This is a draft of some sections for the paper on manifolds with different complete riemannian metrics of nonnegative sectional curvature which have nondiffeomorphic codimension 2 souls. It is mainly limited to the input from geometric topology. **(RS)**

Summary

In the study of existence and classification problems for manifolds generally known as surgery theory (*cf.* [Wall book]), there is a fundamentally important fourfold periodicity which corresponds geometrically to multiplication by the complex plane $\mathbb{CP}^2$. More generally, if m is a positive integer, then the m-fold iterate of this periodicity corresponds to multiplication by the $2m$-dimensional complex projective space $\mathbb{CP}^{2m}$. Formally, one may view this as a cancellation principle; namely, if certain types of objects become equivalent after taking products with $\mathbb{CP}^{2m}$, then the objects themselves are equivalent.

It is well known that such cancellation principles fail in many cases (*cf.* [Kwasik-Schultz 2002]); in particular, there are manifolds N^7 which are homeomorphic but not diffeomorphic to S^7 (*i.e.*, **exotic spheres**) such that $N^7 \times \mathbb{CP}^2$ is diffeomorphic to $S^7 \times \mathbb{CP}^2$ (for example, see [Browder 1968], pp. 38–39), and a similar situation holds if 7 is replaced by most integers of the form $4k-1 \geq 11$ (see the discussion following Corollary Y.2). On the other hand, if we take an exotic 7-sphere which generates the Kervaire-Milnor group of exotic spheres $\Theta_7 \cong \mathbb{Z}_{28}$ (see [Kervaire-Milnor]), then we shall prove the following cancellation theorem.

THEOREM S.1. *Let Σ^7 be an exotic 7-sphere which generates the Kervaire-Milnor group $\Theta_7 \cong \mathbb{Z}_{28}$, and let $m \geq 1$ be an arbitrary positive integer. Then $\Sigma^7 \times \mathbb{CP}^{2m}$ is not diffeomorphic to $S^7 \times \mathbb{CP}^{2m}$.*

In contrast, as noted in [Browder 1968, Thm. 1.6], if N^7 is an arbitrary exotic 7-sphere and m is a nonnegative integer, then $N^7 \times \mathbb{CP}^{2m+1}$ is diffeomorphic to $S^7 \times \mathbb{CP}^{2m+1}$.

Clearly there are several different choices for the generator of Θ_7, but there is a standard choice which we shall call the **standard generator** henceforth. Specifically, this exotic sphere is describable as the boundary of a parallelizable manifold W^8 whose signature is equal to 8 (see [Browder 1972, Chapter V]). As noted in [Grove-Ziller], this standard generator has a (complete) riemannian metric with nonnegative sectional curvature. The examples of complete riemannian n-manifolds which are diffeomorphic but have nondiffeomorphic $(n-2)$-dimensional souls will follow from Theorem 1 and the next result:

THEOREM S.2. *Let ω be a nontrivial complex line bundle over $\mathbb{CP}^q$, where $q \geq 1$, let $E(\omega)$ be its total space, and let N^7 be an arbitrary exotic 7-sphere. Then $N^7 \times E(\omega)$ is diffeomorphic to $S^7 \times E(\omega)$.*

If q is odd then this follows because the products of N^7 and S^7 are diffeomorphic (see above), so the main content of this result involves the cases where q is odd.

We shall prove Theorems 1 and 2 using surgery-theoretic methods, and the Sullivan-Wall exact sequence for the set of homotopy structures $\mathbf{S}^s(S^7 \times \mathbb{CP}^{2m})$ (*e.g.*, see Chapter 10 of [Wall book]), will play a central role. If N^7 is a closed oriented smooth manifold and $h : N^7 \to S^7$ is an orientation-preserving homotopy equivalence, then the product map $h \times \mathbf{id}(\mathbb{CP}^{2m})$ defines a class in the structure set, and $N^7 \times \mathbb{CP}^{2m}$ is diffeomorphic to $S^7 \times \mathbb{CP}^{2m}$ if and only if this class is equal to a class determined by a homotopy self-equivalence of $S^7 \times \mathbb{CP}^{2m}$ (see Section H). Therefore we need to analyze the group $\mathcal{E}(S^7 \times \mathbb{CP}^{2m})$ of homotopy classes of homotopy self-equivalences of $S^7 \times \mathbb{CP}^{2m}$ and determine where such classes lie in the structure set. There is a fundamentally important map on a structure set, known as the *normal invariant*, which takes values in a homotopy-theoretic object, and in general this invariant partially measures the extent to which a homotopy structure is nontrivial. For the manifolds $S^7 \times \mathbb{CP}^{2m}$, we shall prove that a homotopy self-equivalence determines the trivial element of the structure set if and only if its normal invariant is zero. The proof of Theorem S.1 will essentially follow from this and a result of L. Taylor [Taylor] which gives a lower estimate on the number of homotopy structures whose normal invariants are nontrivial. Theorem S.2 follows from a different sort of surgery-theoretic argument which involves the Sullivan-Wall surgery exact sequence for manifolds with boundary.

If we specialize to the case $m = 1$ we can obtain more complete information. Following standard notation, we shall say that two compact smooth manifolds M^n and N^n are *(stably) tangentially homotopy equivalent* if there is a homotopy equivalence $h : M \to N$ such that the tangent bundles τ_M and τ_N satisfy the pullback condition $h^*(\tau_N \oplus \mathbb{R}^k) \cong \tau_M \oplus \mathbb{R}^k$ for some $k \geq 0$. Similar definitions can be formulated for piecewise linear and topological manifolds. The methods of surgery theory are adapted to treat tangentially homotopically equivalent manifolds in [Madsen-Taylor-Williams] (see Section A below).

PROPOSITION S.3. *Let M^{11} be a smooth manifold which is tangentially homotopy equivalent to $S^7 \times \mathbb{CP}^2$. Then M is diffeomorphic to exactly one of the manifolds $S^7 \times \mathbb{CP}^2$, $\Sigma^7 \times \mathbb{CP}^2$, or $(\#^2\Sigma^7) \times \mathbb{CP}^2$, where Σ^7 is the exotic sphere described above and $\#^2 P$ denotes the connected sum of a manifold P with itself.*

The proof of Proposition S.3 is similar to the proof of Theorem S.1; one needs a refinement of the main result in [Taylor] for $S^7 \times \mathbb{CP}^2$, and we shall do this using the approach to proving Taylor's result in [RS 1987].

Proposition S.3 has a curious implication for the following problem:

Question. *Suppose that E is a complete riemannian manifold with nonnegative sectional curvature whose soul is diffeomorphic to M, and suppose that N is another compact manifold which can be realized as the soul of such a riemannian metric on N. How are M and N related?*

In general, it is clear that M and N must be homotopy equivalent, and if the codimension $\dim E - \dim M$ is equal to 2, then it is fairly straightforward to show that M and N must be tangentially homotopy equivalent (see [Madsen-Taylor-Williams] and Section A for more on this notion). On the other hand, this condition is far from sufficient, for it is not difficult to construct examples of tangentially homotopically equivalent manifolds $M = S^k \times \mathbb{CP}^q$ and N such that for all complex line bundles α, β over these manifolds with corresponding Chern classes, the smooth total spaces $E(\alpha \downarrow M)$ and $E(\beta \downarrow N)$ are not diffeomorphic. It is not known when such manifolds admit riemannian metrics with nonnegative sectional curvature, so these examples do not really shed any light on positive sectional curvature questions in riemannian geometry. However, if we combine our preceding results with the nonnegatively curved metrics on exotic spheres constructed by K. Grove and W. Ziller [Grove-Ziller], we obtain the following answer to the stated question when $M = S^7 \times \mathbb{CP}^2$:

THEOREM S.4. *Let M^{11} be a smooth manifold which is homotopy equivalent to $S^7 \times \mathbb{CP}^2$, let ω be a nontrivial complex line bundle over $\mathbb{CP}^2$, and let $E(\omega)$ be its total space. Then M is diffeomorphic to the soul of a complete nonnegatively curved riemannian metric on $S^7 \times E(\omega)$ if and only if M is tangentially homotopy equivalent to $S^7 \times \mathbb{CP}^2$.*

Theorem S.1 also yields the following extension of a main result in [RS 1987].

THEOREM S.5. *Let Σ be the standard generator of Θ_7, and let $k \geq 1$ be odd. Then there is no smooth semifree S^1-action on a homotopy $(2k+7)$-sphere whose fixed point set is diffeomorphic to Σ.*

The corresponding result for $k \geq 5$ is established in [RS 1987, Theorem III]; as noted in that paper and [Browder 1968], one can always realize Σ as a fixed point set of a smooth semifree S^1-action on a homotopy $(2k+7)$-sphere if $k \geq 2$ is even. When $k = 1$ the conclusion is a special case of a result due to Wu-yi Hsiang [Hsiang].

Proof of Theorem S.5. As in [RS 1987] and [Browder 1968], the result will follow if we know that $\Sigma^7 \times \mathbb{CP}^{k-1}$ is not diffeomorphic to $S^7 \times \mathbb{CP}^{k-1}$ for all odd integers $k \geq 3$. But this is precisely the conclusion of Theorem S.1.■

REMARK. Since arbitrary products of suitable exotic 7-spheres and complex projective spaces have metrics with nonnegative sectional curvature, it is natural to ask whether the products of the total spaces $E(\omega)$ with nonnegatively curved exotic 7-spheres or even-dimensional complex projective spaces yield further examples of complete nonnegatively curved manifolds with nondiffeomorphic souls. The answer is negative if one takes a product with an exotic 7-sphere because it is well-known that the product of two such manifolds is diffeomorphic to $S^7 \times S^7$; more generally, product formulas for surgery obstructions imply that if the exotic sphere Σ^{4r-1} bounds a parallelizable manifold and W is an odd-dimensional closed simply connected manifold, then $\Sigma^{4r-1} \times W$ is diffeomorphic to $S^{4r-1} \times W$. However, if one take a product with an even-dimensional complex projective space, it is not at all clear what happens. Attacking this problem would require an analysis of the homotopy self-equivalences of $S^7 \times \mathbb{CP}^{2m} \times \mathbb{CP}^{2n}$ corresponding to the study of homotopy self-equivalences of $S^7 \times \mathbb{CP}^{2m}$ in this paper.

On a more positive note, one can use the examples of this paper to obtain further examples by taking products with the r-torus T^r, where $r \geq 1$ is arbitrary. This is true because the vanishing of the Whitehead groups of $\mathrm{Wh}(\mathbb{Z}^r)$ implies the following fact: *The products of two closed simply connected manifolds M^n and N^n $(n \geq 5)$ with T^r are diffeomorphic if and only if M and N are diffeomorphic* (see [RS 1971] for the case where M and N are homotopy spheres).

Homotopy self-equivalences of products

Although surgery theory in principle yields a diffeomorphism classification for manifolds with a fixed homotopy type, it does so in a slightly indirect manner which is often marginalized in the literature. Since the explicit classification statements are indispensable for our purposes, we shall summarize what is needed and discuss the importance of homotopy self-equivalences in the classification theory. Background references are Wall's book [Wall book] and the anthology of papers on the Hauptvermutung [Ranicki 1995].

Suppose that M^n is a compact smooth manifold, with or without boundary, and assume both M and ∂M are connected. A **simple homotopy structure on** M is a pair (N, f) consisting of a compact smooth manifold N and a simple homotopy equivalence of manifolds with boundary (in other words, a homotopy equivalence of pairs). Two such structures (N_0, f_0) and (N_1, f_1) are said to be equivalent if and only if there is a diffeomorphism $h : N_0 \to N_1$ such that $f_1 \circ h \simeq f_0$, where again the homotopy is a homotopy of pairs. The set of all such equivalence classes is a pointed set we shall denote by $\mathbf{S}^s(M)$. Its base point is the identity on M, and this pointed set fits into an exact Sullivan-Wall surgery exact sequence

$$\cdots \;\to\; L^s_{\dim M+1}\big(\,\pi_1(M), \pi_1(\partial M), w\,\big) \;\to\; \mathbf{S}^s(M) \;\to\; [M, F/O] \;\to\; L^s_{\dim M}\big(\,\pi_1(M), \pi_1(\partial M), w\,\big)$$

which is described in the given references. Specifically, the map

$$\Delta : L^s_{\dim M+1}\big(\,\pi_1(M), \pi_1(\partial M), w\,\big) \;\longrightarrow\; \mathbf{S}^s(M)$$

comes from an action of the group on the left hand side on $\mathbf{S}^s(M)$, and it is called *the action of the Wall group on the base point*, the map $\mathfrak{q}$ from $\mathbf{S}^s(M)$ to $[M, F/O]$ is called the *normal invariant*, and the map σ from $[M, F/O]$ to $L^s_{\dim M}\big(\,\pi_1(M), \pi_1(\partial M), w\,\big)$ is called the *surgery obstruction map*. It is important to recognize that σ is **not necessarily** a homomorphism with respect to the usual abelian group structures on its domain and codomain.

