line. Then the left hand side is empty but the right hand side is the set $\{1\}$.■

(c) $\overline{A - B} = \overline{A} - \overline{B}$

SOLUTION.

This time we shall first give an example where the first set properly contains the second. Take $A = [-1, 1]$ and $B = \{0\}$. Then the left hand side is A but the right hand side is $A - B$. We shall now show that $\overline{A - B} \supset \overline{A} - \overline{B}$ always holds. Given $x \in \overline{A} - \overline{B}$ we need to show that for eacn open set U containing x the intersection $U \cap (A - B)$ is nonempty. Given such an open set U, the condition $x \notin \overline{B}$ implies that $x \in U - \overline{B}$, which is open. Since $x \in \overline{A}$ it follows that

$$A \cap \left(U - \overline{B}\right) \neq \emptyset$$

and since $U - \overline{B} \subset U - B$ it follows that

$$(A - B) \cap U = A \cap (U - B) \neq \emptyset$$

and hence that $x \in \overline{A - B}$.■

19. If $A \subset X$ then the *boundary* $\mathrm{Bd}(A)$ is defined by the expression

$$\mathrm{Bd}(A) = \overline{A} \cap \overline{X - A} \ .$$

(a) Show that $\mathrm{Int}(A)$ and $\mathrm{Bd}(A)$ are disjoint and their union is the closure of A.

SOLUTION.

The interior of A is the complement of $\overline{X - A}$ while the boundary is contained in the latter, so the intersection is empty.■

(b) Show that $\mathrm{Bd}(A)$ is empty if and only if A is both open and closed.

SOLUTION.

($\Longleftarrow$) If A is open then $X - A$ is closed so that $X - A = \overline{X - A}$, and if A closed then $\overline{A} = A$. Therefore

$$\mathrm{Bd}(A) = \overline{A} \cap \overline{X - A} = A \cap (X - A) = \emptyset \ .■$$

($\Longrightarrow$) The set $\mathrm{Bd}(A)$ is empty if and only if $\overline{A}$ and $\overline{X - A}$ are disjoint. Since the latter contains $X - A$ it follows that $\overline{A}$ and $X - A$ are disjoint. Since their union is X this means that $\overline{A}$ must be contained in A, which implies that A is closed. If one reverses the roles of A and $X - A$ in the preceding two sentences, it follows that $X - A$ is also closed; hence A is both open and closed in X.■

(c) Show that U is open if and only if $\mathrm{Bd}(U) = \overline{U} - U$.

SOLUTION.

By definition the boundary is $\overline{U} \cap \overline{X - U}$.

($\Longrightarrow$) If U is open then $X - U$ is closed and thus equal to its own closure, and therefore the definition of the boundary for U reduces to $\overline{U} \cap (X - U)$, which is equal to $\overline{U} - U$.■

($\Longleftarrow$) We shall show that $X - U$ is closed. By definition $\mathrm{Bd}(U) = \mathrm{Bd}(X - U)$, and therefore $\mathrm{Bd}(X - U) = \overline{U} - U \subset X - U$. On the other hand, by part (a) we also know that $(X - U) \cup \mathrm{Bd}(X - U) = \overline{X - U}$. Since both summands of the left hand side are contained in $X - U$, it also follows that the right hand side is contained in $X - U$, which means that $X - U$ is closed in X.■

(d) If U is open is it true that $U = \mathrm{Int}(\overline{U})$? Justify your answer.

SOLUTION.

No. Take $U = (-1, 0) \cup (0, 1)$ as a subset of $\mathbb{R}$. Then the interior of the closure of U is $(-1, 1)$. However, we always have $U \subset \mathrm{Int}(\overline{U})$ because $U \subset \overline{U} \implies U = \mathrm{Int}(U) \subset \mathrm{Int}(\overline{U})$.■

FOOTNOTE.

Sets which have the property described in **19.(d)** are called *regular open sets.*

20. Find the boundary and interior for each of the following subsets of $\mathbb{R}^2$:
(a) $A = \{ \ (x, y) \in \mathbb{R}^2 \ \big| \ y = 0 \ \}$

SOLUTION.

We need to find the closures of A and its complement in $\mathbb{R}^2$. The complement of A is the set of all points whose second coordinate is nonzero. We claim it is open. But if $(x, y) \in \mathbb{R}^2 - A$ then $y \neq 0$ and the set $N_{|y|}\left((x, y) \right) \subset \mathbb{R}^2 - A$. Therefore A is closed. But the closure of the complement of A is all of $\mathbb{R}^2$; one easy way of seeing this is that for all $x \in \mathbb{R}$ we have $\lim_{n\to\infty}(x, 1/n) = (x, 0)$, which means that every point of A is a limit point of $\mathbb{R}^2 - A$. Therefore the boundary of A is equal to $A \cap \mathbb{R}^2 = A$; *i.e.*, every point of A is a boundary point.■

(b) $B = \{ \ (x, y) \in \mathbb{R}^2 \ \big| \ x > 0 \ \text{ and } \ y \neq 0 \ \}$

SOLUTION.

Again we need to find the closures of B and its complement. It will probably be very helpful to draw pictures of this set and the sets in the subsequent portions of this problem. The closure of B turns out to be the set of all points where $x \geq 0$ (find sequences converging to all points in this set that are not in B **!**) and the complement is just the complement of B because the latter is open in $\mathbb{R}^2$. Therefore the boundary of B is $\overline{B} - B$, and this consists of all points such that either $x = 0$ or both $x > 0$ and $y = 0$ hold.■

(c) $C = A \cup B$

SOLUTION.

The first two sentences in part **(b)** apply here also. The set C consists of all points such that $x > 0$ or $y = 0$ The closure of this set is the set of all points such that $x \geq 0$ or $y = 0$, the complement of C is the set of all points such that $x \geq 0$ and $y \neq 0$, and the closure of this complement is the set of all points such that $x \leq 0$. The intersection of the two sets will be the set of all point such that $x = 0$ or both $x < 0$ and $y = 0$ hold.■

(d) $D = \{ \ (x, y) \in \mathbb{R}^2 \ \big| \ x \notin \mathbb{Q} \ \}$

SOLUTION.

The first two sentences in part **(b)** apply here also. Since every real number is a limit of irrational numbers, it follows that the closure of D is all of $\mathbb{R}^2$. The complement of D consists of all points whose first coordinates are rational, and since every real number is a limit of a sequence of rational numbers it follows that the closure of $\mathbb{R}^2 - D$ is also $\mathbb{R}^2$. Therefore the boundary is $\mathbb{R}^2$.■

(e) $E = \{ \ (x, y) \in \mathbb{R}^2 \ \big| \ 0 < x^2 - y^2 < 1 \ \}$

SOLUTION.

The first two sentences in part **(b)** apply here also. In problems of this sort one expects the boundary to have some relationship to the curves defined by changing the inequalities into equations; for this example the equations are $x^2 - y^2 = 0$ and $x^2 - y^2 = 1$. The first of these is a pair of diagonal lines through the origin that make 45 degree angles with the coordinate axes, and the second is a hyperbola going through $(\pm 1, 0)$ with asymptotes given by the lines $x^2 - y^2 = 0$. The closure of E turns out to be the set of points where $0 \leq x^2 - y^2 \leq 1$, and the closure of its complement is the set of all points where either $x^2 - y^2 \leq 0$ or $x^2 - y^2 \geq 1$. The intersection will then be the set of points where $x^2 - y^2$ is either equal to 0 or 1.

For the sake of completeness, here is a proof of the assertions about closures: Suppose that we have a sequence $\{ (x_n, y_n) \}$ in E and the sequence has a limit $(a, b) \in \mathbb{R}^2$. Then

$$a^2 - b^2 \quad = \quad \lim_{n\to\infty} x_n^2 - y_n^2$$

where for each n the n^{th} term on the right hand side lies in the open interval $(0, 1)$, and therefore the limit value on the right hand side must lie in the closed interval $[0, 1]$. Similarly, suppose that we have sequences in the complement of E that converge to some point (a, b). Since the sequence of numbers $z_n = x_n^2 - y_n^2$ satisfies $z_n \in \mathbb{R} - (0, 1)$ and the latter subset is closed, it also follows that

$a^2 - b^2 \in \mathbb{R} - (0,1)$. This proves that the closures of E and its complement are contained in the sets described in the preceding paragraph.

