Tangential thickness of manifolds

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Abstract: A motion of tangential thickness of a manifold introduced. An extensive calculation for the class of lens and fake lens spaces leads to complete classification of these manifolds with thickness 1,2, or 3. On the other hand calculations of tangential thickness in terms of dimension and the rank of the fundamental group show interesting and some what surprising correlations between these invariants.

Given two nonhomeomorphic topological spaces X and Y, it is often interesting or important to specify necessary or sufficient conditions for $X \times R$ and $Y \times R$ to be homeomorphic, where as usual R denotes the real line. More generally, it is also useful to have criteria for determining whether $X \times R^E$ and $Y \times R^k$ are homeomorphic for some $1 < 1$. If X and Y are closed manifolds, the following result, which is due to B. Mazur in the smooth and piecewise linear categories of [**?**, **?**] provides an abstract answer.

In this result CAT refer to the category of smooth, piecewise linear or topological manifolds and a CAT-isomorphism is a diffeomorphism, piecewise linear homeomorphism or homeomorphism respectively:

STABLE EQUIVALENCE THEOREM Let M and N be closed CAT-manifolds. Then $M \times R^k$ and $N \times R^h$ are CAT isomorphic for some $k \geq 1$ if and only if M and N are tangentially homotopy equivalent ie, there is a homotopy equivalence $f = M \to\ N$ such that the pullback of the stable tangent bundle of N is that stable tangent bundle of M)

In fact, if f exists then for some k the map $F \times id_{Rk}$ is properly homotopic to a CAT-isomorphism the topological version of this result follows from [**?**, **?**] if [**?**]

Given two manifolds M and N satisfying the conditions of the stable Equivalence Theorem, it is natural to ask the following:

OPTIMAL VALUE QUESTION *For a given tangential homotopy equivalence $f : M \to N$, what is the least value of k such that $f \times id_{Rk}$ is properly homotopic to a CAT isomorphism?*

The Whitney Embedding and Tubular Neighborhood Theorems imply that $dimN + 1$ is a universal upper bound for k in the smooth category, while standard results in piecewise linear manifolds yield the same result in that category and results of Siebanmann, of [**?**, **?**] [**?**, **?**] imply the analog in the topological category.

in [**?**]he Optional Value Question was considered for linear spherical space forms, and in particular it was shown that if M and N are linear space forms such that $M \times R^2$ is homeomorphic to $N \times R^2$, then M and N are homeomorphic but $M \times R^3$ are not diffeomorphic. These results already reject the relative complexity of this problem.

Our first result concerns a linear lens spaces.

THEOREM 0 Let $f : M \to N$ be tangential homotopy equivalence of lens spaces with prime order fundamental groups. Then $f \times id_{R^3}$ is properly homotopic to a homeomorphism.

The restriction to lens spaces is opposed to fake (homotopy) lens spaces is crucial. It is well know that there are fake lens spaces L_1 and L_2 are not homeomorphic but $L_1 \times R$ and $L_2 \times R$ are diffeomorphic [**?**]

In this paper we shall study the Optimal Value Question for fake lens spaces. We will concentrate on the case of (odd) prime order fundamental groups, although many of our results hold without this restriction Qualitatively, one can describe the results in terms of a concept we shall call tangential thickness. Specifically, two CAT manifolds M and N are said to leave tangential thickness $\leq k$. Given a manifold M let $TT_k(M)$ denote the isomorphism closes of manifolds N such that $\left\{_{M,N}\right\}$ has tangential thickness $\leq k$. One then has an increasing sequence of sets $TT_k(M)$ i.e.

$M = TT_o(M) < TT_1(M) < ... < TT_k(M) < ... < TT(M)$

Here $TT(M)$ is the set of isomorphism classes of manifolds that are tangentially homotopy equivalent to M. This finite sequence stabilizes for $K_0 \geq dimM + 1$ by the Marur's result i.e.
$TT_{k0}(M) = TT_{k_oti}(M) = TT(M)i = 1, 2, 3, ...$
In particular given a manifold M then the classification of all manifolds having the tangential thickness $tt(M) = k$ is equivalent to the computation of the set $TT_k(M) - TT_{k-1}(M)$. We are ready now to state results of this paper.

Theorem 1 Let $M^{2N-1}, N \geq 3$ be a fake lens space (arbitrary fake spherical space form of dimension ≥ 3). Then $TT^{TOP}(M^{2n-1})$ consists of manifolds h-cobordant to M^{2n-1}. These manifolds are classified by $WhTT, (M2n-1)$ via realization of whitehead torsion by h-cobordant i.e. free torsion of $WhTT, (M^{2n-1})$ or M^{2n-1}.

Theorem 2 Let $M^{2n-1}, n \geq 3$ be a fake lens space. Then a manifold N^{2n-1} is in $TT_2^{TOP}(M^{2n-1})$ if and only if $N^{2n-1} \times R$ is properly h-cobordant to $M^{2n-1} \times R$. The set $TT_2^{TOP}(M^{2n-1})$ - $TT_1^{TOP}(M^2n-1)$ is in one-to-one correspondence with $H^0(K_0Z[\prod_1 M^{2n-1})]$. Moreover all possible manifolds in $TT_2^{TOP}(M^{2n-1})$ - $TT_1^{TOP}(M^{2n-1})$ are obtained by a free action of $H_0(K_0 2[_1)M^{2n-1}]$ on $M^{2n-1} \times R$ is the realization of Whitehead torsion by proper h-cobordisms.

Theorem 3. Let $M^{2n-1}, n \geq 3$ be a fake lens space with $\prod_1(M^{2n-1}) \cong 2p, p-$odd prime. The the set