In order to extract a diffeomorphism classification, we need to take the quotient of this structure set by an action of the group $\mathcal{E}^s(M, \partial M)$ of simple homotopy self-equivalences of $(M, \partial M)$. Most of the time we shall be working with simply connected objects, and in such cases all homotopy equivalences are simple, and in such cases we shall write $\mathcal{E}$ instead of $\mathcal{E}^s$. If (N, f) represents a class in $\mathbf{S}^s(M)$ and $h \in \mathcal{E}^s(M, \partial M)$, then the left action of $\mathcal{E}^s(M, \partial M)$ on $\mathbf{S}^s(M)$ is given by ordinary composition of functions:

$$[h] \cdot [N, f] \quad = \quad [N, h \circ f]$$

For our purposes the crucial point is the need to analyze homotopy self-equivalences of a manifold M in order to use surgery in determining whether a manifold homotopy equivalent to M is actually diffeomorphic to M.

Self-equivalences of $S^k \times \mathbb{CP}^q$

We shall need some fairly detailed information about homotopy self-equivalences of $S^7 \times \mathbb{CP}^{2m}$, where $m \geq 1$. The first few steps are fairly elementary:

PROPOSITION H.1. *(i) Let $f : S^k \times \mathbb{CP}^q \to S^k \times \mathbb{CP}^q$ be a homotopy self-equivalence, where $q \geq 1$ and k is odd. Then there is a diffeomorphism $h : S^k \times \mathbb{CP}^q \to S^k \times \mathbb{CP}^q$ such that f and h induce the same automorphism of $H^*(S^k \times \mathbb{CP}^q; \mathbb{Z})$.*

(ii) Let f be above, and assume that f induces the identity on $H^*(S^k \times \mathbb{CP}^q; \mathbb{Z})$. Let $j(S^k) : S^k \to S^k \times \mathbb{CP}^q$ and $j(\mathbb{CP}^q) : \mathbb{CP}^q \to S^k \times \mathbb{CP}^q$ be slice inclusions whose images are subspaces of the form $S^k \times \{y_0\}$ and $\{x_0\} \times \mathbb{CP}^q$ respectively, and let let $\pi(S^k) : S^k \times \mathbb{CP}^q \to S^k$ and $\pi(\mathbb{CP}^q) : S^k \times \mathbb{CP}^q \to \mathbb{CP}^q$ denote projections onto the respective factors. Then the composites $\pi(S^k) \circ f \circ j(S^k)$ and $\pi(\mathbb{CP}^q) \circ f \circ j(\mathbb{CP}^q)$ are homotopic to the corresponding identity mappings.

Proof. (i) Cup product considerations imply that the induced cohomology automorphism f^* of $H^*(S^k \times \mathbb{CP}^q; \mathbb{Z})$ is completely determined by its behavior on the generators in dimensions 2 and k, and it must be multiplication by ± 1 in each case. If χ is the conjugation involution on $\mathbb{CP}^q$, then $\mathbf{id}(S^k) \times \chi$ is multiplication by $+1$ on the k-dimensional generator and multiplication by -1 on the 2-dimensional generator, while if φ is reflection about a standard $(k-1)$- sphere in S^k then $\varphi \times \mathbf{id}(\mathbb{CP}^q)$ is multiplication by -1 on the k-dimensional generator and multiplication by $+1$ on the 2-dimensional generator. Finally, the product of these maps is multiplication by -1 on both generators. Thus every automorphism of $H^*(S^k \times \mathbb{CP}^q; \mathbb{Z})$ is in fact induced by a diffeomorphism.■

Proof of (ii). The composite self-map of S^k induces the identity in cohomology and hence is homotopic to the identity; similarly, the composite self-map of $\mathbb{CP}^q$ also induces the identity in cohomology, and a simple obstruction-theoretic argument shows that this composite must also be homotopic to the identity (the restrictions to $\mathbb{CP}^1$ are homotopic by the cohomological condition, and the obstructions to extending this to a homotopy of the original maps lie in the groups $H^{2i}\big(\mathbb{CP}^q, \mathbb{CP}^1; \pi_{2i}(\mathbb{CP}^q)\big)$, which are all trivial).■

The next step in analyzing the homotopy self-equivalences of $S^7 \times \mathbb{CP}^q$ can be done in a fairly general context. In the discussion below, X and Y will denote arcwise connected finite complexes with base points x_0 and y_0 respectively. If T is an arbitrary compact $\mathbf{T_2}$ space, then $\mathcal{E}(T)$ will denote the group of all homotopy classes of homotopy self-equivalences of T, and $E_1(T)$ will denote the arc component of the identity in the topological monoid of all self-maps of T (with the compact-open topology). If T is homeomorphic to a finite connected simplicial complex, then $E_1(T)$ has the homotopy type of a CW complex by a result of Milnor [Milnor 1959], and by a result of M. Sugawara [Sugawara] there is an inverse-up-to-homotopy map $\rho : E_1(T) \to E_1(T)$ such that the self-maps of $E_1(T)$ given by $c_1(f) = \rho(f) \circ f$ and $c_2(f) = f \circ \rho(f)$ are homotopic to the constant map on $E_1(T)$ whose value is 1_T.

As in Proposition 1, we define slice inclusions $i(X), i(Y) : X, Y \to X \times Y$ and projections $p(X), p(Y) : X \times Y \to X, Y$ by the standard formulas.

A factorization principle

As suggested in the title of this section, we are interested in self-equivalence groups of the form $\mathcal{E}(X \times Y)$, and especially in the subset $\mathcal{E}'(X \times Y)$ of all classes $[f]$ such that $p(X) \circ f \circ j(X) \simeq \mathbf{id}(X)$ and $p(Y) \circ f \circ j(Y) \simeq \mathbf{id}(Y)$. If $T = X$ and Y have torsion free integral cohomology and satisfy the condition

> *if g is a continuous self-map of T such that the induced map g^* in cohomology is the identity, then g is homotopic to the identity,*

then $\mathcal{E}'(X \times Y)$ is the normal subgroup of $\mathcal{E}(X \times Y)$ of all homotopy self-equivalences which induce the identity on integral cohomology; in particular, this holds if X is a sphere and Y is either a sphere or a complex or quaternionic projective space, and therefore $\mathcal{E}'(X \times Y)$ is a normal subgroup in the cases of interest to us.

Regardless of whether or not $\mathcal{E}'(X \times Y)$ is a subgroup of $\mathcal{E}(X \times Y)$, there are two important subsets of $\mathcal{E}'(X \times Y)$ that are subgroups. One of these is the image of a homomorphism $[X, E_1(Y)] \to \mathcal{E}'(X \times Y)$ defined as follows: Given a class in $[X, E_1(Y)]$, choose a base point preserving representative $g : X \to E_1(Y)$; then the adjoint or exponential isomorphism for function spaces

$$\mathfrak{F}\big(A, (B, C)\big) \quad \cong \quad \mathfrak{F}\big(A \times B, C\big)$$

(where denotes the continuous function space with the compact open topology and A, B, C are compact $\mathbf{T_2}$ spaces) implies that g is adjoint to a continuous map $g_\# : X \times Y \to Y$ whose restriction to $\{x_0\} \times Y$ is the identity; furthermore, if g' is homotopic to g then $g'_\#$ is homotopic to $g_\#$. The class $\alpha_X([g])$ is defined to be the class of the homotopy self-equivalence G such that $p(Y) \circ G = g_\#$ and $p(X) \circ G = p(X)$. This class lies in $\mathcal{E}'(X \times Y)$ because (i) the assumption that g is base point preserving implies that $p(Y) \circ G \circ j(Y) = g_\# \circ j(Y) = \mathbf{id}(Y)$, (ii) we also have $p(X) \circ G \circ j(X) = p(X) \circ j(X) = \mathbf{id}(X)$. Basic properties of adjoints imply that α_X is a well-defined homomorphism, and therefore its image is a subgroup of $\mathcal{E}'(X \times Y)$. Similar considerations apply if we interchange the roles of X and Y, and this yields a second homomorphism $\alpha_Y : [Y, E_1(X)] \to \mathcal{E}'(X \times Y)$.

PROPOSITION H.2. *If $\mathcal{E}'(X \times Y)$ is a subgroup of $\mathcal{E}(X \times Y)$, then every element in $\mathcal{E}'(X \times Y)$ can be expressed as a product $\alpha_Y(v)\alpha_X(u)$ for suitable $u \in [X, E_1(Y)]$ and $v \in [Y, E_1(X)]$.*

Since $X \times Y$ and $Y \times X$ are canonically homeomorphic, it also follows that classes in $\mathcal{E}'(X \times Y)$ can be expressed as composites of the form $\alpha_X(u')\alpha_Y(v')$ for suitable u' and v'.

Special cases of Proposition 2 are well-documented in the literature (*cf.* [Levine 1969]).

Proof. Suppose that f represents an element of $\mathcal{E}'(X \times Y)$, and let $g_0 = p(Y) \circ f$. By assumption $g_0|\{x_0\} \times Y$ is homotopic to the identity, and therefore by the Homotopy Extension Property we can deform g_0 to a map g_1 such that $g_1|\{x_0\} \times Y$ is the identity. Let $\overline{g} : X \to \mathfrak{F}(Y, Y)$ be adjoint to g_1; since $\overline{g}(x_0) = \mathbf{id}_Y$ and X is arcwise connected, it follows that the image of $\overline{g}$ is contained in $E_1(Y)$, and by an abuse of language we shall use $\overline{g}$ to denote the associated base point preserving map from X to $E_1(Y)$.

Let $\overline{h} = \rho(\overline{g})$, where ρ is an inverse-up-to-homotopy in $E_1(Y)$, let $h_\# : X \times Y \to Y$ be the adjoint map, let H be the representative for $\alpha_X\big([\overline{h}]\big)$ with $p(Y) \circ H = h_\#$ and $p(X) \circ H = p(X)$. Then the basic properties of the exponential/adjoint function space homeomorphism

$$\mathfrak{F}\big(X, \mathfrak{F}(Y, Y)\big) \quad = \quad \mathfrak{F}(X \times Y, Y)$$

imply that $g_1 \circ H : X \times Y \to Y$ is adjoint to

$$\mu\big(\overline{g}, \overline{h}\big) \quad = \quad \mu\big(\overline{g}, \rho(\overline{g})\big)$$

where $\mu : \mathfrak{F}(Y, Y) \times \mathfrak{F}(Y, Y) \to \mathfrak{F}(Y, Y)$ is the (continuous) composition mapping. Since the map sending φ to $\varphi \circ \rho(\varphi)$ from $E_1(X)$ to itself is nullhomotopic, it follows that $g_1 \circ H$ is homotopic to homotopic to the adjoint of the constant map with value $\mathbf{id}_Y$. Now the adjoint of this constant map is the coordinate projection $p(Y)$, and therefore we see that

$$p(Y) \circ f \circ H \quad = \quad g_0 \circ H \quad \simeq \quad g_1 \circ H \quad \simeq \quad p(Y) \ .$$

This means that $f \circ H$ is represented by a homotopy equivalence K in the image of α_Y, so that the homotopy class $[f]$ of f satisfies $[f] = [K] \circ [H]^{-1}$, where $[K]$ lies in the image of α_Y and $[H]^{-1}$ lies in the image of α_X.■

Homotopy groups of equivariant function spaces

Let G be a compact Lie group which is either abelian or finite for the sake of simplicity, and let V be a finite dimensional orthogonal representation of G such that G acts freely away from the origin (*i.e.*, a *free G-module*). Following [BeS], we shall denote the unit sphere of V by $S(V)$, and we shall let $F_G(V)$ denote the space of G-equivariant self-maps of $S(V)$ with the compact-open topology; when it is necessary to take a basepoint, the default choice will be the identity. As noted in [RS 1973] and [BeS], if W is another free G-module, then there is a stabilization homomorphism from $F_G(V)$ to $F_G(W)$, and the infinite stabilization will be denoted by F_G. The main results of [BeS] yield an isomorphism from F_G to the free infinite loop space $\mathbf{Q}\left(S^d BG_+\right)$, where $\mathbf{Q}(X)$ is the limit of the iterated loop spaces $\Omega^k S^k(X)$, the dimension of G is equal to d, the r-fold (reduced) suspension functor is denoted by S^r, and X_+ denotes the disjoint union of X and a point.