To complete the proof of the closure assertions we need to verify that every point on the hyperbola or the pair of intersecting lines is a limit point of E. Suppose that we are given a point (a,b) on the hyperbola, and consider the sequence of points

$$(x_n,\, y_n) \quad = \quad \left(a - \frac{\sigma(a)}{n},\, b \right)$$

where $\sigma(x)$ is ± 1 depending on whether a is positive or negative (we know that $|a| > 1$ because $a^2 = 1 + b^2$). If we take $z_n = x_n^2 - y_n^2$ as before then $\lim_{n\to\infty} z_n = 1$ and moreover

$$z_n \quad = \quad \left(a - \frac{\sigma(a)}{n} \right)^2 - b^2 \quad = \quad 1 + \frac{1}{n^2} - \frac{2\,a\,\sigma(a)}{n}$$

where the expression on the right hand side is positive if

$$|a| \quad < \quad \frac{1}{2n}\ .$$

To see that this expression is less than 1 it suffices to note that $|a| = \sigma(a) \cdot a$ and

$$\frac{1}{n^2} - \frac{2\,|a|}{n} \quad \leq \quad \frac{1}{n^2} - \frac{2}{n} \quad < \quad 0$$

for all $n \geq 1$. This verifies that every point on the hyperbola is a limit point of E. — Graphically, we are taking limits along horizontal lines; the reader might want to draw a picture in order to visualize the situation.

Suppose now that (a,b) lies on the pair of intersecting lines, so that $a = \pm b$. How do we construct a sequence in E converging to (a,b)? Once again, we take sequences that live on a fixed horizontal line, but this time we choose

$$(x_n,\, y_n) \quad = \quad \left(a + \frac{\sigma(a)}{n},\, b \right)$$

and note that $z_n = x_n^2 - y_n^2$ is equal to

$$\frac{1}{n^2} \quad + \quad \frac{2\,|a|}{n}$$

which is always positive and is also less than 1 if $n > 2|a| + 1$ (the latter follows because the expression in question is less than $(1 + 2|a|)/n$ by the inequality $n^{-2} < n^{-1}$ for all $n \geq 2$). Thus every point on the pair of intersecting lines also lies in $\mathbf{L}(E)$, and putting everything together we have verified that the closure of E is the set we thought it was.■

(f) $F = \{\ (x,y) \in \mathbb{R}^2 \ \big|\ x \neq 0 \ \text{ and } \ y \leq 1/x\ \}$

SOLUTION.

Geometrically, this is the region underneath either the positive or negative branch of the hyperbola $y = 1/x$ with the x-axis removed, and the branches of the hyperbola are included in the set. We claim that the closure of F is the union of F with the y-axis (*i.e.*, the set of points where

$x = 0$). To see that the y-axis is contained in this set, consider a typical point $(0, y)$ and consider the sequence

$$\left(\frac{1}{n}, y\right)$$

whose limit is $(0, y)$; the terms of this sequence lie in the set F for all $y \geq n$, and therefore the y-axis lies in $\mathbf{L}(F)$. Therefore the closure is at least as large as the set we have described. To prove that it is no larger, we need to show that there are no limit points of F such that $x \neq 0$ and $y > 1/x$. But suppose that we have an infinite sequence in F with terms of the form $(x_n.y_n)$ and limit equal to (a, b), where $a \neq 0$. There are two cases depending upon whether a is positive or negative. Whichever case applies, for all sufficiently large values of n the signs of the terms x_n are equal to the sign of a, so we may as well assume that all terms of the sequence have first coordinates with the same sign as a (drop the first finitely many terms if necessary). If $a > 0$ it follows that $x_n > 0$ and $y_n \leq 1/x_n$, which imply $x_n \, y_n \leq 1$ Taking limits we see that $ab \leq 1$ also holds, so that $b \leq 1/a$. On the other hand, if $a < 0$ then it follows that $x_n < 0$ and $y_n \leq 1/x_n$, which imply $x_n \, y_n \geq 1$ Taking limits we see that $ab \geq 1$ also holds, so that $b \leq 1/a$ holds in this case too.

Now we have to determine the closure of the complement of F; we claim it is the set of all points where either $x = 0$ or $x \neq 0$ and $y \geq 1/x$. By definition it contains all points where $x = 0$ or $y > 1/x$, so we need to show that the hyperbola belongs to the set of limit points and if we have a sequence of points of the form (x_n, y_n) in the complement of F with a limit in the plane, say (a, b), then $b \geq 1/a$. Proving the latter proceeds by the same sort of argument given in the preceding paragraph (it is not reproduced here, but it must be furnished in a complete proof). To see that the hyperbola belongs to the set of limit points take a typical point (a, b) such that $a \neq 0$ and $b = 1/a$ and consider the sequence

$$(x_n, y_n) \quad = \quad \left(a, \; \frac{1}{a} + \frac{1}{n}\right)$$

whose terms all lie in the complement of F and whose limit is (a, b).■

Additional exercises

0. Prove or give a counterexample to the following statement: If U and V are disjoint open subsets of a topological space X, then their closures are also disjoint.

SOLUTION.

Let U and V be the open intervals $(-1, 0)$ and $(0, 1)$ respectively. Then their closures are the closed intervals $[-1, 0]$ and $[0, 1]$ respectively, and the intersection of these two sets is $\{0\}$. This counterexample shows that the statement is false.

1. Give an example to show that in a metric space the closure of an open ε disk about a point is not necessarily the set of all points whose distance from the center is $\leq \varepsilon$.

SOLUTION.

Take any set S with the discrete metric and let $\varepsilon = 1$. Then the set of all points whose distance from some particular $s_0 \in S$ is ≤ 1 is all of S, but the open disk of radius 1 centered at s_0 is just the one point subset $\{ s_0 \}$.■

Definition. A subspace D of a topological space X is *dense* if $\overline{D} = X$; equivalently, it is dense if and only if for every nonempty open subset $U \subset X$ we have $U \cap D \neq \emptyset$.

2. For which spaces is X the only dense subset of itself?

SOLUTION.

If X has the discrete topology then every subset is equal to its own closure (because every subset is closed), so the closure of a proper subset is always proper. Conversely, if X is the only dense subset of itself, then for every proper subset A its closure $\overline{A}$ is also a proper subset. Let $y \in X$ be arbitrary, and apply this to $X - \{y\}$. Then it follows that the latter is equal to its own closure and hence $\{y\}$ is open. Since y is arbitrary, this means that X has the discrete topology.■

3. Let U and V be open dense subsets of X. Prove that $U \cap V$ is dense in X.

SOLUTION.

Given a point $x \in X$ and and open subset W such that $x \in W$, we need to show that the intersection of W and $U \cap V$ is nonempty. Since U is dense we know that $W \cap U \neq \emptyset$; let y be a point in this intersection. Since V is also dense in X we know that

$$(U \cap V) \cap W \quad = \quad V \cap (U \cap W) \quad \neq \emptyset$$

and therefore $U \cap V$ is dense. — You should be able to construct examples in the real line to show that the conclusion is not necessarily true if U and V are not open.■

4. A subspace A of a topological space X is said to be *locally closed* if for each $a \in A$ there is an open neighborhood U of a in X such that $U \cap A$ is closed in U. Prove that A is locally closed if and only if A is the intersection of an open subset and a closet set.

SOLUTION.

