BETWEEN LOWER AND HIGHER DIMENSIONS

(in the work of Terry Lawson)

Reinhard Schultz

There are several approaches to summarizing a mathematician's research accomplishments, and each has its advantages and disadvantages. This article is based upon a talk given at Tulane that was aimed at a fairly general audience, including faculty members in other areas and graduate students who had taken the usual entry level courses. As such, it is meant to be relatively nontechnical and to emphasize qualitative rather than quantitative issues. Furthermore, since this was an hour talk, it was not feasible to describe every single piece of published work that Terry Lawson has done or to give the complete sort of listing that one can now retrieve fairly quickly with a search of the databases for *Mathematical Reviews* and *Zentralblatt für Mathematik*. Instead, we shall focus on some ways in which Terry Lawson's work relates to an important thread in geometric topology; namely, the passage from studying problems in a given dimension to studying problems in the next dimension. Much of his work, and most of his best known results and papers, are directly related to such questions.

1. Lower versus higher dimensions

Of course, the concept of *dimension* is central to many geometrical questions, and in the physical world one can have objects of dimension n for $n = 0, 1, 2, 3$. During the nineteenth century, several mathematicians recognized that the methods of coordinate geometry lead to a theory of n-dimensional geometrical objects, where n is an arbitrary nonnegative integer. In particular, the vector space structure on $\mathbb{R}^n$, including the standard inner product, provide a setting in which one can describe an n-dimensional analog of classical Euclidean plane or solid geometry.

Many important n-dimensional geometrical objects are examples of *topological n-manifolds*; formally, these are Hausdorff topological spaces in which every point has an open neighborhood which is homeomorphic to $\mathbb{R}^n$. We shall deal almost exclusively with such objects in this article.

In classical Euclidean geometry, it is clear that things become more complicated when one passes from (say) line geometry to plane geometry or from plane geometry to solid geometry, and it is normal to expect a similar pattern when one goes from n-dimensional objects to $(n+1)$-dimensional objects. This is true in many cases, but one also has the following somewhat unanticipated fact:

> *Sometimes the answers to basic geometrical questions become simpler if the dimension n is sufficiently large. In other words, there are instances where general patterns of results exist if one excludes finitely many exceptional dimensions.*

AN EXAMPLE FROM EUCLIDEAN GEOMETRY. The classification of (solid) regular polyhedra in Euclidean n-space up to similarity illustrates this phenomenon fairly well. If $n = 2$ then the possibilities are given by the usual regular k-gons, where k is an arbitrary integer ≥ 3. On the other hand, if $n = 3$ then the theory is simpler in some ways but more complicated in others. There are only finitely many possibilities, and they are given by the classical *Platonic solids*; namely, the regular triangular pyramid (or tetrahedron), the cube, the octahedron (which can be constructed by taking the centers of the six faces of a cube — alternatively, it is given by gluing together two square pyramids along their bases), the dodecahedron (whose 12 faces are regular pentagons), and the icosahedron (whose 20 faces are equilateral triangles, with 5 containing each vertex). Although the list of objects is finite, some objects on the list are definitely more complicated than 2-dimensional regular polygons.

What happens if we pass to higher dimensions? For all $n \geq 4$ we have analogs of the square and cube which are frequently called *hypercubes*. It is easy to write down such objects in coordinate form; for example, a solid hypercube in $\mathbb{R}^n$ with edges of length 2 is given by all $(x_1, \cdots, x_n) \in \mathbb{R}^n$ such that $-1 \leq x_j \leq 1$ for all j. Similarly, for each $n \geq 4$ we can define analogs of equilateral triangles and tetrahedra that one might call *regular n-simplexes*. Such objects can be constructed recursively as follows: Suppose we have a regular $(n-1)$-simplex Δ_{n-1} in $\mathbb{R}^{n-1}$. For each $h > 0$, we may generate a solid cone in $\mathbb{R}^n$ with base Δ_{n-1} and altitude h by taking all points in bR^n which lie on closed segments joining points in Δ_{n-1} to $h \cdot \mathbf{b}$, where $\mathbf{b}$ is the geometrical center of Δ_{n-1}. Straightforward compuations imply this solid cone will be a regular polyhderon if we choose

$$h = \sqrt{\frac{n+1}{2n}}$$

and if $n = 3$ this reduces to a standard construction for a regular tetrahedron starting with an equilateral triangle. Finally, there is a regular polyhedron which is dual to the hypercube; specifically, the vertices are given by the centers of the faces of the hypercube, and the points are all convex combinations of the vertices (*i.e.*, linear combinations whose coefficients are nonnegative and add up to 1). If $n = 3$ this reduces to the octahedron.