Results of I. M. James [James 1963] yield a close relationship between the homotopy groups of the space $E_1(\mathbb{CP}^q)$ which arose previously and the homotopy groups of $F_{S^1}(\mathbb{C}^{q+1})$, where $q \geq 2$. There is an obvious continuous homomorphism from $F_{S^1}(\mathbb{C}^{q+1})$ to $E_1(\mathbb{CP}^q)$ given by passage to quotients, and the results of [James 1963] show that this map induces isomorphisms from $\pi_k\left(F_{S^1}(\mathbb{C}^{q+1})\right)$ to $\pi_k\left(E_1(\mathbb{CP}^q)\right)$ for all $k \geq 2$ (it is not difficult to compute the fundamental groups in both cases, but this is not needed for our purposes). This will allow us to use the methods and results of [RS 1973] and [BeS] in studying $\pi_k\left(E_1(\mathbb{CP}^q)\right)$ for $k \geq 2$. The main tools will be the homotopy spectral sequences described in [RS 1973, Sections 1 and 5] and the relations among them. These spectral sequences arise from the long exact homotopy sequences associated to standard filtrations of function spaces and certain classical Lie groups. In the case of $F_{S^1}(\mathbb{C}^{q+1})$, the filtration is given by the submonoids $\mathrm{Filt}^{(2p-1)} = \mathrm{Filt}^{(2p)}$ of functions whose restrictions to the standard subspheres $S^{2q-2p+1} \subset S^{2q+1}$ are the standard inclusions, where $0 \leq p \leq q$; by convention, the submonoid of filtration $2q+1$ is the entire space. For the group U_{q+1}, a similar filtration is given by the standardly embedded unitary groups U_{p+1}, where p runs through the same set of values. There is an obvious inclusion of U_{q+1} in $F_{S^1}(\mathbb{C}^{q+1})$ which is compatible with these filtrations, and the results of [RS 1973, Section 5] relate the spectral sequences for the homotopy groups of these spaces. Other results in [RS 1973] describe spectral sequence mappings corresponding to the stabilization maps $F_{S^1}(\mathbb{C}^{q+1}) \subset F_{S^1}(\mathbb{C}^{q+2})$ given by taking double suspensions. In all cases, the terms $E^1_{s,t}$ are the relative homotopy groups $\pi_{s+t}(\mathrm{Filt}^{(s)}, \mathrm{Filt}^{(s-1)})$, and the latter turn out to be canonically isomorphic to certain homotopy groups of spheres.

Some of our computations could be done more efficiently if we used relationships between these spectral sequences and others which follow from [BeS], but we shall work entirely with the spectral sequences of [RS 1973] to avoid additional digressions.

One important step in [RS 1987, Section 4] was the computation of $\pi_7\left(F^1_S(\mathbb{C}^q)\right)$ where $q \geq 4$. This homotopy group was shown to be isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$, where the infinite cyclic summand corresponds to the image of $\pi_7(U_{q+1}) \cong \mathbb{Z}$. We shall need the corresponding description of $\pi_7\left(F^1_S(\mathbb{C}^3)\right)$; as we shall see, there are both similarities and differences between this group and the stable groups where $q \geq 4$.

NOTATION. The spectral sequences for the homotopy groups of the unitary group U_{q+1} which appear in [RS 1973, Theorem 5.2] will be denoted by $E^r_{s,t}(U_{q+1})$, and the previously discussed spectral sequences for the homotopy groups of the spaces $F_{S^1}(\mathbb{C}^{q+1})$ (which are called $G\mathbb{C}_{q+1}$ in [RS 1973]) will will be denoted by $E^r_{s,t}(G\mathbb{C}_{q+1})$; we are using the notation $G\mathbb{C}_{q+1}$ for the equivariant function space instead of $F_{S^1}(\mathbb{C}^{q+1})$ in order to simplify the notation in the next two results.

We shall begin with a list of formulas for differentials in some of these spectral sequences. All notation for elements in the homotopy groups of spheres is the same as in Toda's book [Toda].

PROPOSITION C.1. *In the preceding spectral sequences, one has the following differentials:*

(1) *The differential $d^2_{5,0}(U_3) : \pi_5(S^5) = \mathbb{Z} \to \pi_4(S^3) = \mathbb{Z}_2$ sends the generator of the domain to the generator of the codomain (which is the Hopf map $\eta_4 : S^4 \to S^3$).*

(2) *The differential $d^2_{5,0}(G\mathbb{C}_3) : \pi_5(S^5) = \mathbb{Z} \to \pi_6(S^5) = \mathbb{Z}_2$ sends the generator of the domain to the generator of the codomain (which is the Hopf map $\eta_6 : S^6 \to S^5$).*

(3) *The differential $d^2_{5,0}(G\mathbb{C}_3) : \pi_7(S^5) = \mathbb{Z}_2 \to \pi_8(S^5) = \mathbb{Z}_{24}$ sends the generator of the domain to the unique element in of the codomain of order 2. In particular, this differential is injective.*

(4) *For each $q \geq 3$, the generator of $E^2_{1,6}(G\mathbb{C}_{q+1}) = \mathbb{Z}_2$ is a permanent cycle, and the associated class in $\pi_7 \left(F_{S^1} \right)$ is the unique element of order 2 in the latter.*

Proof(s). The validity of (1) follows because this is the only choice of differential which is compatible with the fact that $\pi_4(U_3) = 0$, and (2) then follows because the map from $\pi_4(S^3) = E^2_{3,1}(U_3)$ to $\pi_6(S^5) = E^2_{3,1}(G\mathbb{C}_3)$ is given by double suspension [RS 1973, Thm. 5.2, p. 70], and this map is bijective [Toda, Proposition 5.1, p. 39]. Formula (3) now follows because $\pi_7(S^5)$ is generated by the square of the Hopf map from S^7 to S^5 [Toda, Proposition 5.2, p. 42]; this means we can use the composition operations in the spectral sequence (see [RS 1973, Prop. 1.4, p. 54] to show that the image of the generator in $E^2_{5,2}(G\mathbb{C}_3) = \pi_7(S^5)$ maps to the composition cube of the Hopf map in $E^2_{3,3}(G\mathbb{C}_3) = \pi_8(S^5)$, and this maps onto the unique element of order two by the Toda formula $\eta^3 = 4\nu$ (see [Toda, formula (5.5), p. 42; here $\nu \in \pi^{\mathbf{S}}_3$ is the stabilized Hopf map).

For the sake of brevity, we shall use some computations from [RS 1987, Section 4] to verify (4). We start with the case $q \geq 3$. First of all, the stabilization maps for spectral sequences from [RS 1973, Theorem 3.2, p. 64] imply that all of the differentials $d^2_{5,0}(G\mathbb{C}_q) : \pi_{2q+1}(S^{2q+1}) = \mathbb{Z} \to \pi_{2q+2}(S^{2q+1}) = \mathbb{Z}_2$ must be nontrivial, and likewise for the differentials $d^2_{5,0}(G\mathbb{C}_{q+1}) : \pi_{2q+3}(S^{2q+1}) = \mathbb{Z}_2 \to \pi_{2q+4}(S^{2q+1}) = \mathbb{Z}_{24}$. It follows that if $q \geq 4$ then $E^\infty_{s,7-s}(G\mathbb{C}_{q+1})$ is trivial unless $s = 1$ or $s = 7$, and the group $E^\infty_{7,0}(G\mathbb{C}_{q+1}) = \mathbb{Z}$ is isomorphic to $E^\infty_{7,0}(U_{q+1}) = \mathbb{Z}$. Furthermore, since $E^2_{1,6} = \mathbb{Z}_2$, it follows that if $E^\infty_{1,6} = 0$ then the canonical map from $\pi_7(U_q)$ to $\pi_7 \left(F_{S^1}(\mathbb{C}^{q+1}) \right)$ is an isomorphism; the results of [RS 1987] show this is false, and therefore we must have $E^2_{1,6} = E^\infty_{1,6} = \mathbb{Z}_2$. Furthermore, the nonzero class in the latter generates the torsion subgroup. This completes the argument for $q \geq 3$. To recover the case $q = 2$, it suffices to note that the stabilization map from $\pi_{11}(S^5)$ to $\pi^{\mathbf{S}}_6$ is bijective (see [Toda, Prop. 5.11, p. 46]), and therefore the mapping properties of spectral sequences imply that $E^2_{1,6} = E^\infty_{1,6}$ also holds in this case.■

Our computation for $\pi_7 \left(F_{S^1}(\mathbb{C}^3) \right)$ is less complete than the result in the stable case, but it will suffice for our purposes.

PROPOSITION C.2. *The group $\pi_7 \left(F_{S^1}(\mathbb{C}^3) \right)$ is finite, its image in $\pi_7(F_{S^1})$ is the torsion subgroup of the latter, and the kernel of the stabilization homomorphism from $\pi_7 \left(F_{S^1}(\mathbb{C}^3) \right)$ to $\pi_7(F_{S^1})$ is of order at most 2.*

Proof. We know that $E^\infty_{s,7-s}(G\mathbb{C}_3) = 0$ if $s \neq 1, 3$ and $E^\infty_{1,6} = \mathbb{Z}_2$, where the latter maps stably into $\pi_7(F_{S^1})$. Therefore the only uncertainty involves the group $E^\infty_{3,4}(G\mathbb{C}_3)$. The results of [Toda, Prop. 5.8, p. 43] imply that the latter equals $\mathbb{Z}_2$ and that $E^\infty_{3,4}(G\mathbb{C}_{q+1}) = 0$ for $q \geq 3$. This means that any nonzero element of $E^\infty_{3,4}(G\mathbb{C}_3)$ would correspond to an element of $E^\infty_{1,6}(G\mathbb{C}_{q+1}) = \mathbb{Z}_2$ for all $q \geq 3$. On the other hand, we know that the groups $E^\infty_{1,6}(G\mathbb{C}_{q+1}) = \mathbb{Z}_2$ map isomorphically under stabilization for all $q \geq 2$, and therefore it follows that if there is a nonzero class in $E^\infty_{3,4}(G\mathbb{C}_3)$, then there is also a class of this type which represents a stably trivial element of $\pi_7\left(F_{S^1}(\mathbb{C}^3)\right)$. The bound on the order of the stabilization homomorphism now follows from the preceding estimates for the groups $E^{\infty}_{s,t}(G\mathbb{C}_3)$ when $s > 1$. ∎

Tangential structure sets

For our purposes it is useful to consider a well-known variant of the ordinary surgery exact sequence for tangential homotopy equivalences [Madsen-Taylor-Williams]; a **tangential homotopy structure** on a manifold X is a triple (M, f, Φ) such that (M, f) is a homotopy structure in the previous sense and Φ is a map of the total spaces of the (stable) tangent bundles $E(\tau_M \oplus \mathbb{R}^k) \to E(\tau_X \oplus \mathbb{R}^k)$ (for some $k \geq 0$) such that the following diagram is commutative:

$$\begin{array}{ccc} E(\tau_M \oplus \mathbb{R}^k) & \xrightarrow{\Phi} & E(\tau_X \oplus \mathbb{R}^k) \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

The correct equivalence relation for such structures (M, f, Φ) and (N, g, Ψ) is stated in [Madsen-Taylor-Williams]; informally, one allows stabilization by taking direct sums with trivial bundles, and one relates the stable tangent bundles of M and N using the vector bundle isomorphism of tangent spaces $\mathbf{T}(h) : \mathbf{T}(M) \to \mathbf{T}(N)$ associated to a diffeomorphism h by taking the derivative on each fiber. We shall denote the set of equivalence classes of tangential homotopy structures on M by $\mathbf{S}^{s,t}(M)$; there is a natural map from the latter to $\mathbf{S}^s(M)$ obtained by forgetting the bundle data.