$(\Longleftarrow)$ If $A = E \cap U$ where E is closed and U is open then for each $a \in A$ one can take U itself to be the required open neighborhood of a. $(\Longrightarrow)$ Given $a \in A$ let U_a be the open set containing a such that $U_a \cap A$ is closed in U_a. This implies that $U_a - A$ is open in U_a and hence also in X. Let $U = \cup_a U_a$. Then by construction $A \subset U$ and

$$U - A \quad = \quad \bigcup_{\alpha} \, (U_a - A)$$

is open in X. If we take

$$E \quad = \quad X - \overline{U - A}$$

then E is closed in X and $A = U \cap E$ where U is open in X and E is closed in X.■

5. **(a)** Suppose that D is dense in X, and let $A \subset X$. Give an example to show that $A \cap D$ is not necessarily dense in A.

SOLUTION.

Let X be the real numbers, let D be the rational numbers and let $A = X - D$. Then $A \cap D = \emptyset$, which is certainly not dense in X.■

(b) Suppose that $A \subset B \subset X$ and A is dense in B. Prove that A is dense in $\overline{B}$.

SOLUTION.

If $x \in \overline{B}$ and U is an open set containing x, then $U \cap B \neq \emptyset$. Let b be a point in this intersection. Since $b \in U$ and A is dense in B it follows that $A \cap U \neq \emptyset$ also. But this means that A is dense in $\overline{B}$.■

6. Let E be a subset of the topological space X. Prove that every closed subset $A \subset E$ is also closed in X if and only if E itself is closed in X.

SOLUTION.

$(\implies)$ If $A = E$ then E is closed in itself, and therefore the first condition implies that E is closed in X. $(\impliedby)$ If E is any subset of X and A is closed in E then $A = U \cap E$ where U is closed in X. But we also know that E is closed in X, and therefore $A = U \cap E$ is also closed in X. (Does all of this sound familiar? The exercise is essentially a copy of an earlier one with with "closed" replacing "open" everywhere.) ■

7. Given a topological space X and a subset $A \subset X$, explain why the closure of the interior of A does not necessarily contain A.

SOLUTION.

Consider the subset A of $\mathbb{R}$ consisting of $(0,1) \cup \{2\}$. The closure of its interior is $[0,1]$.■

8. If U is an open subset of X and B is an arbitrary subset of X, prove that $U \cap \overline{B} \subset \overline{U \cap B}$.

SOLUTION.

Suppose $x \in U \cap B \subset U \cap \overline{B}$. Then the inclusion $U \cap V \subset \overline{U \cap B}$ shows that $x \in \overline{U \cap B}$. Since $\overline{B} = B \cup \mathbf{L}(B)$ the resulting set-theoretic identity

$$U \cap \overline{B} \quad = \quad (U \cap B) \;\cap\; (U \cap \mathbf{L}(B))$$

implies that we need only verify the inclusion

$$(U \cap \mathbf{L}(B)) \quad \subset \quad \overline{U \cap B}$$

and it will suffice to verify the stronger inclusion statement

$$(U \cap \mathbf{L}(B)) \quad \subset \quad \mathbf{L}(U \cap B)\ .$$

Suppose that $x \in U \cap \mathbf{L}(B)$, and let W be an open subset containing x. Then $W \cap U$ is also an open subset containing X, and since $x \in \mathbf{L}(B)$ we know that

$$\Big(U \cap W \;-\; \{x\}\Big) \;\cap B \quad \neq \quad \emptyset\ .$$

But the expression on the left hand side of this display is equal to

$$\Big(W - \{x\}\Big) \;\cap\; U \;\cap\; B$$

and therefore the latter is nonempty, which shows that $x \in \mathbf{L}(U \cap B)$ as required.■

9. If X is a topological space, then the *Kuratowski closure axioms* are the following properties of the operation $A \to \mathbf{CL}(A)$ sending $A \subset X$ to its closure $\overline{A}$:

(C1) $A \subset \mathbf{CL}(A)$ for all $A \subset X$.

(C2) $\mathbf{CL}\,(\mathbf{CL}(A)\,) = \mathbf{CL}(A)$

(C3) $\mathbf{CL}(A \cup B) = \mathbf{CL}(A) \cup \mathbf{CL}(B)$ for all $A,\ B \subset X$.

(C4) $\mathbf{CL}(\emptyset) = \emptyset$.

Given an arbitrary set Y and a operation $\mathbf{CL}$ assigning to each subset $B \subset Y$ another subset $\mathbf{CL}(B) \subset Y$ such that $\mathbf{(C1)} - \mathbf{(C4)}$ all hold, prove that there is a unique topology $\mathbf{T}$ on Y such that for all $B \subset Y$, the set $\mathbf{CL}(B)$ is the closure of B with respect to $\mathbf{T}$.

SOLUTION.

In order to define a topological space it is enough to define the family $\mathcal{F}$ of closed subsets that satisfies the standard properties: It contains the empty set and Y, it is closed under taking arbitrary intersections, and it is closed under taking the unions of two subsets. If we are given tne abstract operator $\mathbf{CL}$ as above on the set of all subsets of Y let $\mathcal{F}$ be the family of all subsets A such that $\mathbf{CL}(A) = A$. We need to show that this family satisfies the so-called standard properties mentioned in the second sentence of this paragraph.

The empty set belongs to $\mathcal{F}$ by $\mathbf{(C4)}$, and Y does by $\mathbf{(C1)}$ and the assumption that $\mathbf{CL}(A) \subset Y$ for all $A \subset Y$, which includes the case $A = Y$. If A and B belong to $\mathcal{F}$ then the axioms imply

$$A \cup B \quad = \quad \mathbf{CL}(A) \cup \mathbf{CL}(B) \quad = \quad \mathbf{CL}(A \cup B)$$

(use $\mathbf{(C3)}$ to derive the second equality).

The only thing left to check is that $\mathcal{F}$ is closed under arbitrary intersections. Let $\mathcal{A}$ be a set and let $\{ A_\alpha \}$ be a family of subsets in $\mathcal{F}$ indexed by all $\alpha \in \mathcal{A}$; by assumption we have $\mathbf{CL}(A_\alpha) = A_\alpha$ for all α, and we need to show that

$$\mathbf{CL}\left(\bigcap_\alpha A_\alpha\right) \quad = \quad \bigcap_\alpha A_\alpha \ .$$

By $\mathbf{(C2)}$ we know that

$$\mathbf{CL}\left(\bigcap_\alpha A_\alpha\right) \quad \subset \quad \bigcap_\alpha \mathbf{CL}\,(A_\alpha)$$

and thus we have the chain of set-theoretic inclusions

$$\bigcap_\alpha A_\alpha \quad \subset \quad \mathbf{CL}\left(\bigcap_\alpha A_\alpha\right) \quad \subset \quad \bigcap_\alpha \mathbf{CL}\,(A_\alpha) \quad = \quad \bigcap_\alpha A_\alpha$$

which shows that all sets in the chain of inclusions are equal and hence that if $D = \cap_\alpha A_\alpha$, then $D = \mathbf{CL}(D)$.■

FOOTNOTE.

Exercise 21 on page 102 of Munkres is a classic problem in point set topology that is closely related to the closure operator on subsets of a topological space: Namely, if one starts out with a fixed subset and applies a finite sequence of closure and (set-theoretic) complement operations, then one obtains at most 14 distinct sets, and there are examples of subsets of the real line for which this upper bound is realized. Some hints for working this exercise appear in the following web site:

`http://www.math.ou.edu/∼nbrady/teaching/f02-5853/hint21.pdf`

10. Suppose that X is a space such that $\{p\}$ is closed for all $x \in X$ (this includes all metric spaces), and let $A \subset X$. Prove the following statements:

(a) $\mathbf{L}(A)$ is closed in X.

SOLUTION.

