# Classical transcendental curves

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In his writings on coordinate geometry, Descartes emphasized that he was only willing to work with curves that could be defined by algebraic equations. He was not able to find such equations for some important curves from classical Greek geometry, including the quadratrix/trisectrix of Hippias (*c.* 460 B.C.E.– *c.* 400 B.C.E.), and he excluded them from his setting by describing them as "mechanical." Several decades later, Leibniz took a different view of the situation, recognizing that curves that are not definable by algebraic equations can — and in fact **should** — also be studied effectively using the methods of coordinate geometry and calculus.

Even though the Leibniz viewpoint is now universally accepted in analytic geometry and calculus, one can still ask whether certain classical Greek curves in the plane with no reasonably simple description by an algebraic equation are indeed not definable by an algebraic equation $F(x, y) = 0$, where $F$ is a nontrivial polynomial in $x$ and $y$ with real coefficients. The purpose of this note is to prove that two important examples have no description of this type. One is the quadratrix/trisectrix of Hippias, and the other is the Archimedean spiral, which is given in polar coordinates by $r = \theta$ and in rectangular coordinates by

$$\mathbf{s}(t) \quad = \quad (t\cos t,\ t\sin t) \ .$$

In order to analyze such curves, we shall need some background results which state that the so-called *elementary transcendental functions* do not satisfy equations of the form $F(x,\ f(x)\,) \equiv 0$, where $F$ is a nontrivial polynomial in two variables. This property is reflected by the use of the word "transcendental" to describe these functions, but proofs of such results do not appear in standard analytic geometry and calculus texts for several reasons (in particular, the mathematical level of such proofs is well above the levels suitable for basic courses in single variable calculus, and the results themselves are not needed for the usual applications of calculus to problems in other subjects). The following online reference contains explicit statements and proofs that exponential functions, logarithmic functions, and the six standard trigonometric functions do not satisfy the types of algebraic equations described above:

`http://math.ucr.edu/∼res/math144/transcendentals.pdf`

The discussion here will be at about the same mathematical level, using some input from advanced undergraduate and beginning graduate algebra courses together with standard results from calculus and differential equations courses. The following companion document contains figures related to the discussion here:

`http://math.ucr.edu/∼res/math153/transcurves2.pdf`

## 1. Algebraic curves and analytic parametrizations

Our goal is to prove a useful result which shows that if certain types of parametrized curves satisfy an algebraic equation near a point then they do so everywhere.

**PROPOSITION.** *Suppose that we are given a parametrized curve*

$$\mathbf{s}(t) \quad = (\, x(t),\, y(t)\,)$$

*defined on an open interval $J$ containing $t_0$, where the parametric equations are real analytic functions on $J$. If there is a strongly nontrivial polynomial $F(x,y)$ such that $F \circ \mathbf{s}(t) = 0$ for all $t$ sufficiently close to $t_0$, then this equation holds for all $t \in J$.*

**Proof.** Standard considerations show that the composite function $F \circ \mathbf{s}(t)$ is real analytic on $J$ (use Fact 2 in Section I.4 of `transcendentals`), and it is zero on some subinterval $(t_0 - \delta, t_0 + \delta)$. Therefore, by Fact 3 from the section cited in the previous sentence we know that $F \circ \mathbf{s}(t)$ is zero everywhere on $J$.■

## 2. The Archimedean Spiral

As noted above, the standard equation defining this curve **AS** in polar coordinates is $r = \theta$, where $\theta \geq 0$, and this yields the parametrization $\mathbf{s}(t) = (t\cos t,\, t\sin t)$, where $t \geq 0$. We shall show that there is no nonzero point $\mathbf{p}$ on this curve for which one can find an open neighborhood $U$ containing $\mathbf{p}$ and a strongly nontrivial polynomial $G$ such that $G \equiv 0$ for all points on $\mathbf{AS} \cap U$.

By the proposition in the preceding section and the existence of real analytic parametric equations for **AS**, if one can find a point, open neighborhood and polynomial as above, then it follows that $G \equiv 0$ on all points of **AS**. Denote the polynomial in question by $G(x,y) = \sum_{i,j}\ a_{i,j} x^i y^j$.