One then has an analog of the surgery exact sequence

$$\cdots \;\to\; L^s_{\dim X+1}\big(\pi_1(X), w\big) \;\to\; \mathbf{S}^{s,t}(X) \;\to\; [X, F] \;\to\; L^s_{\dim X}\big(\pi_1(X), w\big)$$

(assuming here that X is has not boundary) in which F is the stabilized monoid $\lim_{n\to\infty} G_{n+1}$, where G_{n+1} maps to G_{n+2} by taking unreduced suspensions. If M is connected, then by [Wh] there is an isomorphism of sets

$$[M, F] \;\cong\; \{M, S^0\}$$

where the right hand side is the stable homotopy classes of maps from M into S^0, but it is important to recognize that this map is usually not a homomorphism with respect to the standard algebraic structures on the domain and codomain. We shall be particularly interested in the map $\mathfrak{q}^t : \mathbf{S}^{s,t}(X) \to [X, F]$, which is a refinement of the normal invariant and will be denoted by $\mathfrak{q}^t$. Finally, there is an expected sort of commutative diagram whose rows are the tangential and ordinary structure sets, and whose vertical arrows are the corresponding forgetful maps or identities, and this diagram fits into an interlocking exact sequence associated to homotopy classes of maps from X to the spaces in the fibration sequence $F \to F/O \to BO$; for example, there is an associated map from $[X, O]$ to $\mathbf{S}^{s,t}(X)$ whose value is given by $(X, 1_X, \Phi)$, where Φ is a stable vector bundle automorphism of $X \times \mathbb{R}^k$ associated to a class in $[X, O_k]$ for k sufficiently large, and one has the following exact sequence:

$$\cdots \;\longrightarrow\; [M, O] \;\longrightarrow\; \mathbf{S}^{s,t}(M) \;\longrightarrow\; \mathbf{S}^s(M) \;\longrightarrow\; [M, BO] \;\cong\; \widetilde{KO}(M)$$

Up to a canonical choice of sign, the final map in this exact sequence takes the class represented by (X, f) to the difference of the stable vector bundles τ_M and $g^*\tau_X$, where g is a homotopy inverse to f.

We shall need to take all of this one step further and consider tangential structure sets on manifolds with boundary such that the associated map on the boundary is a diffeomorphism; in contrast to our default hypotheses, we do not necessarily assume that the boundary is connected. The relative counterparts of ordinary structure sets are standard constructions, and the analog for the tangential case fits into the following exact sequence:

$$\cdots \;\rightarrow\; L_{\dim X+1}\big(\pi_1(X), w\big) \;\rightarrow\; \mathbf{S}^{s,t}(X \text{ rel } \partial X) \;\rightarrow\; [(X/\partial X), F] \;\rightarrow\; L_{\dim X}\big(\pi_1(X), w\big)$$

As before, there is an expected sort of commutative diagram whose rows are the tangential and ordinary structure sets, and whose vertical arrows are the corresponding forgetful maps or identities. In particular, if Y is a closed manifold then one can consider the bounded product manifolds $(D^k \times Y, S^{k-1} \times Y)$, and familiar homotopy-theoretic considerations imply that the relative structure sets

$$\mathbf{S}^{s,t}_k(Y) \;\;:=\;\; \mathbf{S}^{s,t}(D^k \times Y \text{ rel } S^{k-1} \times Y)$$

have group structures ("track addition"), which are abelian if $k \geq 2$, and in such cases all the maps in the relative surgery sequences are group homomorphisms; in fact, one can use "spacification" techniques as in [Rourke 1968] to show that the surgery sequence comes from the exact homotopy sequence associated to some fibration.

The preceding machinery is useful in the study of normal invariants a homotopy self-equivalence of $S^k \times \mathbb{CP}^q$ which comes from a class $\alpha \in \pi_k(F_{S^1}\big(\mathbb{C}^{q+1})\big)$. Since the associated homotopy self-equivalences of $S^k \times \mathbb{CP}^q$ are orbit space maps arising from equivariant homotopy self-equivalences of $S^k \times S^{2q+1}$, it follows that if $\widetilde{f}$ is an equivariant self-map of representing α which projects to f on $S^k \times \mathbb{CP}^q$, then the balanced product construction $\widetilde{f} \times_{S^1} \mathbf{id}(\mathbb{C})$ defines a vector bundle isomorphism of the canonical line bundle on $S^k \times \mathbb{CP}^q$ which covers f, and since the stable tangent bundle of $\mathbb{CP}^q$ is stably a direct sum of $(q+1)$ copies of the canonical line bundle (*e.g.*, see [Milnor-Stasheff]), it follows that we obtain an **explicit** tangential homotopy structure on $S^k \times \mathbb{CP}^q$ which refines the ordinary homotopy structure $(S^k \times \mathbb{CP}^q, f)$. As in [RS 1978] this passes to a homomorphism

$$\psi : \pi_k\big(\, F_{S^1}(\mathbb{C}^{q+1})\big) \;\longrightarrow\; \mathbf{S}^{s,t}_k(\mathbb{CP}^{q+1})$$

and we can compose this with the refined normal invariant mapping into

$$\big[\,(D^k \times \mathbb{CP}^q / S^{k-1} \times \mathbb{CP}^q,\; F\,\big] \;\;\cong\;\; \{S^k\mathbb{CP}^q_+, S^0\}\;;$$

by standard homotopy-theoretic considerations, the displayed map is actually an isomorphism of groups when $k \geq 1$.

PROPOSITION A.1. *In the preceding setting, suppose that $k + 2q \equiv 3 \bmod 4$ and we are given a homotopy self-equivalence f of $S^k \times \mathbb{CP}^q$ which comes from an equivariant homotopy self-equivalence of $S^k \times S^{2q+1}$ defined by an element α of finite order in the group $\pi_k\big(\, F_{S^1}(\mathbb{C}^{q+1})\big)$, and assume that the refined normal invariant of f in $\{S^k\mathbb{CP}^q_+, S^0\}$ is trivial. Then f is homotopic to a diffeomorphism.*

In general, if we are given a homotopy self-equivalence f of a manifold X which is normally cobordant to the identity (compare [Cappell-Shaneson], [Kwasik-Schultz 1989, 1995], [Masuda-Schultz]), it is very difficult to determine whether f is homotopic to a diffeomorphism, and the preceding result gives a simple condition under which we can answer such a question affirmatively. The assumption that α have finite order is far less restrictive than it might seem, for we know that all elements of the homotopy group $\pi_k\big(\, F_{S^1}(\mathbb{C}^{q+1}\big)$ can be expressed as the sum of an element of finite order with an element in the image of $\pi_k(U_{q+1})$ which is either zero or of infinite order (*cf.* [BeS, Theorem 11.1] in the stable case; in the unstable case the homotopy groups are finite by [RS 1973]).

Proof. Since the map from $\pi_k\big(F_{S^1}(\mathbb{C}^{q+1})\big)$ to $\mathbf{S}^{s,t}_k(\mathbb{CP}^q)$ is a homomorphism, it follows immediately that the image of α in the latter has finite order. On the other hand, consider the following portion of the tangential surgery sequence:

$$\{S^{k+1}\mathbb{CP}^q)_+, S^0\} \ \to \ L_{k+2q+1}(\{1\}) \cong \mathbb{Z} \ \to \ \mathbf{S}^{s,t}_k(\mathbb{CP}^q) \ \to \{S^k\mathbb{CP}^q_+, S^0\}$$

We know that $\{X, S^0\}$ is finite if X is a connected finite complex, and therefore the map from $\mathbb{Z}$ into the structure set must be a monomorphism. But this means that if an element in the tangential structure set has a trivial refined normal invariant, then this element must be trivial. This applies to the class α, and therefore it follows that the image of α in the ordinary relative structure set $\mathbf{S}^s_k(\mathbb{CP}^q)$ must also be trivial.

For the final step of the proof, we shall need the standard decomposition of S^k as the union of the upper and lower hemispheres UD^k and LD^k; by construction, their intersection is the equatorial sphere $S^{k-1} \subset S^k$. There is a canonical map Γ from

$$\mathbf{S}^s_k(\mathbb{CP}^q) \quad \cong \quad \mathbf{S}^s(UD^k \times \mathbb{CP}^q \ \mathrm{rel}\ S^{k-1} \times \mathbb{CP}^q)$$

to $\mathbf{S}^s(S^k \times \mathbb{CP}^q)$, which is obtained by attaching a copy of $LD^k \times \mathbb{CP}^q$ along the boundary. Explicitly, if we are give a representative

$$f : (W, \partial W) \ \longrightarrow \ (UD^k \times \mathbb{CP}^q, S^{k-1} \times \mathbb{CP}^q)$$

of a relative structure, so that the boundary map ∂f is a diffeomorphism, then we take

$$V \quad = \quad W \ \cup_{\partial f} \ LD^k \times \mathbb{CP}^q$$

and let $g : V \to S^k \times \mathbb{CP}^q$ be the well-defined map which is given by f on W and the identity on $LD^k \times \mathbb{CP}^q$. Since this construction sends the identity to itself and the original map α maps to zero in the domain, it follows that $\Gamma(\alpha)$ in the codomain must be homotopic to a diffeomorphism.∎

COROLLARY A.2. *Let $\alpha \in \pi_7\big(F_{S^1}(\mathbb{C}^3)\big)$ be an element which determines a nontrivial class of $E^\infty_{3,4}$ of the spectral sequence for the homotopy group in question and stabilizes to zero in $\pi_7(F_{S^1})$. Then the image of α in $\mathbf{S}^s(S^k \times \mathbb{CP}^2)$ is trivial.*

Proof. As in [RS 1975] we have the following commutative diagram, in which the vertical arrow on the left is induced by the stabilization map from $F_{S^1}(\mathbb{C}^3)$ to $F_{S^1}(\mathbb{C}^{q+1})$ for $q \geq 3$ and the vertical arrow on the right is induced by the inclusion map from $\mathbb{CP}^2$ to $\mathbb{CP}^q$:

$$\begin{array}{ccc}
\pi_k\left(F_{S^1}(\mathbb{C}^3)\right) & \xrightarrow[\text{normal invt.}]{\text{refined}} & \{S^k\mathbb{CP}^2_+,\ S^0\} \\
\downarrow & & \uparrow \\
\pi_k\left(F_{S^1}(\mathbb{C}^{q+1})\right) & \xrightarrow[\text{normal invt.}]{\text{refined}} & \{S^k\mathbb{CP}^q_+,\ S^0\}
\end{array}$$

Since the image of α under stabilization is trivial. it follows that the top horizontal arrow maps α to zero, so that α satisfies the condition in the preceding proposition.∎

An additivity formula

The following identity is fairly elementary, but it will be useful later in this paper.

THEOREM A.3. *Let X be a closed smooth manifold of dimension ≥ 3, let $k \geq 2$ be an integer, let $\mathbf{S}^s_k(X)$ be the relative structure set for the manifold $(D^k \times X, S^{k-1} \times X)$ as described above, let*

$$\Gamma : \mathbf{S}^s_k(X) \longrightarrow \mathbf{S}^s(S^k \times X)$$

be the canonical map defined as above by gluing on copies of $LD^k \times X$ along the boundary, and let $\psi : \pi_k\big(E_1(X)\big) \to \mathbf{S}^s_k(X)$ be the map constructed as above. Then the following diagram is commutative:

$$\begin{array}{ccccc}
\pi_k\,(\,E_1(X)\,) \times \mathbf{S}^s_k(X) & \xrightarrow{\psi \times \mathbf{id}} & \mathbf{S}^s_k(X) \times \mathbf{S}^s_k(X) & \xrightarrow{\text{addition}} & \mathbf{S}^s_k(X) \\
\big\downarrow \alpha \times \Gamma & & & & \big\downarrow \Gamma \\
\mathcal{E}(S^k \times X) \times \mathbf{S}^s(S^k \times X) & \xrightarrow[\text{action}]{\text{standard}} & \mathbf{S}^s(S^k \times X) & \xrightarrow{=} & \mathbf{S}^s(S^k \times X)
\end{array}$$

The map "standard action" is given by the action of the self-equivalence group on the structure set.