It suffices to show that $\mathbf{L}(\mathbf{L}(A)) \subset \mathbf{L}(A)$. Suppose that $x \in \mathbf{L}(\mathbf{L}(A))$. Then for every open set U containing x we have $(U - \{x\}) \cap A \neq \emptyset$, so let y belong to this nonempty intersection. Since one point subsets are closed, it follows that $U - \{x\}$ is an open set containing y, and therefore we must have

$$\left(U - \{x,\, y\}\right) \;\cap\; A \;\;\neq\;\; \emptyset$$

and therefore the sets $U - \{x\}$ and A have a nonempty intersection, so that $x \in \mathbf{L}(A)$.■

(b) For each point $b \in \mathbf{L}(A)$ and open set U containing b, the intersection $U \cap A$ is infinite.

SOLUTION.

If all one point subsets of X are closed, then all finite subsets of X are closed, and hence the complements of all finite subsets of X are open. We shall need this to complete the proof.

Suppose that the conclusion is false; *i.e.*, the set $(U - \{b\}) \cap A$ is finite, say with exactly n elements. If F denotes this finite intersection, then by the preceding paragraph $V = U - F$ is an open set, and since $x \notin F$ we also have $x \in V$. Furthermore, we have $(V - \{x\}) \cap A = \emptyset$; on the other hand, since $b \in \mathbf{L}(A)$ we also know that this intersection is nonempty, so we have a contradiction. The contradiction arose from the assumption that $(U - \{b\}) \cap A$ was finite, so this set must be infinite.■

11. Suppose that X is a set and that $\mathbf{I}$ is an operation on subsets of X such that the following hold:

(i) $\mathbf{I}(X) = X$.
(ii) $\mathbf{I}(A) \subset A$ for all $A \subset X$.
(iii) $\mathbf{I}(\mathbf{I}(A)) = \mathbf{I}(A)$ for all $A \subset X$.
(iv) $\mathbf{I}(A \cap B) = \mathbf{I}(A) \cap \mathbf{I}(B)$.

Prove that there is a unique topology $\mathbf{T}$ on X such that $U \in \mathbf{T}$ if and only if $\mathbf{I}(A) = A$.

SOLUTION.

The first and second conditions respectively imply that X and the empty set both belong to $\mathbf{T}$. Furthermore, the fourth condition implies that the intersection of two sets in $\mathbf{T}$ also belongs to $\mathbf{T}$, so it remains to verify the condition on unions. Suppose that A is a set and $U_\alpha \in \mathbf{T}$ for all $\alpha \in A$. Then we have

$$\bigcap_\alpha U_\alpha \;\;=\;\; \bigcap_\alpha \mathbf{I}(U_\alpha) \;\;\subset\;\; \mathbf{I}\left(\bigcap_\alpha U_\alpha\right) \;\;\subset\;\; \bigcap_\alpha U_\alpha$$

where the third property implies the right hand containment; the chain of inequalities implies that $\cup_\alpha ! U_\alpha$ belongs to $\mathbf{T}$, and therefore it follows that the latter is a topology for X such that a set U is open if and only if $\mathbf{I}(U) = U$.■

12. If X is a topological space and $A \subset X$ then the *exterior* of X, denoted by $\mathrm{Ext}(X)$, is defined to be $X - \overline{A}$. Prove that this construction has the following properties:

(a) $\mathrm{Ext}(A \cup B) = \mathrm{Ext}(A) \cap \mathrm{Ext}(B)$.

SOLUTION.

By definition $\mathrm{Ext}(A \cup B)$ is equal to

$$X - \overline{A \cup B} \;\;=\;\; X - \left(\overline{A} \cup \overline{B}\right) \;\;=\;\; \left(X - \overline{A}\right) \cap \left(X - \overline{B}\right)$$

which again by definition is equal to $\mathrm{Ext}(A) \cap \mathrm{Ext}(B)$.■

(b) $\mathrm{Ext}(A) \cap A = \emptyset$.

SOLUTION.

Since $A \subset \overline{A}$ it follows that $X - \overline{A} \subset X - A$ and hence $\mathrm{Ext}(A) \cap A \subset (X - A) \cap A = \emptyset$.■

(c) $\mathrm{Ext}(\emptyset) = X$.

SOLUTION.

The empty set is closed and therefore $\mathrm{Ext}(\emptyset) = X - \emptyset = X$.■

(d) $\mathrm{Ext}(A) \subset \mathrm{Ext}\,(\mathrm{Ext}\,(X - A\,)\,)$.

SOLUTION.

What is the right hand side? It is equal to $X - \overline{B}$ where $B = X - \overline{X - A}$. Note that $B = \mathrm{Int}(A)$. Therefore the right hand side may be rewritten in the form

$$X - \overline{(\mathrm{Int}(A)\,)}\ .$$

We know that $\mathrm{Int}(A) \subset A$ and likewise for their closures, and thus the reverse implication holds for the complements of their closures. But the last containment relation is the one to be proved.■

13. Let A_1 and A_2 be subsets of a topological space X, and let B be a subset of $A_1 \cap A_2$ that is closed in both A_1 and A_2 with respect to the subspace to topologies on each of these sets. Prove that B is closed in $A_1 \cup A_2$.

SOLUTION.

We may write $B = A_i \cap F_i$ where F_i is closed in X. It follows that $B = B \cap F_2 = A_1 \cap F_1 \cap F_2$ and $B = B \cap F_1 = A_2 \cap F_2 \cap F_1 = A_2 \cap F_1 \cap F_2$. Therefore

$$B = B \cup B \quad = \quad \Big(A_1 \cap F_1 \cap F_2\Big) \cup \Big(A_2 \cap F_1 \cap F_2\Big) \quad - \quad \Big(A_1 \cup A_2\Big) \ \cap \ \Big(F_1 \cap F_2\Big)$$

which shows that B is closed in $A_1 \cup A_2$.■

Note that the statement and proof remain valid if "closed" is replaced by "open."■

14. Suppose that A is a closed subset of a topological space X and B is the closure of $\mathrm{Int}(A)$. Prove that $B \subset A$ and $\mathrm{Int}(B) = \mathrm{Int}(A)$.

SOLUTION.

The first follows because $\mathrm{Int}(A) \subset A$, closure preserves set-theoretic inclusion, and $A = \overline{A}$. To prove the second statement, begin by noting that the first set is contained in the second because $B \subset A$. The reverse inclusion follows because $B = \overline{\mathrm{Int}(A)} \supset \mathrm{Int}(A)$ implies

$$\mathrm{Int}(B) \supset \mathrm{Int}\,(\,\mathrm{Int}(A)\,) = \mathrm{Int}(A)\ .■$$

15. Let X be a topological space and let $A \subset Y \subset X$.

(a) Prove that the interior of A with respect to X is contained in the interior of A with respect to Y.

SOLUTION.

The set $\mathrm{Int}_X(A)$ is an open subset of X and is contained in A, so it is also an open subet of Y that is contained in A. Since $\mathrm{Int}_Y(A)$ is the maximal such subset, it follows that $\mathrm{Int}_X(A) \subset \mathrm{Int}_Y(A)$.■

(b) Prove that the boundary of A with respect to Y is contained in the intersection of Y with the boundary of A with respect to X.

SOLUTION.

It will be convenient to let $\mathrm{CL}_U(B)$ denote the closure of B in U in order to write things out unambiguously.

By definition $\mathrm{Bd}_Y(A)$ is equal to $\mathrm{CL}_Y(A) \cap \mathrm{CL}_Y(Y - A)$, and using the formula $\mathrm{CL}_Y(B) = \mathrm{CL}_X(B) \cap Y$ we may rewrite $\mathrm{Bd}_Y(A)$ as the subset $\mathrm{CL}_X(A) \cap \mathrm{CL}_X(Y - A) \cap Y$. Since $Y - A \subset X - A$ we have $\mathrm{CL}_X(Y - A) \subset \mathrm{CL}_X(X - A)$, and this yields the relation

$$\mathrm{Bd}_Y(A) = \mathrm{CL}_X(A) \cap \mathrm{CL}_X(Y - A) \cap Y \subset \mathrm{CL}_X(A) \cap \mathrm{CL}_X(X - A) = \mathrm{Bd}_X(Y)$$

that was to be established.■

(c) Give examples to show that the inclusions in the preceding two statements may be proper (it suffices to give one example for which both inclusions are proper).