CLAIM 1: *Each vertical line $x = b$ and each horizontal line $y = c$ meets* **AS** *in infinitely many points.*

This seems clear if one sketches the curve (see Figure 1 in the file `transcurves2.pdf`), and on the coordinate axes we know that the points $(\,(-1)^k k\pi, 0)$ lie on both the spiral and the $x$-axis while the points $(0, (-1)^k(k + \frac{1}{2})\pi\,)$ lie on both the spiral and the $y$-axis. Since the problem is symmetric in $x$ and $y$, we shall only prove the statement regarding horizontal lines other than the $x$-axis, so that $c \neq 0$; the argument in the vertical case is similar.

The first step is to notice that if $(x,c)$ lies on **AS** and $c \neq 0$ and $(x,c) = \mathbf{s}(t)$, then we have

$$t^2 \quad = \quad x^2 \;+\; c^2 \qquad \text{and} \qquad \cot t \quad = \quad \frac{x}{c}\,.$$

This is illustrated in Figure 2 from the file `transcurves2.pdf`. We need to prove that, for each nonzero real number $c$, there are infinitely many values of $X$ which solve the resulting equation in $x$:

$$x \quad = \quad c \cdot \cot \sqrt{x^2 + c^2}$$

If we make the change of variables $u = \sqrt{x^2 + c^2}$, this equation can be rewritten in the form $c\cot u = \sqrt{u^2 - c^2}$, and the goal translates to showing that there are infinitely many solutions to this curve for which $u > |c|$. In the discussion below, $k$ will denote an arbitrary positive integer such that $k > |c|/\pi$.

CLAIM 2: *If $h(u) = c \cot u - \sqrt{u^2 - c^2}$, then for each $k$ as above there is a real number $u_k$ such that $k\pi < u_k < (k+1)\pi$ and $h(u_k) = 0$.*

The proof of this is similar to the proof that there are infinitely many solutions to the equation $\tan x = x$. Let $\varepsilon$ be the sign of $c$. Then one has the following one-sided limit formulas:

$$\lim_{u \to k\pi\,+} \varepsilon\, h(u) \;=\; +\infty \qquad\qquad \lim_{u \to (k+1)\pi\,-} \varepsilon\, h(u) \;=\; -\infty$$

It follows tht there is some real number $u_k$ between $k\pi$ and $(k+1)\pi$ such that $\varepsilon\, h(u_k) = 0$, and the claim follows because $\varepsilon\, h(u) = 0$ if and only if $h(u) = 0$.

*Completion of the proof that* **AS** *is not algebraic.* Suppose that $G$ is a polynomial in two variables such that $G(x, y) = 0$ for all $(x, y)$ on **AS**. For each real number $c$ we know there are infinitely many $x$ such that $G(x, c) = 0$. If $G(x, y) = \sum_{i,j} \ a_{i,j} x^i y^j$, then for each $i$ left hand side is thus a polynomial in $x$ which has infinitely many roots, and therefore the coefficients $\sum_j \ a_{i,j} c^j$ must be zero for all $i$. Define polynomials $h_i(y) = \sum_j \ a_{i,j} y^j$. Since $h_i(c) = 0$ for all $c$, it follows that for each $i$ we have $a_{i,j} = 0$ for all $j$; but this means that all the coefficients $a_{i,j}$ must vanish. This completes the proof that **AS** is not algebraic (in fact, it is not algebraic even if one restricts to some open interval of the positive real line).■

## 3. The Quadratrix/Trisectrix of Hippias

The standard equation for this curve is $y = x \cot x$ for $x \neq 0$; this limit of the right hand side as $x \to 0$ is equal to 1 (*e.g.*, this follows from L'Hospital's Rule or more elementary considerations), so it is customary to add the point $(0, 1)$ so that the curve becomes continuous for $|x| < \pi$. This curve has infinitely many disconnected pieces, but in classical Greek mathematics the portion of the curve receiving attention was the connected piece in the first quadrant with $0 < x < \frac{1}{2}\pi$ (see Figure 3 in `transcurves.pdf`.

Our objective is to prove that (the coordinates for) the points on this piece of the curve do not satisfy a strongly nontrivial polynomial equation in two variables. But this follows immediately by combining the results in Sections II.2 and I.4 in `transcendentals.pdf` with the following simple observation: *If the function $f$ defined on an open interval $J$ is transcendental on $J$, then so are the functions $x^m \cdot f$ for all $m > 0$.* These considerations imply that $x \cot x$ is transcendental on the interval $0 < x < \frac{1}{2}\pi$, and from this it follows that the coordinates on this piece of the curve do not satisfy a strongly polynomial equation.■