Proof. This will follow quickly if we choose representatives for classes in $\pi_k\,(\,E_1(X)\,)$ and $\mathbf{S}^s_k(X)$ carefully. The map from the first group to the second is defined by taking a base point preserving representative $S^k \to E_1(X)$ and composing it with the collapsing map from D^k to $D^k/S^{k-1} \cong S^k$. In fact, we can choose this representative to be constant on the (image of) a small disk D^* contained in the interior of the annulus

$$A \;=\; \{\, \mathbf{v} \in D^k \mid \tfrac{1}{2} \;\leq\; |\mathbf{v}| \;\leq\; 1 \,\}$$

so assume we have done so. Next, we can choose the homotopy structure so that it induces a map of triads

$$(M; M_0, M_1) \;\longrightarrow\; \big(D^k \times X; \tfrac{1}{2}D^k \times X, A \times X\big)$$

so that the map on M_1 is a diffeomorphism. For these choices one can check directly that the action of the homotopy self-equivalence on a homotopy structure is given by the addition operation in the structure set.■

Normal invariants

In a previous section we identified an exotic sphere Σ^7 which can be viewed as a standard generator for the Kervaire-Milnor group $\Theta_7 \cong \mathbb{Z}_{28}$. If we choose an orientation preserving homeomorphism $h : \Sigma^7 \to S^7$, then $h \times \mathbf{id}(\mathbb{CP}^{2m})$ defines an element σ of the structure set $\mathbf{S}^s(S^7 \times \mathbb{CP}^{2m})$, and since Σ^7 bounds a parallelizable manifold by definition, it follows that the normal invariant of this homotopy structure is trivial. If $\Sigma^7 \times \mathbb{CP}^{2m}$ is diffeomorphic to $S^7 \times \mathbb{CP}^{2m}$, then σ and the standard base point of the structure set (namely, the class of the identity on $S^7 \times \mathbb{CP}^{2m}$) both lie in the same orbit of the action of $\mathcal{E}(S^7 \times \mathbb{CP}^{2m})$, so we clearly need some information on the orbit of the standard base point under the action of $\mathcal{E}(S^7 \times \mathbb{CP}^{2m})$. The main result can be stated very simply:

THEOREM N.1. (Dichotomy Principle) *Let f be a homotopy self-equivalence of $S^7 \times \mathbb{CP}^{2m}$; then the image of (the homotopy class of) f in the structure set $\mathbf{S}^s(S^7 \times \mathbb{CP}^{2m})$ is trivial if and only if its normal invariant is trivial.*

In other words, **either** *f is homotopic to a diffeomorphism* **or else** *f is not even normally cobordant to the identity*.

Note. The results in the final section of [Masuda-Schultz] illustrate the complexity of determining the class of an arbitrary homotopy self-equivalence $g : M \to M$ in the corresponding structure set $\mathbf{S}^s(M)$.

Dichotomy principles

Following James [James 1954], we shall use G_{k+1} to denote $\mathfrak{F}(S^k, S^k)$, and its identity component will be denoted by SG_{k+1}. Likewise, the submonoid of base point preserving self-maps of degree 1 will be denoted by SF_k. The results of [James 1954] and the existence of classifying spaces for connected topological monoids then yields an exact sequence of spaces

$$S^k \longrightarrow BSF_k \longrightarrow BSG_{k+1} \ .$$

Furthermore, results of G. Whitehead [Wh] imply that the space SF_k has the homotopy type of the component of the constant map in the iterated loop space $\Omega^k S^k$, and therefore $\pi_n(SF_k) \cong \pi_{n+k}(S^k)$ for all $n \geq 1$. Note that unreduced suspension defines continuous monoid homomorphisms $G_{k+1} \to G_{k+2}$ and $SG_{k+1} \to SG_{k+2}$. Furthermore, the limiting object SF satisfies $\pi_n(SF) \cong \pi_n^{\mathbf{S}}$, where the codomain is the n^{th} stable homotopy group of spheres, and the limiting object also fits into an exact sequence of topological spaces

$$SF_\infty \longrightarrow F/O \longrightarrow BSO \ .$$

In fact, this sequence is infinitely deloopable [Boardman-Vogt], but we shall not need this fact.

PROPOSITION N.2. (Dichotomy property) *Let X be a closed connected smooth k-manifold, let $n \geq 2$ such that $n+k \geq 5$, let $u : X \to SG_{n+1}$ be continuous, and let $f : X \times S^n \to X \times S^n$ denote the homotopy self-equivalence of X which is determined by the adjoint map $u_\# : X \times S^n \to S^n$. Then either f is homotopic to a diffeomorphism, or else f is not normally cobordant to the identity. In the first case, the diffeomorphism extends to a diffeomorphism of $X \times D^{n+1}$.*

Similar results hold in the piecewise linear and topological categories with the corresponding notions of (piecewise linear and topological) homeomorphisms, bundle isomorphisms and normal invariants; the proofs in all cases are formally analogous.

Proof. The most important point is that every homotopy self-equivalence of S^n extends to D^{n+1} by the cone construction, which implies that f extends to a homotopy self-equivalence F of $X \times D^{n+1}$ and hence yields a homotopy structure on $X \times D^{n+1}$. By Wall's $\pi - \pi$ Theorem [Wall book, Chapter 3], the map F is homotopic to a diffeomorphism if and only if its normal invariant is trivial. Since the restriction map from $[X \times D^{n+1}, F/O] \cong [X, F/O]$ to $[X \times S^n, F/O]$ is split injective and the normal invariant of F maps to the normal invariant of f by restriction, it follows that the normal invariant of f is trivial if and only if the normal invariant of F is trivial, and in this case both F and f must be homotopic to diffeomorphisms.■

One step in the preceding argument is important enough to be cited explicitly; we shall give a strengthened form of this result.

COROLLARY N.3. *In the setting above, the normal invariant of f lies in the image of $[X, F/O]$ in $[X \times S^n, F/O]$. Furthermore, if A is a closed subset of X and the restriction of the original class is trivial in $[A, SG_{n+1}]$, then the restrictions of the normal invariant to A and $A \times S^n$ are also trivial.*

Proof. The first sentence is an immediate consequence of the proposition. To prove the second part, recall that SG_{n+1} has the homotopy type of a CW complex, and therefore it follows that if the restriction of $u : X \to SG_{n+1}$ to A is nullhomotopic, then the same is true if A is replaced by an open neighborhood U (of A). General considerations imply that there is a closed neighborhood B of A such that $B \subset U$ and B is a manifold with boundary (which contains A in its interior); by the Homotopy Extension Property we may replace u with some v in the same homotopy class such that the restriction of v to B is constant (with value 1_X). Let g be the homotopy self-equivalence of $X \times S^n$ formed from the adjoint of g. Then g maps $B \times S^n$ to itself by the identity, and it also maps $\overline{X - B}$ to itself. Standard restriction properties of normal invariants imply that the restriction of the normal invariant of g to $B \times S^n$ is trivial, and this implies the same conclusion for the restriction to $A \times S^n$.■

There is a similar result for homotopy self-equivalences of $S^7 \times \mathbb{CP}^{2m}$ which come from $\pi_7\big(E_1(\mathbb{CP}^2)\big) \cong \pi_7\big(F_{S^1}(\mathbb{C}^3)\big)$.

PROPOSITION N.4. *Let f be a homotopy self-equivalence of $S^7 \times \mathbb{CP}^{2m}$ which comes from an element of $\pi_7\big(F_{S^1}(\mathbb{C}^3)\big)$. Then either f is homotopic to a diffeomorphism or else f has a nontrivial normal invariant.*

Proof. If $m \geq 2$ then this is given by [RS 1987, Prop. 4.2]. Suppose now that $m = 1$. Then by Proposition H.2 the map f is homotopic to $f_1 \circ f_2$, where each factor has order at most 2, the map f_1 comes from an element of $E^{\infty}_{1,6} \subset \pi_7\big(F_{S^1}(\mathbb{C}^3)\big)$ and f_2 is stably trivial. In the preceding section we saw that f_2 is homotopic to a diffeomorphism, and the results of Section C and [RS 1987] show that either f_1 is homotopic to the identity or else it maps to the element of order 2 in $\pi_7(F_{S^1} \cong \mathbb{Z} \oplus \mathbb{Z}_2)$ and its normal invariant is nontrivial.■

Skeletal filtrations

Let T be a contravariant functor defined from the homotopy category of pointed finite cell complexes to the category of abelian groups. If X is a pointed finite cell complex and $u \in T(X)$, then we shall say that u *has skeletal filtration* $\geq k$ if the restriction of u to the k-skeleton X_k is trivial, and we shall say that the skeletal filtration of a class is equal to k if the class has skeletal filtration $\geq k$ but does not have skeletal filtration $\geq k + 1$; see [Mosher-Tangora] for more on this concept.

The standard Cellular Approximation Theorem for continuous maps of CW complexes implies that the skeletal filtration of a class in $T(X)$ does not depend upon the choice of cell decomposition; in fact, it follows that the sets $T^{\langle k \rangle}(X)$ of elements with skeletal filtration $\geq k$ are subgroups and define a filtration of T by subfunctors.

We are interested in the skeletal filtrations of normal invariants associated to homotopy self-equivalences of $S^7 \times \mathbb{CP}^q$ which come from the group $[\mathbb{CP}^q, SG_8] = [\mathbb{CP}^q, E_1(S^7)]$. The first step is fairly elementary:

LEMMA N.5. *The group* $[\mathbb{CP}^2, SG_8]$ *is trivial.*

Proof. We know that $[\mathbb{CP}^2, S^7]$ is trivial because the dimension of $\mathbb{CP}^2$ is less than the connectivity of S^7, so by the exact sequence of spaces $SF_7 \to SG_8 \to S^7$ and the isomorphism

$$[\mathbb{CP}^2, SF_7] \quad \cong \quad [S^7\mathbb{CP}^2, S^7]$$

it will suffice to show that the latter vanishes. Now $S^7\mathbb{CP}^2$ is the mapping cone of $S^5\eta_2$, where $\eta_2 : S^3 \to S^2$ is the Hopf map, and therefore we have the following exact sequence:

$$\pi_{11}(S^7) \quad \longrightarrow \quad [S^7\mathbb{CP}^2, S^7] \quad \longrightarrow \quad \pi_9(S^7) \xrightarrow{(S^5\eta)^*} \quad \pi_{10}(S^7)$$

Fundamental results on the homotopy groups of spheres (from [Toda, Ch. V]) imply that $\pi_{11}(S^7) = 0$, $\pi_9(S^7) \cong \mathbb{Z}_2$, and $(S^5\eta)^*$ is injective; combining these, we see that $[\mathbb{CP}^2, SF_7]$ must be trivial, and as noted above the conclusion of the lemma follows from this.■

COROLLARY N.6. *Suppose that f is a homotopy self-equivalence of $S^7 \times \mathbb{CP}^q$ $(q \geq 2)$ which comes from an element of $[\mathbb{CP}^q, SG_8]$. If the normal invariant of f is nontrivial, then its filtration is an even number strictly greater than* 4.

Proof. We might as well consider the normal invariant for the extended homotopy self-equivalence of $D^8 \times \mathbb{CP}^q$. Since $\mathbb{CP}^q$ has cells only in even dimensions, it follows that the filtration of a nontrivial element cannot be odd and hence must be even. Since $[\mathbb{CP}^2, SG_8]$ is trivial, it follows that the restriction of f to $S^7 \times \mathbb{CP}^2$ is the identity, and as in Corollary N.3 it follows that the restriction of the normal invariant to $[\mathbb{CP}^2, F/O]$ is trivial, so that the skeletal filtration of the normal invariant is at least 4.■

Combining the preceding results with others from earlier sections, we obtain the following strong conclusion about the homotopy structures of $\mathbf{S}^s(S^7 \times \mathbb{CP}^{2m})$ which come from elements of $\mathcal{E}'(S^7 \times \mathbb{CP}^{2m})$.