SOLUTION.

One obvious class of examples for **(a)** is given by taking A to be a nonempty subset that is not open and to let $Y = A$. Then the interior of A in X must be a proper subset of A but the interior of A in itself is simply A.

Once again, the best way to find examples where BOTH inclusions are proper is to try drawing a few pictures with pencil and paper. Such drawings lead to many examples, and one of the simplest arises by taking $A = [0, 1] \times \{0\}$, $Y = \mathbb{R} \times \{0\}$ and $X = \mathbb{R}^2$. For this example the interior inclusion becomes $\emptyset \subset (0, 1) \times \{1\}$ and the boundary inclusion becomes $\{0, 1\} \times \{0\} \subset [0, 1] \times \{0\}$. The details of verifying these are left to the reader.■

16. Let X be a topological space and let D be a subspace. A point a is called a *frontier point* of $D \subset X$ if every open set containing a contains at least one point of D and at least one point of $X - D$. Prove that a either belongs to D or is a limit point of D.

SOLUTION.

If a does not belong to D and U is an open set containing a, then $D \cap A = (D - \{a\}) \cap A$, and the condition on A implies that the first, hence also the second, intersection is nonempty. But this is just the definition of the set of limit points of D.■

II.3 : Continuous functions

Problems from Munkres, § 18, pp. 111 – 112

2. Suppose that $f : X \to Y$ is continuous and $A \subset X$. If x is a limit point of A, is $f(x)$ a limit point of $f[A]$?

SOLUTION.

Not necessarily. If f is constant then $f[A]$ has no limit points.■

6. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

SOLUTION.

Let $f(x) = x$ if x is rational and 0 if x is irrational. Then f is continuous at 0 because $|x| < \varepsilon \implies |f(x| < \varepsilon$. We claim that f is not continuous anywhere else. What does it mean in terms of δ and ε for f to be discontinuous at x? For some $\varepsilon > 0$ there is no $\delta > 0$ such that $|t - x| < \delta \implies |f(t) - f(x)| < \varepsilon$. Another way of putting this is that for some ε and all $\delta > 0$ sufficiently small, one can find a point t such that $|t - x| < \delta$ and $|f(t) - f(x)| \geq \varepsilon$.

There are two cases depending upon whether $x \neq 0$ is rational or irrational.

The rational case. Let $\varepsilon = |x|/2$ and suppose that $\delta < |x|$. Then there is an irrational number y such that $|y - x| < \delta$, and $|f(y) - f(x)| = |x| > \varepsilon$. Therefore f is not continuous at x.■

The irrational case. The argument is nearly the same. Let $\varepsilon = |x|/2$ and suppose that $\delta < |x|/4$. Then there is a rational number y such that $|y - x| < \delta$, and $|f(y) - f(x)| = |f(y)| > 3|x|/4 > \varepsilon$. Therefore f is not continuous at x.■

8.(a) [*Only for the special case $X = \mathbb{R}$ where the order topology equals the standard topology.*], Let $f : X \to \mathbb{R}$ be continuous. Show that the set of all points where $f(x) \leq g(x)$ is closed in X.

SOLUTION.

See the first proof of Additional Exercise 1 below.■

9.(c) An indexed family of sets $\{ A_\alpha \}$ is said to be *locally finite* if for each point $x \in X$ there is an open neighborhood that is disjoint from all but finitely many sets in the family.

Suppose that f is a set-theoretic function from X to Y such that for each α the restriction $f|A_\alpha$ is continuous. If each set is closed and the family is locally finite, prove that f is continuous.

SOLUTION.

The idea is to find an open covering by sets U_β such that each restriction $f|U_\beta$ is continuous; the continuity of f will follow immediately from this. Given $x \in X$, let U_x be an open subset containing x that is disjoint from all but finitely many closed subsets in the given family. Let $\alpha(1), \cdots \alpha(k)$ be the indices such that $U_x \cap A_\alpha = \emptyset$ unless $\alpha = \alpha(j)$ for some j. Then the subsets $A_{\alpha(j)} \cap U_x$ form a finite closed covering of the latter, and our assumptions imply that the restriction of f to each of these subsets is continuous. But this implies that the restriction of f to the open subset U_a is also continuous, which is exactly what we wanted to prove.■

Additional exercises

1. Give examples of continuous functions from $\mathbb{R}$ to itself that are neither open nor closed.

SOLUTION.

The easiest examples are those for which the image of $\mathbb{R}$ is neither open nor closed. One example of this sort is

$$f(x) = \frac{x^2}{x^2+1}$$

whose image is the half-open interval $(0,1]$.■

2. Let X be a topological space, and let $f,\ g : X \to \mathbb{R}$ be continuous. Prove that the functions $|f|$, $\max(f,g)$ [whose value at $x \in X$ is the larger of $f(x)$ and $g(x)$] and $\min(f,g)$ [whose value at $x \in X$ is the smaller of $f(x)$ and $g(x)$] are all continuous. [*Hints:* If $h : X \to \mathbb{R}$ is continuous, what can one say about the sets of points where $h = 0$, $h < 0$ and $h > 0$? What happens if we take $h = f - g$?]

FIRST SOLUTION.

First of all, if $f : X \to \mathbb{R}$ is continuous then so is $|f|$ because the latter is the composite of a continuous function (absolute value) with the original continuous function and thus is continuous.

We claim that the set of points where $f \geq g$ is closed in X and likewise for the set where $g \geq f$ (reverse the roles of f and g to get this conclusion). But $f \geq g \Longleftrightarrow f - g \geq 0$, and the latter set is closed because it is the inverse image of the closed subset $[0,\infty)$ under the continuous mapping $f - g$.

Let A and B be the closed subsets where $f \geq g$ and $g \geq f$ respectively. Then the maximum of f and g is defined by f on A and g on B, and since this maximum function is continuous on the subsets in a finite closed covering of X, it follows that the global function (the maximum) is continuous on all of X. Similar considerations work for the minimum of f and g, the main difference being that the latter is equal to g on A and f on B.■

SECOND SOLUTION.

First of all, if $f : X \to \mathbb{R}$ is continuous then so is $|f|$ because the latter is the composite of a continuous function (absolute value) with the original continuous function and thus is continuous. One then has the following formulas for $\max(f,g)$ and $\min(f,g)$ that immediately imply continuity:

$$\max(f,g) \quad = \quad \frac{f+g}{2} + \frac{|f-g|}{2}$$

$$\min(f,g) \quad = \quad \frac{f+g}{2} - \frac{|f-g|}{2}$$

Verification of these formulas is a routine exercises that is left to the reader to fill in; for each formula there are two cases depending upon whether $f(x) \leq g(x)$ or vice versa.■

3. Let $f : X \to Y$ be a set-theoretic mapping of topological spaces.

(a) Prove that f is open if and only if $f[\,\mathrm{Int}(A)\,] \subset \mathrm{Int}(\,f[A]\,)$ for all $A \subset X$ and that f is closed if and only if $\overline{f[A]} \subset f[\,\overline{A}\,]$ for all $A \subset X$.

SOLUTION.