Of course, one immediate question is whether there are any other examples, and this was answered by results of Ludwig Schläfli from the middle of the nineteenth century (however, his work was first published posthumously near the end of that century). In particular, he showed that there are *three additional examples* if $n = 4$, but *no additional examples* if $n \geq 5$. All but one of the examplex for $n = 4$ are analogs of 3-dimensional Platonic solids.

This illustrates the earlier comment about simplifications for sufficiently large dimensions; if we agree that the 2- and 3-dimensional cases are understood, then we see that the 4-dimensional case is more complicated than the 3-dimensional situation and in all dimensionss $n \geq 5$ there is a uniform pattern of behavior which is simpler to describe than in either dimension 3 or 4.

SIMILAR PATTERNS IN ALGEBRA. Such patterns also arise very often in group theory. For example, for each integer n consider the alternating group A_n of all even permutations on n letters. A basic result of group theory states that A_n has no nontrivial normal subgroups (either the identity group or the group itself) for all n except $n = 4$. For lower values of n there is no room to squeeze in any nonzero proper subgroups at all, while if $n \geq 5$ there is enough room to perform certain algebraic constructions which force a nontrival normal subgroup to be the whole group.

Other examples arise at deeper levels of group theory; in each case one has a very systematic conclusion provided one avoids a finite list of exceptional values. For example, one can consider the automorphism group for the symmetric group Σ_n on n letters, and a natural question is whether this group has any automorphisms besides the standard inner automorphisms; in this case there are no other automorphisms with the exception of $n = 6$, in which case there is an additional "outer" automorphism. Another illustration of systematic behavior with finitely many exceptions is the classification of compact simply connected Lie groups (which can be written down very directly proved a numerical invariant called the *rank* is greater than 8), and yet another is the classification of finite simple groups (which involves 26 exceptional cases — not that in this case the orders in the exceptional cases are often astronomical, so the notion of "sufficiently large" is not in the very small ranges we have seen thus far).

COUNTERPARTS IN GEOMETRIC TOPOLOGY. Here is a question that is simple to formulate:

> *For a fixed value of n, which finite abelian groups can arise as the fundamental groups of compact (unbounded) n-manifolds?*

If $n \leq 2$ one can answer this using the well-known classification theory for manifolds in these dimensions; no finite groups can be realized if $n = 1$, and only finite groups of orders 1 and 2 can be realized if $n = 2$. Fundamental results of C. D. Papakyriakopoulos in 3-dimensional topology imply that a finite group G can be realized if $n = 3$ if and only if G is cyclic. On the other hand, if $n \geq 4$ then one has enough geometric "room" to show that *every* finite abelian group can be realized.

Similar patterns appear elsewhere in geometric topology. Often one sees that everything can be described fairly systematically if $n \geq M$ for some small value of M (which is generally equal to 4, 5 or 6), and for all sufficiently small values of n (usually $n \leq 2$) everything is fairly well understood but usually for entirely different reasons. In particular, if $n = 1$ everything is usually extremely straightforward, and our understanding geometric topology in dimension 2 is fairly complete based upon advances from the first part of the twentieth century. If $n = 3$, there are many new phenomena to consider, and it appears that 3-dimensional topology will be in a fairly definitive form within the next ten years.

On the other hand, if $n \geq 5$, then several breakthroughs involving work from the nineteen forties to seventies have laid a very solid foundation for studying n-manifolds, with a few loose ends if $n = 5$. On the other hand, our understanding of the case $n = 4$ is still only partial despite some revolutionary advances during the past three decades.

Much of Terry Lawson's mathematical work has been devoted to issues involving the relation of 4-manifold theory to the theory of manifolds in higher dimensions. I shall concentrate on two themes runing through many of his papers; the first mainly involves work up to the early nineteen eighties, and the second mainly involves work after that point.

2. Shadows of higher dimensions : Stabilizations and bisections