PROPOSITION N.7. *Suppose that we are given a homotopy equivalence f of $S^7 \times \mathbb{CP}^{2m}$ (where $m \geq 1$) which represents an element of $\mathcal{E}'(S^7 \times \mathbb{CP}^{2m})$, and express f as a composite $f_1 \circ f_2$, where f_2 comes from $[\mathbb{CP}^{2m}, SG_8]$ and f_1 comes from $\pi_7\big(F_{S^1}(\mathbb{C}^{2m+1})\big)$. Then the normal invariant of f is trivial if and only if the normal invariants of f_1 and f_2 are trivial.*

Proof. The starting point for this argument is the following standard formula for normal invariants of composites (see [Ranicki 2009] or [RS 1971]).

$$\mathfrak{q}(g \circ h) \quad = \quad \mathfrak{q}(g) \ + \ (g^*)^{-1} \, \mathfrak{q}(h)$$

The sum is taken with respect to the usual direct sum operation on F/O. The "if" direction follows immediately from this formula, so for the rest of this argument we shall look at the "only if" direction.

We begin with one simple consequence of the composition formula: If $g \circ h$ has trivial normal invariant, then either the normal invariants of g and h are both trivial or else these normal invariants are both nontrivial.

The results of the previous section imply that $\mathfrak{q}(f_1) = 0$ if and only if f_1 is homotopic to a diffeomorphism, and in fact the results of [RS 1987] imply that if $\mathfrak{q}(f_1) \neq 0$ then its filtration is equal to 9 (see the proof of Proposition 4.2 on p. 197). On the other hand, the results of this section show that either $\mathfrak{q}(f_2) = 0$ or else its filtration is even; general considerations then imply the same conclusion for $(f_1^*)^{-1}\mathfrak{q}(f_2)$. It follows that if $\mathfrak{q}(f_1) \neq 0$ and $\mathfrak{q}(f_2) \neq 0$, then $\mathfrak{q}(f_1 \circ f_2)$ is a nonzero element whose skeletal filtration is equal to the smaller of 9 and the filtration of $\mathfrak{q}(f_2) \neq 0$, which is even and hence not equal to 9. Therefore, if the normal invariant of $f_1 \circ f_2$ is trivial, then the normal invariants of both f_1 and f_2 must be trivial.■

Proof of Theorem N.1. Let g be a homotopy self-equivalence of $S^7 \times \mathbb{CP}^2$. By Proposition H.1 we know that $g = f \circ h$, where h is a self-diffeomorphism and f represents an element of $\mathcal{E}'(S^7 \times \mathbb{CP}^{2m})$. This means that $(S^7 \times \mathbb{CP}^{2m}, g)$ and $(S^7 \times \mathbb{CP}^{2m}, f)$ determine the same element in $\mathbf{S}^s(S^7 \times \mathbb{CP}^2)$. By Proposition H.2 we know that f is homotopic to $f_1 \circ f_2$, where f_2 comes from $[\mathbb{CP}^{2m}, SG_8]$ and f_1 comes from $\pi_7\big(F_{S^1}(\mathbb{C}^{2m+1})\big)$. We need to show that if f does not determine the trivial element in the structure set, then $\mathfrak{q}(f)$ is nontrivial. By the previous results of this section, we know that the corresponding statement holds for both f_1 and f_2. Furthermore, we also know that $\mathfrak{q}(f) = \mathfrak{q}(f_1 \circ f_2)$ is trivial if and only if $\mathfrak{q}(f_1)$ and $\mathfrak{q}(f_2)$ are trivial. Therefore, if $\mathfrak{q}(f)$ is trivial then both f_1 and f_2 must be homotopic to diffeomorphisms, and consequently we know that f must also be homotopic to a diffeomorphism.■

More surgery sequence computations

The Kervaire-Milnor groups Θ_n have important subgroups ∂P_{n+1} consisting of all exotic spheres which bound parallelizable manifolds. These groups are finite cyclic groups whose orders are known in almost all cases; in particular, if $n = 4r - 1 \geq 7$ is congruent to 3 mod 4, then the order θ_r of ∂P_{4r} is given by the formula

$$a_r 2^{2r-2}(2^{2r-1} - 1)\,\mathrm{numerator}(B_r/4r)$$

where a_r is 2 if r is odd and 1 if r is even, and B_r is the corresponding Bernoulli number. Basic results in number theory imply that the last factor is odd and is equal to a product of irregular primes.

As noted in [Kervaire-Milnor] it is well known that $\Theta_7 = \partial P_8$ and $\Theta_{11} = \partial P_{12}$; By the displayed formula, the orders of these groups are $28 = 4 \cdot 7$ and $992 = 32 \cdot 31$ respectively.

Given an integer d, for each integer $r \geq 2$ the results of [Kervaire-Milnor] show that there is a unique oriented homotopy sphere $\Sigma^{4r-1}(d)$ which bounds a parallelizable manifold whose signature is equal to $8d$. In particular, $\Sigma^7(1)$ is the standard generator for Θ_7 that we described earlier. It also follows immediately that $\Sigma^{2r-1}(0) = S^{4r-1}$ and $\Sigma^{2r-1}(d_1 + d_2) \cong \Sigma^{2r-1}(d_1)\#\Sigma^{2r-1}(d_2)$.

If M is a closed simply connected smooth manifold of dimension $4r-1 \geq 5$, then the exactness of the surgery sequence implies that if the orientation-preserving homotopy equivalence $f : N \to M$ is normally cobordant to the identity, then N is orientation-preservingly diffeomorphic to a connected sum $M\#\Sigma^{4r-1}(d)$ for some integer d. We shall need some refinements of this observation.

As in [RS 1987], much of the following discussion is based upon a well known consequence of the product formula for surgery obstructions (*cf.* [Browder 1968]):

PROPOSITION Y.1. *Given an oriented exotic sphere Σ, let h generically denote an orientation-preserving homeomorphism from Σ to S^n with the standard orientation. Then for all integers d,r and m such that $d \geq 2$ and $m \geq 1$ the homotopy structures $h\times\mathbf{id} : \Sigma^{4r-1}(d)\times\mathbb{CP}^{2m} \to S^{4r-1}\times\mathbb{CP}^{2m}$ and $\mathbf{id}\#h : S^{4r-1} \times \mathbb{CP}^{2m}\#\Sigma^{4r+4m-1}(d) \to S^{4r-1} \times \mathbb{CP}^{2m}\#S^{4r+4m-1} \cong S^{4r-1} \times \mathbb{CP}^{2m}$ are the image of d times the generator of $L_{4r+4m}(\{1\}) = \mathbb{Z}$ under the action map Δ from the latter to $\mathbf{S}^s(S^{4r-1} \times \mathbb{CP}^{2m})$.*■

Surgery sequence inertia groups

The preceding result has a well-known curious implication (compare [Browder 1968]).

COROLLARY Y.2. *The standard homotopy equivalence*

$$\mathbf{id}\#h : S^7 \times \mathbb{CP}^2\#\Sigma^{11} \longrightarrow S^7 \times \mathbb{CP}^2\#S^{11} \cong S^7 \times \mathbb{CP}^2$$

is homotopic to a diffeomorphism.

This follows from simple numerical considerations; since the orders of ∂P_8 and ∂P_{12} are 28 and 992 respectively, one knows that 28 and 992 times the generator of $L_{12}(\{1\}) = \mathbb{Z}$ map to the trivial element of the structure set under Δ. Since the set of all elements d such that $\Delta(d) =$ [trivial] is a subgroup, and we know in the given case this subgroup contains both 28 and 992 times the generator, it follows that the subgroup must also contain 4 times the generator.■

In fact, using the displayed formula for θ_{m+2} one can generalize this to products with $\mathbb{CP}^{2m}$ such that m is not divisible by 3, for the formula shows that θ_{m+2} is divisible by 7 if and only if $m \equiv 0 \bmod 3$.

The preceding corollary can be rephrased to state that the exotic sphere lies in a subgroup of Θ_{11} called the *homotopy inertia group* of $S^7 \times \mathbb{CP}^2$; in particular, this terminology is used in both [RS 1987] and [Masuda-Schultz]. It will be convenient to work with a variant of the homotopy inertia group in this paper.

Definition. Let M be a closed smooth manifold of dimension $n \geq 5$. The **surgery sequence inertia group** of M, written $I_\Delta(M)$, is the set (subgroup) of all elements $\alpha \in L^s_{n+1}(\pi_1(M), w)$ such that $\Delta(\alpha)$ is the trivial element of $\mathbf{S}^s(M)$.

If M is a closed orientable manifold and $n = \dim M \equiv 3 \bmod 4$, then $I_\Delta(M)$ is always an infinite subgroup. Specifically, if $n = 4r - 1$ and

$$i_* : \mathbb{Z} = L_{4r}(\{1\}) \longrightarrow L^s_{4r}(\mathbb{Z})$$

is the (split injective) map induced by the constant homomorphism, then $I_\Delta(M)$ contains the image of $\theta_r\mathbb{Z}$. In contrast, if $\pi = \pi_1(M)$ is finite, then there is a body of results related to the so-called oozing conjecture (see [Milgram] or [HMTW]) which imply that many classes in the groups $L_{n+1}(\pi)$ never lie in the subgroups $I_\Delta(M)$.

For the sake of completeness, here is the relationship between $I_\Delta(M)$ and the homotopy inertia group $I^h(M)$.

FACT Y.3. *If M is an arbitrary closed oriented smooth manifold, then $I^h(M) \cap \partial P_{n+1}$ is contained in $I_\Delta(M)$, and if M is simply connected then equality holds.*∎

We shall not need this result (see [Brumfiel] for the simply connected case), but we shall need some to explain how some computations involving homotopy inertia groups from [RS 1987] can be rephrased in our present setting. By the exactness of the surgery sequence, the group $I_\Delta(M)$ is the image of the surgery obstruction map $\sigma_1 : [SM, F/O] \to L^s_{n+1}(\pi_1(M))$; we suppress w because we are working with oriented manifolds. As before, we know that this mapping is additive. If n is congruent to 3 mod 4 and we compose σ_1 with the map of Wall groups

$$C_* : L^s_{n+1}(\pi_1(M)) \longrightarrow L_{n+1}(\{1\}) \cong \mathbb{Z}$$

then the value of the composite $\varphi = C_* \circ \sigma_1$ is essentially a signature difference and hence is computable by means of characteristic classes. Specifically, if we take a representative (ξ, t) of a class in $[SM, F/O]$ where ξ is a vector bundle and t is a fiber homotopy trivialization of (the associated sphere bundle of) ξ, then $\varphi(\xi, t)$ is given by formula (2.2) in [RS 1987]; in this setting one does not need to work modulo θ_n or to worry about how to define the top Pontryagin class (the latter is already given). We then have the following basic result:

PROPOSITION Y.4. *Let M be a closed smooth oriented manifold of dimension $4r-1 \geq 7$, and let $J \subset L^s_{4r}(M)$ be the image of $L_{4r}(\{1\})$ under the map induced by the inclusion $\{1\} \to \pi_1(M)$. Then $J \cap I_\Delta(M) \subset L_{4r}(\{1\})$ is contained in the image of φ, and equality holds if M is simply connected.*∎

For the sake of comparing approaches, we note that the image of the subgroup $J \cap I_\Delta(M)$ under the natural surjection from $L_{4r}(\{1\})$ to ∂P_{4r} is equal to $I^h(M) \cap \partial P_{4r}$.

In this setting, the result of L. Taylor on homotopy inertia groups [Taylor] can be reformulated as follows; the proof of this result is essentially identical to the proof of Theorem 2.1 in [RS 1987]:

THEOREM Y.5. *If M is a closed oriented smooth manifold of dimension $4r - 1 \geq 7$, then the image of φ is contained in $2\mathbb{Z} \subset L_{4r}(\{1\}) = \mathbb{Z}$.*■

Theorem Y.5 is the best possible result on the index of the image of φ in $L_{4r}(\{1\})$ for general choices of M; as noted in many places (*e.g.*, [RS 1987 Example 2, p. 190]), it is well-known that the image is exactly equal to $2\mathbb{Z}$ if $M = S^3 \times \mathbb{CP}^{2m-1}$ for all $m \geq 1$. However, there are many cases in which one can improve upon this result very substantially (for example, the main result of [Browder 1965] gives the optimally strong conclusion $I_\Delta(M) = \theta_r Ł_{4r}(\{1\})$ if M is a simply connected spin manifold of dimension $4r = 1 \geq 7$). In this paper we are interested in proving a slight improvement of Theorem Y.5 for $M = S^7 \times \mathbb{CP}^2$.