Suppose that f is open. Then $\mathrm{Int}(A) \subset A$ implies that $f[\,\mathrm{Int}(A)\,]$ is an open set contained in $f[A]$; since $\mathrm{Int}(\,f[A]\,)$ is the largest such set it follows that $f[\,\mathrm{Int}(A)\,] \subset \mathrm{Int}(\,f[A]\,)$.■

Conversely, if the latter holds for all A, then it holds for all open subsets U and reduces to $f[U] \subset \mathrm{Int}(\,f[U]\,)$. Since the other inclusion also holds (every set contains its interior), it follows that the two sets are equal and hence that $f[U]$ is open in Y.■

Suppose now that f is closed. Then $A \subset \overline{A}$ implies that $f[A] \subset f[\overline{A}]$, so that the latter is a closed subset containing $f[A]$. Since $\overline{f[A]}$ is the smallest such set, it follows that $\overline{f[A]} \subset f[\,\overline{A}\,]$.■

Conversely, if the latter holds for all A, then it holds for all closed subsets F and reduces to $\overline{f[F]} \subset f[F]$. Once again the other inclusion also holds (each set is contained in its closure), and therefore the two sets are equal and $f[F]$ is closed in Y.■

(b) Using this and other results from the course notes, prove that f is closed if and only if $\overline{f[A]} = f[\,\overline{A}\,]$ for all $A \subset X$ and f is continuous and open if and only if $f^{-1}\,[\,\mathrm{Int}(B)\,] = \mathrm{Int}(\,f^{-1}[B]\,)$ for all $B \subset Y$.

SOLUTION.

To see the statement about continuous and closed maps, note that if f is continuous then for all $A \subset X$ we have $f[\,\overline{A}\,] \subset \overline{f[A]}$ (this is the third characterization of continuity in the course notes), while if f is closed then we have the reverse inclusion. This proves the $(\Longrightarrow)$ direction. To prove the reverse implication, split the set-theoretic equality into the two containment relations given in the first sentence of this paragraph. One of the containment relations implies that f is continuous and the other implies that f is closed.■

To see the statement about continuous and open maps, note that f is continuous if and only if for all $B \subset Y$ we have $f^{-1}\,[\,\mathrm{Int}(B)\,] \subset \mathrm{Int}\left(f^{-1}[B]\right)$ (this is the sixth characterization of continuity in the course notes). Therefore it will suffice to show that f is open if and only if the reverse inclusion holds for all $B \subset Y$. Suppose that f is open and $B \subset Y$. Then by our characterization of open mappings we have

$$f\left(\,\mathrm{Int}\left(f^{-1}[B]\,\right)\,\right) \subset \mathrm{Int} f\left[\,f^{-1}[B]\,\right] \subset \mathrm{Int}(B)$$

and similarly if we take inverse images under f; but the containment of inverse images extends to a longer chain of containments:

$$\mathrm{Int}\left(f^{-1}[B]\right) \quad\subset\quad f^{-1}\left[\,f\left[\mathrm{Int}\left(f^{-1}[B]\right)\,\right]\,\right] \quad\subset\quad \mathrm{Int}\left(f^{-1}[B]\right) \quad\subset$$

$$f^{-1}\left[f\left[\mathrm{Int}\left(f^{-1}[B]\,\right)\,\right]\,\right] \quad\subset f^{-1}\left[\mathrm{Int}(B)\right]$$

This proves the $(\Longrightarrow)$ implication. What about the other direction? If we set $B = f[A]$ the hypothesis becomes

$$\mathrm{Int}\left(f^{-1}[\,f[A]\,]\right) \quad\subset\quad f^{-1}\left[\mathrm{Int}(\,f[A]\,)\right]$$

and if we take images over f the containment relation is preserved and extends to yield

$$f\left[\,\mathrm{Int}\left(\,f^{-1}[\,f[A]\,]\right)\,\right] \quad\subset\quad f\left[\,f^{-1}\left[\mathrm{Int}(\,f[A]\,)\right]\,\right] \quad\subset\quad \mathrm{Int}(\,f[A]\,)\ .$$

Since $A \subset f^{-1}[\,f[A]\,]$ the left hand side of the previous inclusion chain contains $f[\,\mathrm{Int}(A)\,]$, and if one combines this with the inclusion chain the condition $f[\,\mathrm{Int}(A)\,] \subset \mathrm{Int}(\,f]A]\,)$, which characterizes open mappings, is an immediate consequence.■

4. A mapping of topological spaces $f : X \to Y$ is said to be *light* if for each $y \in Y$ the subspace $f^{-1}[\,\{x\}\,]$ inherits the discrete topology (every subset is both open and closed). Prove that the composite of two continuous light mappings is also light (of course, it is also continuous).

SOLUTION.

Suppose that $f : X \to Y$ and $g : Y \to Z$ are both light mappings. Then for each $z \in Z$ we have

$$(g\circ f)^{-1}\ \left[\,\{z\}\,\right] \quad=\quad f^{-1}\ \left[\,g^{-1}[\{z\}]\,\right]$$

and we shall denote this set by E_z. If we now write $g^{-1}[\{z\}]$ as a union of its pairwise disjoint one-point subsets $\{a\}$ then it follows by continuity that each set $f^{-1}[\{a\}]$ is open and closed in E_z. But now we also know that for each $b \in f^{-1}[\{a\}]$ the one-point subset $\{b\}$ is open and closed in the latter. Combining these, we see that for each $b \in E_z$ the one-point set $\{b\}$ is both open and closed in E_z, so that E_z is discrete and hence $g \circ f$ is light.■

5. If $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ unless $x = y = 0$ and $f(0, 0) = 0$, show that $r : \mathbb{R}^2 \to \mathbb{R}$ is not continuous at $(0, 0)$. [*Hint:* Consider the behavior of f on straight lines through the origin.]

SOLUTION.

Consider the line defined by the parametric equations $x(t) = at$, $y(t) = bt$ where a and b are not both zero; the latter is equivalent to saying that $a^2 + b^2 > 0$. The value of the function $f(at, bt)$ for $t \neq 0$ is given by the following formula:

$$f(at, bt) = \frac{a^2t^2 - b^2t^2}{a^2t^2 + b^2t^2} = \frac{a^2 - b^2}{a^2 + b^2}$$

If $a = 0$ or $b = 0$ then this expression reduces to 1, while if $a = b = 1$ this expression is equal to 0. Therefore we know that for every $\delta > 0$ and $t < \delta/4$ the points $(t, 0)$ and (t, t) lie in the open disk of radius δ about the origin and the values of the function at these points are given by $f(t, 0) = 1$ and $f(t, t) = 0$. If the function were continuous at the origin and its value was equal to L, then we would have that $|L|$, $|L - 1| < \varepsilon$ for all $\varepsilon > 0$. No such number exists, and therefore the function cannot be made continuous at the origin.■

6. Let $f(x, y) = 2\,x^2y/(x^4 + y^2)$ unless $x = y = 0$ and $f(0, 0) = 0$, and define $\varphi(t) = (t, at)$ and $\psi(t) = (t, t^2)$.

(a) Show that $\lim_{t\to 0} f\circ\varphi(t) = 0$; *i.e.*, f is continuous on every line through the origin.

SOLUTION.

Direct computation shows that $f(t, at)$ is equal to

$$\frac{2\,a\,t^3}{t^4 + a^2t^2} \quad = \quad \frac{2\,a\,t}{t^2 + a^2}$$

and the limit of this expression as $t \to 0$ is zero **provided** $a \neq 0$. Strictly speaking, this is not enough to get the final conclusion, for one also has to analyze the behavior of the function on the x-axis and y-axis. But for the nonzero points of the x-axis one has $f(t, 0) = 0$ and for the nonzero points of the y-axis one has $f(0, t) = 0$.■

(b) Show that $\lim_{t\to 0} f\circ\psi(t) \neq 0$ and give a rigorous argument to explain why this and the preceding part of the exercise imply f is not continuous at $(0, 0)$.

SOLUTION.

Once again, we can write out the composite function explicitly:

$$f(t, t^2) = \frac{2\,t^4}{t^4 + t^4} \quad = \quad 1 \qquad (\text{provided} \quad t \neq 0)$$

The limit of this function as $t \to 0$ is clearly 1.

One could give another $\varepsilon - \delta$ proof to show the function is not continuous as in the preceding exercise, but here is another approach by contradiction: Suppose that f is continuous at the origin.

