TANGENTIAL THICKNESS OF HOMOTOPY LENS SPACES

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Given two nonhomeomorphic topological spaces X and Y, it is often interesting or important to specify necessary or sufficient conditions for $X \times \mathbb{R}$ and $Y \times \mathbb{R}$ to be homeomorphic, where as usual $\mathbb{R}$ denotes the real line. More generally, it is also useful to have criteria for determining whether $X \times \mathbb{R}^l$ and $Y \times \mathbb{R}^k$ are homeomorphic for some $k > 1$. If X and Y are closed manifolds, the following result, which is due to B. Mazur in the smooth and piecewise linear (PL) categories [16, 17], provides an abstract answer. and to R. Kirby and L. Siebenmann in the topological category [12], provides an abstract answer. In this result CAT refer to the category of smooth, piecewise linear or topological manifolds and a CAT-isomorphism is a diffeomorphism, piecewise linear homeomorphism or homeomorphism respectively:

Theorem 1. STABLE EQUIVALENCE THEOREM. *Let M and N be closed CAT-manifolds. Then $M \times \mathbb{R}^k$ and $N \times \mathbb{R}^k$ are* CAT *isomorphic for some $k \geq 1$ if and only if M and N are stably tangentially homotopy equivalent.*

For the sake of completeness, we note that two manifolds are stably tangentially homotopy equivalent if and only if they are homotopy equivalent such that the stable tangent bundle of one pulls back to the stable tangent bundle of the other; other words, the direct sum of the tangent bundle with a trivial line bundle on the codomain pulls back to the corresponding bundle on the domain under the homotopy equivalence. As shown in [4], there are pairs of homotopy equivalent manifolds such that the homotopy equivalence is stably tangential but the unstable tangent bundle of the codomain does not pull back to the unstable tangent bundle of the domain.

In fact, if f exists then for some k the map $f \times \mathrm{id}(\mathbb{R}^k)$ is properly homotopic to a CAT-isomorphism; the topological version of this result follows from [**?**, **?**] if [**?**]

Given two manifolds M and N satisfying the conditions of the stable Equivalence Theorem, it is natural to ask the following:

OPTIMAL VALUE QUESTION. *For a given tangential homotopy equivalence $f : M \to N$, what is the least value of k such that $f \times \mathrm{id}(\mathbb{R}^k)$ is properly homotopic to a* CAT *isomorphism?*

If n is the common dimension of M and N, the standard embedding and stable tubular neighborhood theorems for CAT-manifolds imply that $n + 1$ is a universal upper bound for k in each category.

In [13] special cases of the Optional Value Question were considered for linear spherical space forms in the topological category, and in particular it was shown that if M and N are linear space forms such that $M \times \mathbb{R}^2$ is homeomorphic to $N \times \mathbb{R}^2$, then M and N are homeomorphic but $M \times \mathbb{R}^3$ and $N \times \mathbb{R}^3$ are not diffeomorphic. These results already reflect the relative complexity of this problem.

Our first result concerns a linear lens spaces.

Theorem 2. *Let $f : M \to N$ be tangential homotopy equivalence of linear lens spaces with prime order fundamental groups. Then $f \times id_{R^3}$ is properly homotopic to a homeomorphism.*

The restriction to linear lens spaces as opposed to homotopy lens spaces is crucial. It is well know that there are fake lens spaces L_1 and L_2 are not homeomorphic but $L_1 \times \mathbb{R}$ and $L_2 \times \mathbb{R}$ are diffeomorphic [**?**]
In this paper we shall study the Optimal Value Question for homotopy lens spaces. We will concentrate on the case of (odd) prime order fundamental groups, although many of our results hold without this restriction. Qualitatively, one can describe the results in terms of the concept we call **tangential thickness**. Specifically, two CAT manifolds M and N are said to have tangential thickness $\leq k$ if the following holds: Given a manifold M, let $\mathbf{TT}_k(M)$ denote the isomorphism classes of manifolds N such that $\{M, N\}$ has tangential thickness $\leq k$, and let $\mathbf{TT}(M)$ denote the isomorphism classes of manifolds that are stably tangentially homotopy equivalent to M. One then has an increasing sequence of sets $TT_k(M)$:

$$\{\text{class}(M)\} \;=\; \mathbf{TT}_0(M) \;\subset\; \mathbf{TT}_1(M) \;\subset\; \cdots \;\subset\; \mathbf{TT}_k(M) \;\subset\; \mathbf{TT}(M)$$

The sequence stabilizes for $k \geq \dim M + 1$ by Mazur's result, so that

$$\mathbf{TT}_k(M) = \mathbf{TT}_{k+i}(M) = \mathbf{TT}(M)\ , \qquad i \;=\; 1,\; 2,\; 3,\; \cdots\ .$$

In particular, given a manifold M then the classification of all manifolds having the tangential thickness $tt(M) = k$ is equivalent to the computation of the set $\mathbf{TT}_k(M) - \mathbf{TT}_{k-1}(M)$. We are ready now to state results of this paper.

Theorem 3. *Let $M^{2N-1}, N \geq 3$ be a homotopy lens space (arbitrary fake spherical space form of dimension ≥ 3). Then $\mathbf{TT}^{TOP}(M^{2n-1})$ consists of manifolds h-cobordant to M^{2n-1}. These manifolds are classified by $Wh(\,\pi_1(M2n-1)\,)$ via realization of Whitehead torsion by h-cobordant i.e. free torsion of $Wh(\,\pi_1(M^{2n-1})$ on M^{2n-1}.*

Theorem 4. *Let $M^{2n-1}, n \geq 3$ be a fake lens space. Then a manifold N^{2n-1} determines an element of $\mathbf{TT}_2^{TOP}(M^{2n-1})$ if and only if $N^{2n-1} \times \mathbb{R}$ is properly h-cobordant to $M^{2n-1} \times \mathbb{R}$. The difference set*

$$\mathbf{TT}_2^{TOP}(M^{2n-1}) - \mathbf{TT}_1^{TOP}(M^2n-1)$$

is in one-to-one correspondence with the nontrivial elements of $H^0\big(\,\mathbb{Z}_2; K_0(\mathbb{Z}[\pi_1(M^{2n-1})])\,\big)$. Moreover, all possible manifolds in $\mathbf{TT}_2^{TOP}(M^{2n-1}) - \mathbf{TT}_1^{TOP}(M^{2n-1})$ are obtained by a free action of $H^0\big(\,\mathbb{Z}_2; K_0(\mathbb{Z}[\pi_1(M^{2n-1})])\,\big)$. Moreover, all on $M^{2n-1} \times \mathbb{R}$ as the realization of Whitehead torsion by proper h-cobordisms.

<u>Theorem 3</u>. Let $M^{2n-1}, n \geq 3$ be a fake lens space with $\prod_1(M^{2n-1}) \cong 2p, p-$odd prime. The the set

TO BE COMPLETED

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