THEOREM Y.6. *The subgroup $I_\Delta(S^7 \times \mathbb{CP}^2)$ has index 4 in $L_{12}(\{1\}) = \mathbb{Z}$.*

Proof. We have already seen that I_Δ contains $4\mathbb{Z}$, so we shall focus on proving the reverse containment.

The following computational result, which is a strengthening of [RS 1987, Sublemma 2.3], contains the crucial information needed to prove that $I_\Delta \subset 4\mathbb{Z}$.

LEMMA Y.7. *If ξ is a fiber homotopically trivial vector bundle over the suspension of $S^7 \times \mathbb{CP}^2$, then for each positive integer m the m^{th} Pontryagin class $p_m(\xi)$ is divisible by $2j_{4m}$, where j_{4m} is the order of the image of the J-homomorphism in dimension $4m - 1$.*

If we combine this fact with the proof of Theorem 2.1 in [RS 1987], then the required divisibility condition in Theorem Y.6 follows directly.■

Proof of Lemma Y.7. Since the suspension of $S^7 \times \mathbb{CP}^2$ splits homotopically into a wedge $S^8 \vee S\mathbb{CP}^2 \vee S^8\mathbb{CP}^2$ and $\widetilde{KO}(S\mathbb{CP}^2)$ is trivial (because $\pi_3(BO) = \pi_5(BO) = 0$), it suffices to prove that the conclusion holds if the suspension of $S^7 \times \mathbb{CP}^2$ is replaced by S^8 and $S^8\mathbb{CP}^2$. The argument will use a classical result of Bott (see [Bott-Milnor]) on the divisibility of the image of the Hurewicz map from $\pi_{4m}(BO) \cong \mathbb{Z}$ to $H_{4m}(BO;\mathbb{Z})$. Specifically, the image of a generator is divisible by $a_m \cdot (2m-1)!$, where $a_m = 2$ if m is odd and 1 if m is even.

Suppose that ξ is a (stably) fiber homotopically trivial vector bundle over S^8. Then Bott's result implies that $p_2(\xi)$ is divisible by $6 \cdot j_8$, so the conclusion of the proposition is true for vector bundles over S^8. Next, suppose that ξ is a (stably) fiber homotopically trivial vector bundle over $S^8\mathbb{CP}^2$. By Bott Periodicity and the results of [Adams-Walker], the group $\widetilde{KO}(S^8\mathbb{CP}^2)$ is infinite cyclic, and if $z : S^8\mathbb{CP}^2 \to S^{12}$ is the collapsing map of degree 1, then the map

$$z^* : \mathbb{Z} \cong \widetilde{KO}(S^{12}) \longrightarrow \widetilde{KO}(S^8CP^2) \cong \mathbb{Z}$$

is multiplication by ± 2. If Ω is a generator, then the Adams Conjecture impies that the stably fiber homotopically trivial vector bundles are the elements of $\widetilde{KO}(S^8\mathbb{CP}^2)$ which are divisible by $8t$ for some integer t for some integer t. Since $2\Omega \in \text{Image}\, z^*$, it follows from Bott's result that $p_3(8t\Omega)$ is divisible by $\frac{1}{2} \cdot 2 \cdot 5! \cdot 8t = 64\,(15t)$. Since $j_3 = 8v$ for some odd integer v, this means that $p_3(8t\Omega)$ is divisible by $8j_3$ times an odd integer.■

Applications to classification problems

The following result integrates many of the different ideas and results in this and earlier sections.

THEOREM Y.8. *Let d and m be integers with $m \geq 1$, and suppose that $\Sigma^7(d) \times \mathbb{CP}^{2m}$ is diffeomorphic to $S^7 \times \mathbb{CP}^{2m}$. Then d lies in the surgery sequence inertia group $I_\Delta(S^7 \times \mathbb{CP}^{2m})$.*

Proof. It will suffice to show that if $\Sigma^7(d) \times \mathbb{CP}^{2m}$ is diffeomorphic to $S^7 \times \mathbb{CP}^{2m}$ then $\Delta(d)$ is the trivial element of $\mathbf{S}^s(S^7 \times \mathbb{CP}^{2m})$.

If the two manifolds in question are diffeomorphic, then $\Delta(d)$ is the homotopy structure associated to some homotopy self-equivalence f of $S^7 \times \mathbb{CP}^m$. Since $\Delta(d)$ has a trivial normal invariant, it follows that $\mathfrak{q}(f)$ must also be trivial. However, by the results of the preceding section we know that f is homotopic to a diffeomorphism if and only if $\mathfrak{q}(f) = 0$, and it follows that the class of $[f]$, which equals $\Delta(d)$, must also be trivial.∎

Theorem Y.8 yields the main results on the diffeomorphism classification for products of exotic spheres and complex projective spaces.

THEOREM Y.9. *In the setting of the previous result, we have the following:*

(i) For every $m \geq 1$, the manifolds $\Sigma^7(1) \times \mathbb{CP}^{2m}$ and $S^7 \times \mathbb{CP}^{2m}$ are not diffeomorphic.

(ii) For all integers d, the manifolds $\Sigma^7(d) \times \mathbb{CP}^2$ and $\Sigma^7(d') \times \mathbb{CP}^2$ are diffeomorphic if and only if $d \equiv \pm\, d' \bmod 4$.

Note that this result includes the conclusions of both Theorem S.1 and Proposition S.3.

Proof. The first statement follows from Theorem Y.5 because the generator of $L_{4m+8}(\{1\}) = \mathbb{Z}$ does not lie in $I_\Delta(S^7 \times \mathbb{CP}^{2m})$. To prove the second statement, observe that the two manifolds in question are diffeomorphic if $d \equiv d' \bmod 4$ by Theorem Y.6, and since $\Sigma^7(d)$ is orientation-reversingly diffeomorphic to $\Sigma^7(-d)$ it also follows that $\Sigma^7(d) \times \mathbb{CP}^{2m}$ and $\Sigma^7(-d) \times \mathbb{CP}^{2m}$ are orientation-reversingly diffeomorphic.

Conversely, suppose that $\Sigma^7(d) \times \mathbb{CP}^2$ and $\Sigma^7(d') \times \mathbb{CP}^2$ are diffeomorphic by some diffeomorphism φ. If this diffeomorphism is orientation-reversing, then there is an orientation-preserving diffeomorphism from $\Sigma^7(d) \times \mathbb{CP}^2$ to $\Sigma^7(-d') \times \mathbb{CP}^2$, and therefore it will suffice to show that if $\Sigma^7(d) \times \mathbb{CP}^2$ and $\Sigma^7(d') \times \mathbb{CP}^2$ are orientation- reversingly diffeomorphic then $d \equiv d' \bmod 4$.

Let h and h' be orientation-preserving homeomorphisms from $\Sigma^7(d)$ and $\Sigma^7(d')$ to S^7. Then the two homotopy structures

$$\big(\Sigma^7(d) \times \mathbb{CP}^2, h \times \mathbf{id}\big)\ , \qquad \big(\Sigma^7(d') \times \mathbb{CP}^2, h' \times \mathbf{id}\big)$$

are related by the action of the orientation-preserving homotopy self-equivalence

$$(h' \times \mathbf{id})^\circ \varphi^\circ (h^{-1} \times \mathbf{id})\ .$$

We first note that if there is such a an orientation-preserving homotopy self-equivalence, then we can also find a homotopy self-equivalence of $S^7 \times \mathbb{CP}^2$ which is the identity on cohomology and also sends $\Delta(d)$ to $\Delta(d')$. Every orientation-preserving homotopy self-equivalence of $S^7 \times \mathbb{CP}^2$ induces the same map in cohomology as either the identity or $\mathbf{id}(S^7) \times \chi$, where χ is the conjugation diffeomorphism of $\mathbb{CP}^2$. Clearly the maps $\mathbf{id}(\Sigma^7) \times \chi$ are self-diffeomorphisms of $\mathbf{id}(\Sigma^7) \times \mathbb{CP}^2$ for all exotic spheres Σ, and therefore $h \times \mathbf{id}$ and $h \times \chi$ determine the same class in the structure set. It follows that if f is an orientation-preserving self-equivalence of $S^7 \times \mathbb{CP}^2$ which sends $\Delta(d)$ to $\Delta(d')$, then $f^\circ(\mathbf{id} \times \chi)$ also has this property. One of these self-equivalences induces the identity on cohomology, so pick the one that does.

The results of Sections H and N show that if f is a homotopy self-equivalence of $S^7 \times \mathbb{CP}^2$ which induces the identity on cohomology, then f must come from an element of $\pi_7(E_1(\mathbb{CP}^2)) \cong \pi_7\big(F_{S^1}(\mathbb{C}^3)\big)$. Thus all the relevant data lift back to the relative structure set $\mathbf{S}^s_7(\mathbb{CP}^2)$, so that we can use Theorem A.3 to analyze the action of f on classes in the structure set. We first claim that the class of f in $\pi_7\big(F_{S^1}(\mathbb{C}^3)\big)$ must stabilize to zero in $\pi_7(F_{S^1})$; if it did not, then its normal invariant would be nontrivial, and hence it would follow that the normal invariant of $\Delta(d')$ — which is obtained by the action of f on $\Delta(d)$ would be nonzero (since the normal invariant of $\Delta(d)$ is trivial. However, we know that the normal invariant of $\Delta(d')$ is trivial, so this cannot happen. Thus we are left with the possibility that f might be nontrivial but stably trivial. We can combine Theorem A.3 and C.2 to show that the image of f in the relative structure set is trivial and that its action on $\Delta(d)$ is to send the latter to itself. Putting this all together, we have seen that if the two products are orientation-preservingly diffeomorphic, then $\Delta(d)$ and $\Delta(d')$ must determine the same element in the structure set, and by Theorem Y.6 this means that $d \equiv d' \bmod 4$.■

Proof of Proposition S.3

We already know that $\Sigma^7(d)\times\mathbb{CP}^2$ and $\Sigma^7(d')\times\mathbb{CP}^2$ are diffeomorphic if and only if $d \equiv \pm\, d' \bmod 4$, so each such product is diffeomorphic to exactly one of the manifolds $S^7\times\mathbb{CP}^2$, $\Sigma^7(1)\times\mathbb{CP}^2$ or $\Sigma^7(2)\times\mathbb{CP}^2$. Therefore it is only necessary to prove that if M^{11} is tangentially homotopy equivalent to $S^7\times\mathbb{CP}^2$ then M is diffeomorphic to one of these products.

If $h : M^{11} \to S^7\times\mathbb{CP}^2$ is a tangential homotopy equivalence, then its refined normal invariant is an element of

$$\{S^7\times\mathbb{CP}^2, S^0\} \;\cong\; \pi_7^{\mathbf{S}} \oplus \{\mathbb{CP}^2, S^0\} \oplus \{S^7\mathbb{CP}^2, S^0\}\ .$$

We know that $\mathbb{CP}^2$ is the mapping cone of the Hopf map $\eta_3 : S^3 \to S^2$, and hence the second and third factors fit into the following exact sequences, in which the maps at the far right are given by composition with η or one of its suspensions:

$$\pi_4^{\mathbf{S}} \to \{\mathbb{CP}^2, S^0\} \to \pi_2^{\mathbf{S}} \to \pi_3^{\mathbf{S}}\ , \quad \pi_{11}^{\mathbf{S}} \to \{S^7\mathbb{CP}^2, S^0\} \to \pi_9^{\mathbf{S}} \to \pi_{10}^{\mathbf{S}}$$

These are easily computed using [Toda]. Since $\pi_4^{\mathbf{S}} = 0$ and compositon with η induces a monomorphism from $\pi_2^{\mathbf{S}}$ to $\pi_3^{\mathbf{S}}$ (by the identity $\eta^3 = 4\nu$), it follows that $\{\mathbb{CP}^2, S^0\} = 0$. Also, we know that $\pi_{11}^{\mathbf{S}}$ is generated by the image of the J-homomorphism from $\pi_{11}(O)$ to $\pi_{11}^{\mathbf{S}}$, while $\pi_9^{\mathbf{S}} \cong \mathbb{Z}_2^3$ such that the kernel of $\eta^*\pi_9^{\mathbf{S}} \to \pi_{10}^{\mathbf{S}}$ is the third summand and the image of the J-homomorphism is the first summand. Therefore the image of $\{S^7\times\mathbb{CP}^2, S^0\}$ in $[S^7\times\mathbb{CP}^2, F/O]$ is isomorphic to $\mathbb{Z}_2$, and in fact it corresponds to the second summand of $\pi_9^{\mathbf{S}}$.