Since φ and ψ are continuous functions, it follows that the composites $f \circ \varphi$ and $f \circ \psi$ are continuous at $t = 0$, and that their values at zero are equal to $f(0,0)$. What can we say about the latter? Using $f \circ \varphi$ we compute it out to be zero, but using $f \circ \psi$ it computes out to $+1$. This is a contradiction, and it arises from our assumption that f was continuous at the origin.■

II.4 : Cartesian products

Problems from Munkres, § 18, pp. 111 – 112

10. If $f : A \to B$ and $g : C \to D$ are continuous, prove that the product map $f \times g : A \times C \to B \times D$ is continuous.

SOLUTION.

This is worked out and generalized in the course notes.■

11. Let $F : X \times Y \to Z$ be a set-theoretic function. We say that F is *continuous in each variable separately* if for each $y_0 \in Y$ the map $h : X \to Z$ defined by $h(x) = F(x, y_0)$ is continuous, and for each $x_0 \in X$ the map $k : Y \to Z$ defined by $k(y) = F(x_0, y)$ is continuous. Show that if F is continuous then F is continuous in each variable separately.

SOLUTION.

Consider the maps $A(y_0) : X \to X \times Y$ and $B(x_0) : Y \to X \times Y$ defined by $[A(y_0)](x) = (x, y_0)$ and $[B(x_0)](y) = (x_0, y)$. Each of these maps is continuous because its projection onto one factor is an identity map and its projection onto the other is a constant map. The maps h and k are composites $F \circ A(y_0)$ and $f \circ B(x_0)$ respectively; since all the factors are continuous, it follows that h and k are continuous.■

FOOTNOTE. The next problem in Munkres gives the standard example of a function from $\mathbb{R}^2 \to \mathbb{R}$ that is continuous in each variable separately but not continuous at the origin. See also the solution to Additional Exercise 5 below. was worked out previously.

Additional exercises

1. (*"A product of products is a product"*) Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of nonempty sets, and let $\mathcal{A} = \cup\{\mathcal{A}_\beta \mid \beta \in \mathcal{B}\}$ be a partition of $\mathcal{A}$. Construct a bijective map of $\prod\{A_\alpha \mid \alpha \in \mathcal{A}\}$ to the set

$$\prod_\beta \{\prod\{A_\alpha \mid \alpha \in \mathcal{A}_\beta\}\ \} .$$

If each A_α is a topological space and we are working with product topologies, prove that this bijection is a homeomorphism.

SOLUTION.

The basic idea is to give axioms characterizing cartesian products and to show that they apply in this situation.

LEMMA. *Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of nonempty sets, and suppose that we are given data consisting of a set P and functions $h_\alpha : P \to A_\alpha$ such that for* **EVERY** *collection of data $(S, \{f_\alpha : S \to A_\alpha\})$ there is a unique function $f : S \to P$ such that $h_\alpha \circ f = f_\alpha$ for all α. Then there* *is a unique* $1-1$ *correspondence* $\Phi : \prod_\alpha A_\alpha \to P$ *such that* $h_\alpha \circ \Phi$ *is the projection from* $\prod_\alpha X_\alpha$ *onto* A_α *for all* α.

If we suppose in addition that each A_α *is a topological space, that* P *is a topological space, that the functions* h_α *are continuous and the unique map* f *is always continuous, then* Φ *is a homeomorphism to* $\prod_\alpha A_\alpha$ *with the product topology.*

Sketch of the proof of the lemma. The existence of Φ follows directly from the hypothesis. On the other hand, the data consisting of $\prod_\alpha A_\alpha$ and the coordinate projections π_α also satisfies the given properties. Therefore we have a unique map Ψ going the other way. By the basic conditions the two respective composites $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are completely specified by the maps $\pi_\alpha \circ \Phi \circ \Psi$ and $h_\alpha \circ \Phi \circ \Psi$. Since $\pi_\alpha \circ \Phi = h_\alpha$ and $h_\alpha \circ \Phi = \pi_\alpha$ hold by construction, it follows that $\pi_\alpha \circ \Phi \circ \Psi = \pi_a lpha$ and $h_\alpha \circ \Phi \circ \Psi = h_\alpha$ and by the uniqueness property it follows that both of the composites $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are identity maps. Thus Φ is a 1–1 correspondence.

Suppose now that everything is topologized. What more needs to be said? In the first place, The product set with the product topology has the unique mapping property for continuous maps. This means that both Φ and Ψ are continuous and hence that Φ is a homeomorphism.■

Application to the exercise. We shall work simultaneously with sets and topological spaces, and morphisms between such objects will mean set-theoretic functions or continuous functions in the respective cases.

For each β let $_\beta$ denote the product of objects whose index belongs to $\mathcal{A}_\beta$ and denote its coordinate projections by p_α. The conclusions amount to saying that there is a canonical morphism from $\prod_\beta P_\beta$ to $\prod_\alpha A_\alpha$ that has an inverse morphism. Suppose that we are given morphisms f_α from the same set S to the various sets A_α. If we gather together all the morphisms for indices α lying in a fixed subset $\mathcal{A}_\beta$, then we obtain a unique map $g_\beta : S \to P_\beta$ such that $p_\alpha \circ g_\beta = f_\alpha$ for all $\alpha in \mathcal{A}_\beta$. Let $q_\beta : \prod_\gamma P_\gamma \to P_\beta$ be the coordinate projection. Taking the maps g_β that have been constructed, one obtains a unique map $F : S \to \prod_\beta P_\beta$ such that $q_\beta \circ F = g_\beta$ for all β. By construction we have that $p_\alpha \circ q_\beta \circ F = f_\alpha$ for all α. If there is a unique map with this property, then $\prod_\beta P_\beta$ will be isomorphic to $\prod_\alpha A_\alpha$ by the lemma. But suppose that θ is any map with this property. Once again fix β. Then $p_\alpha \circ q_\beta \circ F = p_\alpha \circ q_\beta \circ \theta = f_\alpha$ for all $\alpha \in \mathcal{A}_\beta$ implies that $q_\beta \circ F = q_\beta \circ \theta$, and since the latter holds for all β it follows that $F = \theta$ as required.■

2. *Non-Hausdorff topology.* In topology courses one is ultimately interested in spaces that are Hausdorff. However, there are contexts in which certain types of non-Hausdorff spaces arise (for example, the Zariski topologies algebraic geometry, which are defined in most textbooks on that subject).

(a) A topological space X is said to be *irreducible* if it cannot be written as a union $X = A \cup B$, where A and B are proper closed subspaces of X. Show that X is irreducible if and only if every pair of nonempty open subsets has a nonempty intersection. Using this, show that an open subset of an irreducible space is irreducible.

SOLUTION.

First part. We shall show that the negations of the two statements in the second sentence are equivalent; *i.e.*, The space X is reducible (not irreducible) if and only if some pair of nonempty open subsets has a nonempty intersection. This follows because X is reducible $\iff$ we can write $X = A \cup B$ where A and B are nonempty proper closed subsets $\iff$ we can find nonempty proper closed subsets A and B such that $(X - A) \cap (X - B) = \emptyset$ $\iff$ we can find nonempty proper open subsets U and V such that $U \cap V = \emptyset$ (take $U = X - A$ and $V = Y - B$).

Second part. The empty set is irreducible (it has no nonempty closed subsets), so suppose that W is a nonempty open subset of X where X is irreducible. But if U and V are nonempty open subspaces of W, then they are also nonempty open subspaces of X, which we know is irreducible.

Therefore by the preceding paragraph we have $U \cap V \neq \emptyset$, which in turn implies that W is irreducible (again applying the preceding paragraph).■

(b) Show that every set with the indiscrete topology is irreducible, and every infinite set with the finite complement topology is irreducible.

SOLUTION. For the first part, it suffices to note that a space with an indiscrete topology has no nonempty proper closed subspaces. For the second part, note that if X is an infinite set with the finite complement topology, then the closed proper subsets are precisely the finite subsets of X, and the union of two such subsets is always finite and this is always a proper subset of X. Therefore X cannot be written as the union of two closed proper subsets.■

(c) Show that an irreducible Hausdorff space contains at most one point.

SOLUTION.

If X is a Hausdorff space and $u, v \in X$ then one can find open subsets U and V such that $u \in U$ and $v \in V$ (hence both are nonempty) such that $U \cap V = \emptyset$. Therefore X is not irreducible because it does not satisfy the characterization of such spaces in the first part of **(a)** above.■

3. Let $f : X \to Y$ be a map, and define the *graph* of f to be the set Γ_f of all points $(x, y) \in X \times Y$ such that $y = f(x)$. Prove that the map $x \to (x, f(x))$ is a homeomorphism from X to Γ_f if and only if f is continuous.

SOLUTION.

Let $\gamma : X \to \Gamma_f$ be the set-theoretic map sending x to $f(x)$. We need to prove that f is continuous $\Longleftrightarrow$ γ is a homeomorphism. Let $j : \Gamma_f \to X \times Y$ be the inclusion map.

($\Longrightarrow$) If f is continuous then γ is continuous. By construction it is 1–1 onto, and a continuous inverse is given explicitly by the composite $\pi_X \circ j$ where π_X denotes projection onto X.■

($\Longleftarrow$) If γ is a homeomorphism then f is continuous because it may be written as a composite $\pi_Y \circ j \circ \gamma$ where each factor is already known to be continuous.■

4. Let X be a topological space that is a union of two closed subspaces A and B, where each of A and B is Hausdorff in the subspace topology. Prove that X is Hausdorff.

SOLUTION.

Since A and B are closed in X we know that $A \times A$ and $B \times B$ are closed in $X \times X$. Since A and B are Hausdorff we know that the diagonals Δ_A and Δ_B are closed in $A \times A$ and $B \times B$ respectively. Since "a closed subset of a closed subset is a closed subset" it follows that Δ_A and Δ_B are closed in $X \times X$. Finally $X = A \cup B$ implies that $\Delta_X = \Delta_A \cup \Delta_B$, and since each summand on the right hand side is closed in $X \times X$ it follows that the left hand side is too. But this means that X is Hausdorff.■

EXAMPLE.

Does the same conclusion hold if A and B are open? No. Consider the topology on $\{1, 2, 3\}$ whose open subsets are the empty set and all subsets containing 2. Then both $\{1, 3\}$ and $\{2\}$ are Hausdorff with respect to the respective subspace topologies, but there union — which is X — is not Hausdorff because all open sets contain 2 and thus one cannot find nonempty open subsets that are disjoint.■

5. Let A be some nonempty set, let $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ and $\{Y_\alpha \mid \alpha \in \mathcal{A}\}$ be families of topological spaces, and for each $\alpha \in A$ suppose that $f_\alpha : X_\alpha \to Y_\alpha$ is a homeomorphism. Prove that the product map

$$\prod_\alpha f_\alpha \ : \ \prod_\alpha X_\alpha \longrightarrow \prod_\alpha Y_\alpha$$

is also a homeomorphism. [*Hint:* What happens when you take the product of the inverse maps?]

SOLUTION.

For each α let $g_\alpha = f_\alpha^{-1}$. Then we have

$$\prod_\alpha f_\alpha \circ \prod_\alpha g_\alpha \;=\; \prod_\alpha (f_\alpha \circ g_\alpha) \;=\; \prod_\alpha \mathrm{id}(Y_\alpha) \;=\; \mathrm{id}\big(\prod_\alpha Y_\alpha\big)$$

and we also have

$$\prod_\alpha g_\alpha \circ \prod_\alpha f_\alpha \;=\; \prod_\alpha (g_\alpha \circ f_\alpha) \;=\; \prod_\alpha \mathrm{id}(X_\alpha) \;=\; \mathrm{id}\big(\prod_\alpha X_\alpha\big)$$

so that the product of the inverses $\prod_\alpha g_\alpha$ is an inverse to $\prod_\alpha f_\alpha$.■

6. Let X be a topological space and let $T : X \times X \times X \to X \times X \times X$ be the map that cyclically permutes the coordinates: $T(x, y, z) = (z, x, y)$ Prove that T is a homeomorphism. [*Hint:* What is the test for continuity of a map into a product? Can you write down an explicit formula for the inverse function?]

SOLUTION.

Let π_i be projection onto the i^{th} factor for $i = 1, 2 3$. The map T is continuous if and only if each $\pi_i \circ T$ is continuous. But by construction we have $\pi_1 \circ T = \pi_3$, $\pi_2 \circ T = \pi_1$, and $\pi_3 \circ T = \pi_2$, and hence T is continuous.

We can solve directly for T^{-1} to obtain the formula $T^{-1}(u, v, w) = (v, w, u)$. We can prove continuity by looking at the projections on the factors as before, but we can also do this by checking that $T^{-1} = T^2$ and thus is continuous as the composite of continuous functions.■

7. Let $\alpha, \beta \in \{1, 2, \infty\}$, and let $|\cdots|_\alpha$ and $|\cdots|_\beta$ be the norms for $\mathbb{R}^n$ that were described in Section II.1.

(a) Explain why there are positive constants m and M (depending upon α and β) such that

$$m \cdot |x|_\beta \;\leq\; |x|_\alpha \;\leq\; M \cdot |x|_\beta$$

for all $x \in \mathbb{R}^n$.

SOLUTION.

This is a special case of the estimates relating the $\mathbf{d}_1$, $\mathbf{d}_2$ and $\mathbf{d}_\infty$ metrics.■

(b) Explain why the interior of the closed unit disk with $\mathbf{d}_\alpha$ radius 1 in $\mathbb{R}^n$ is the set of all x such that $|x|_\alpha < 1$ and the frontier is the set of all x such that $|x|_\alpha = 1$.

SOLUTION.

The set of points such that $|x|_\alpha < 1$ is open, and if $|x|_\alpha = 1$ is open and U is an open neighborhood of x, then U contains the points $(1 \pm t) \cdot x$ for all sufficiently small values of $|t|$.■

(c) Prove that there is a homeomorphism h from $\mathbb{R}^n$ to itself such that $|h(x)|_\beta = |x|_\alpha$ for all x. [*Hints:* One can construct this so that $h(x)$ is a positive multiple of x. It is necessary to be a little careful when checking continuity at the origin.]

SOLUTION.

Define $h(x) = |x|_\alpha \cdot |x|_\beta^{-1} \cdot x$ if $X \neq 0$ and $h(0) = 0$. By (a), the mapping h is continuous with respect to the α-norm if and only if if is continuous with respect to the β-norm, and in fact each norm is continuous with respect to the other. It follows immediately that h is continuous if $x \neq 0$. To see continuity at 0, it suffices to check that if $\varepsilon > 0$ then there is some $\delta > 0$ such that $|x|_\alpha < \delta$ implies $|h(x)|_\alpha = |x|_\beta < \varepsilon$. Since $|x|_\beta \leq |x|_\alpha / m$, we can take $\delta = m \cdot \varepsilon$.

To see that h is a homeomorphism, let k be the map constructed by interchanging the roles of α and β in the preceding discussion. By construction k is an inverse function to h, and the preceding argument together with the inequality $|x|_\alpha \leq M \cdot |x|_\beta$ imply that k is continuous.■

(d) Prove that the hypercube $[-1, 1]^n$ is homeomorphic to the unit disk of all points $x \in \mathbb{R}^n$ satisfying $\sum x_i^2 = 1$ such that the frontier of the hypercube is mapped onto the unit sphere.

SOLUTION.

In the preceding discussion we have constructed a homeomorphism which takes the set of all points satisfying $|x|_\beta \leq 1$ to the set defined by $|x|_\alpha \leq 1$, and likewise if the inequality is replaced by equality. If $\beta = \infty$ then the domain is the hypercube and the set of points with $|x|_\beta = 1$ is its frontier, and if $\alpha = 2$ then the codomain is the ordinary unit disk and the set of points where $|x|_\alpha = 1$ is the unit sphere which bounds that disk.■