In fact, the nonzero class is the normal invariant for the homotopy self-equivalence f of $S^7\times\mathbb{CP}^2$ which comes from a stably nontrivial element of $\pi_7\big(F_{S^1}(\mathbb{C}^3)\big)$; this follows directly from the formula for the normal invariant. Therefore we have shown that the homotopy structures on $S^7\times\mathbb{CP}^2$ which come from tangential homotopy equivalences all have the form $[f_0]^e\cdot\Delta(d)$, where f_0 is the previously described homotopy equivalence, $e = 0$ or 1, and d is some integer. Since the underlying manifold for such a structure is $\Sigma^7(d)\times\mathbb{CP}^2$ for some d, we know that M^{11} must be diffeomorphic to one of these examples.■

Diffeomorphisms of total spaces

If M^k and N^k are closed smooth simply connected manifolds and $k \geq 5$, then a well known argument shows that $M^k \times \mathbb{R}^2$ and $N^k \times \mathbb{R}^2$ are diffeomorphic if and only if M^k and N^k are diffeomorphic (**Proof:** If such a diffeomorphism exists, then the image of $M \times D^2$ is contained in some subset of the form $N \times rD^2$, where rD^2 is a disk of sufficiently large radius r about the origin, and $N \times rD^2 - \mathrm{Int}\, M \times D^2$ is an h-cobordism between $M \times S^1$ and $N \times S^1$; since $\mathrm{Wh}(\mathbb{Z})$ is trivial, it follows that these two manifolds must be diffeomorphic, and as before this conclusion implies that M and N must also be diffeomorphic.). In contrast, the main result of this section is the following:

THEOREM T.1. *Suppose that $k \geq 2$, $q \geq 1$, let Σ^{4k-1} is a homotopy sphere bounding a parallelizable manifold, and let ω be a nontrivial complex line bundle over $\mathbb{CP}^q$. If $E(\omega)$ denotes the total space of ω, then $\Sigma^{4k-1} \times E(\omega)$ is diffeomorphic to $S^{4k-1} \times E(\omega)$.*

Throughout the discussion below, $D(\omega)$ will denote the associated unit disk bundle (with respect to some Hermitian metric) and $S(\omega)$ will denote the sphere bundle, which is just $\partial D(\omega)$.

Proof. Since the proof is simpler when the Chern class of ω is a generator of $H^2(\mathbb{CP}^q;\mathbb{Z})$, we shall first prove the result in these cases. In this case $S(\omega) \cong S^{2q+1}$, so that both $S(\omega)$ and $D(\omega)$ are simply connected and the inclusion $S(\omega) \subset D(\omega)$ defines an isomrophism of fundamental groups. Therefore we can apply Wall's $\pi - \pi$ Theorem to conclude that the normal invariant mapping

$$\mathfrak{q} : \mathbf{S}^s\big(S^{4k-1} \times D(\omega)\big) \longrightarrow [S^{4k-1} \times D(\omega), F/O] \cong [S^{4k-1} \times \mathbb{CP}^q, F/O]$$

is bijective. Consider the structure given by the product of an orientation-preserving homeomorphism $\Sigma^{4k-1} \to S^{4k-1}$ with the identity on $D(\omega)$. Since Σ bounds a parallelizable manifold, the normal invariant of this structure is trivial, and therefore it follows that the associated homotopy equivalence of pairs must be homotopic to a diffeomorphism.

Assume now that the Chern class of ω is $\pm\, d$ times the generator of $H^2(\mathbb{CP}^q;\mathbb{Z})$, where $d > 1$ is an integer. In this case $S(\omega)$ is a lens space with fundamental group $\mathbb{Z}_d$, so that the map

$$\pi_1\big(S(\omega)\big) \longrightarrow \pi_1\big(D(\omega)\big)$$

is just the constant homomorphism $\mathbb{Z}_d \to \{1\}$. In this case the appropriate surgery exact sequence is given by the first line in the diagram below; the maps from the first to the second line are given by passing to boundaries, and for all the surgery obstruction groups the orientation map from π_1 to $\mathbb{Z}_2$ is trivial.

$$\begin{array}{ccccc} L_2^s\big(\mathbb{Z}_d \to \{1\}\big) & \longrightarrow & \mathbf{S}^s\big(D(\omega)\big) & \longrightarrow & [S^7 \times D(\omega), F/O] \\ \downarrow & & \downarrow & & \downarrow \\ L_2^1(\mathbb{Z}_d) & \longrightarrow & \mathbf{S}^s\big(S(\omega)\big) & \longrightarrow & [S^7 \times S(\omega), F/O] \end{array}$$

The relative Wall group $L_2^s(\mathbb{Z}_d \to \{1\})$ fits into a familiar type of exact sequence from [Wall book]:

$$\cdots\ L_2^s(\mathbb{Z}_d) \to L_2(\{1\}) \to L_2^s\big(\mathbb{Z}_d \to \{1\}\big) \to L_1^s(\mathbb{Z}_d)\ \cdots$$

It will suffice to show that $L_2^s\big(\mathbb{Z}_d \to \{1\}\big)$ is trivial, and we shall use the exact sequence to do this. First of all, results from [Wall 1976] show that $L_1^s(\mathbb{Z}_d) = 0$, so that the map from $\mathbb{Z}_2 \cong L_2(\{1\})$ to $L_2^s\big(\mathbb{Z}_d \to \{1\}\big)$ is onto. Next, since the map $\mathbb{Z}_d \to \{1\}$ has an obvious one-sided inverse (inclusion of the identity) and Wall groups are functorial with respect to group homomorphisms, we know that the map $L_2^s(\mathbb{Z}_d) \to L_2(\{1\})$ is split surjective, so that the map from $\mathbb{Z}_2 \cong L_2(\{1\})$ to $L_2^s\big(\mathbb{Z}_d \to \{1\}\big)$ is the zero homomorphism. Combining these, we see that the relative Wall group must be trivial, and as noted above this suffices to complete the proof of the theorem.■

REFERENCES

J. F. Adams. On the groups $J(X)$. I. Topology **2** (1963), 181–195.

J. F. Adams and G. Walker. On complex Stiefel manifolds. Proc. Cambridge Philos. Soc. **61** (1965), 81–103.

J. C. Becker and R. E. Schultz, Equivariant function spaces and stable homotopy theory. I. Comment. Math. Helv. **49** (1974), 1–34.

J. M. Boardman and R. M. Vogt. Homotopy-everything H-spaces. Bull. Amer. Math. Soc. **74** (968), 1117-1122.

R. H. Bott and J. W. Milnor. On the parallelizability of the spheres. Bull. Amer. Math. Soc. **64** (1958), 87–89.

W. Browder. On the action of $\Theta^n(\partial\pi)$, Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), Princeton Mathematical Series No. 27, pp. 23–36. Princeton University Press, Princeton, 1965.

W. Browder. Surgery and the theory of differentiable transformation groups, Proceedings of the Conference on Transformation Groups (Tulane Univ., 1967), pp. 3–46. Springer-Verlag, Berlin-*etc.*, 1968.

W. Browder. Surgery on Simply Connected Manifolds, Ergeb. der Math. (2) Bd. 65. Springer-Verlag, Berlin-*etc.*, 1972.

G. Brumfiel. Homotopy equivalences of almost smooth manifolds. Comment. Math. Helv. **46** (1971), 381–407.

S. Cappell and J. Shaneson. On 4-dimensional s-cobordisms II. Comment. Math. Helv. **64** (1989) 338–347.

K. Grove and W. Ziller. Curvature and symmetry of Milnor spheres. Ann. of Math. **152** (2000), 331–367.

I. Hambleton, R. J. Milgram, L. Taylor and B. Williams. Surgery with finite fundamental group. Proc. London Math. Soc. (3) **56** (1988), 349–379.

P. J. Hilton. Generalizations of the Hopf invariant. Colloque de topologie algébrique, Louvain, 1956, pp. 9–27. G. Thone, Liège; Masson & Cie., Paris, 1957.

W.-Y. Hsiang. On the unknottedness of the fixed point set of differentiable circle group actions on spheres — P. A. Smith conjecture. Bull. Amer. Math. Soc. **70** (1964), 678–680.

I. M. James. On the iterated suspension. Quart. J. Math. Oxford (2) **5** (1954), 1–10.

I. M. James. The space of bundle maps. Topology **2** (1963), 45–59.

M. Kervaire and J. Milnor. Groups of homotopy spheres. Ann. of Math. **77** (1963), 504–537.

S. Kwasik and R. Schultz. On s-cobordisms of metacyclic prism manifolds. Invent. Math. **97** (1989), 523–552.

S. Kwasik and R. Schultz. Visible surgery, 4-dimensional s-cobordisms, and related questions in geometric topology. K-Theory **9** (1995), 323–352.

S. Kwasik and R. Schultz. Multiplicative stabilization and transformation groups, Current Trends in Transformation Groups, K-Monographs in Mathematics Vol. 7, pp. 147–165. Kluwer Academic Publishers, Dordrecht, NL, 2002.

J. Levine, Self-equivalences of $S^n \times S^k$. Trans. Amer. Math. Soc. **143** (1969), 523–543.

I. H. Madsen and R. J. Milgram. The Classifying Spaces for Surgery and Cobordism of Manifolds, Annals of Mathematics Studies No. 92. Princeton Mathematical Press, Princeton, 1979.

I. H. Madsen, L. R. Taylor and E. B. Williams. Tangential homotopy equivalence. Comment. Math. Helv. **55** (1980), 445–484.

M. Masuda and R. Schultz. On the nonuniqueness of equivariant connected sums, J. Math. Soc. Japan **51** (1999), 413–435.

R. J. Milgram. Surgery with finite fundamental group. II. The oozing conjecture. Pacific J. Math. **151** (1991), 117–150.

J. Milnor. On spaces having the type of a CW complex. Trans. Amer. Math. Soc. **90** (1958), 272–280.

J. Milnor and J. Stasheff. Characteristic Classes, Annals of Mathematics Studies Vol. 76. Princeton Univ. Press, Princeton, 1974.

R. Mosher and M. Tangora. Cohomology Operations and Applications in Homotopy Theory. Dover Publications, New York, 2008.

A. Ranicki *et al*. The Hauptvermutung Book: A Collection of Papers on the Topology of Manifolds, K-Monographs in Mathematics Vol. 1. Kluwer Academic Publishers, Dordrecht, NL, 1996.

A. Ranicki, A composition formula for manifold structures. Pure Appl. Math Quart. **5** (Hirzebruch 80th birthday issue, 2009), 701–727. Online preprint version:
`http://arxiv.org/PS_cache/math/pdf/0608/0608705v2.pdf`

C. P. Rourke. The Hauptvermutung according to Casson and Sullivan, pp. 129–164 in [Ranicki Hauptvermutung].

R. Schultz. On the inertia group of a product of spheres, Trans. Amer. Math. Soc. **156** (1971), 137–153.

R. Schultz. Homotopy decomposition of equivariant function spaces I: Spaces of principal bundle maps, Math. Zeitschrift **131** (1973), 49–75.

R. Schultz. Differentiable group actions on homotopy spheres I: Differential structure and the knot invariant, Invent. Math. **31** (1975), 105–128.

R. Schultz. Smooth actions of small groups on exotic spheres, Proc. AMS Sympos. Pure Math. **32** Part 1 (1978), 155–161.

R. Schultz. Homology spheres as stationary sets of circle actions, Mich. Math. J. **34** (1987), 83–100.

M. Sugawara. A condition that a space is an H-space. Math. J. Okayama Univ. **6** (1957), 109–129.

L. R. Taylor. Divisibility of the index with applications to inertia groups and $H^*(G/O)$. Notices Amer. Math. Soc. **21** (1974), A–424.

C. T. C. Wall. Classification of Hermitian forms. VI: Group rings. Ann. of Math. **103** (1976), 1–80.

C. T. C. Wall. Surgery on compact manifolds (Second edition; edited and with a foreword by A. A. Ranicki). Mathematical Surveys and Monographs, No. 69. American Mathematical Society, Providence, RI, 1999.

G. W. Whitehead. On products in homotopy groups. Ann. of Math. (2) **47** (1946), 460–475.