SOLUTIONS TO ADDITIONAL EXERCISES FOR II.1 AND II.2

Here are the solutions to the additional exercises in `triangle-exercises.pdf`. Illustrations to accompany these solutions are given in the online file

`trianglefigures.pdf`

in the course directory.

C1. Supppose first that X is one of the vertices A, B, C. In these cases the conclusion follows because $A \in AB \cap AC$ and each of these lines contains an infinite number of points on the triangle (namely, all points of $[AB]$ and $[AC]$ respectively), $B \in AB \cap BC$ and each of these lines contains an infinite number of points on the triangle (namely, all points of $[AB]$ and $[BC]$ respectively), and finally $C \in BC \cap AC$ and each of these lines contains an infinite number of points on the triangle (namely, all points of $[BC]$ and $[AC]$ respectively).

Suppose now that X is not a vertex. Without loss of generality we may assume that $X \in (AB)$, for the remaining cases where $X \in (BC)$ or $X \in (AC)$ follow by interchanging the roles of A, B, C in the argument we shall give.

If $X \in (AB)$ and $L = AB$, then clearly $X \in L$ and L contains infinitely many points of the triangle because it contains $[AB]$. From now on, suppose that $X \neq AB$.

If $L = XC$, then X and C lie on both L and the triangle; we claim that no other point of L satisfies these conditions. Suppose to the contrary that there is such a third point Y; there are three cases depending upon whether Y lies on AB, BC or AC. If $Y \in AB$, then both L and AB contain the distinct points X and Y, so that $L = XY$; but we are assuming that X, Y, C are collinear, and this contradicts our even more basic assumption that A, B, C are noncollinear (this is implicit in asserting the existence of ΔABC). Therefore the line XC only meets the triangle in two points.

Now suppose that $C \notin L$ and $L \neq AB$; we need to show that L and ΔABC have at most one other point in common besides X. By Pasch's Theorem there is a second point Y on L which lies on either (BC) or (AC); in either case there we claim that there is no third point in $L \cap \Delta ABC$. Since $X \in AB$ is not one of the vertices and the lines BC and AC meet AB in B and A respectively, it follows that X lies on neither of these lines. Therefore the line $L = XY$ meets ΔABC in two sides and cannot contain any of the vertices. If there were a third point, it would lie on one of (AB), (BC) or (AC). By Exercises II.2.8 we know that L cannot contain points of all three sides,, and if the third point were in (AB) it would follow that $L = AB$. On the other hand, the line L cannot contain X and two points from either (AC) or (BC), for in that case L would be equal to AC or BC and also contain $X \in (AB)$, so that L would also be equal to AB. Thus the existence of a third point leads to a contradiction if $L \neq AB, XC$, and hence no such point can exist, so that all lines through X except AB meet the triangle in two points.■

C2. The most difficult parts of this proof were done in the preceding exercise. Let $\mathbf{T}$ be equal to $\Delta ABC = \Delta DEF$. By the preceding exercise, since $\mathbf{T} = \Delta ABC$ we know that $\{A, B, C\}$ is the set $\mathbf{V}$ of all points X in $\mathbf{T}$ such that two lines through X contain at least three points of $\mathbf{T}$, and likewise $\{D, E, F\}$ is the set $\mathbf{V}$ of all points X in $\mathbf{T}$ such that two lines through X contain at least three points of $\mathbf{T}$. Therefore we have $\{A, B, C\} = \mathbf{V} = \{D, E, F\}$.■

C3. Let H_1 and H_2 denote the two half-planes associated to L. Then each of the points A, B, C lies on exactly one of the subsets L, H_1, H_2.

Before we split the argument into cases using the previous sentence, we make a general observation. Since $X \in L$ lies in the interior of ΔABC, by the Crossbar Theorem we know that $(BX$ and (AC) have a point Y in common. This point cannot be X because a point cannot lie in both the interior of ΔABC and one of the three sides AB, BC, AC (look at the definition of interior). Since $Y \in (BC$, it follows that either $B * X * Y$ or $B * Y * X$ is true; however, if the latter were true, then B and X would lie on opposite sides of the line $AY = AC$, contradicting the assumption on X. Therefore we must have $B * X * Y$.

We claim that all three vertices cannot lie in either H_1 or H_2. If they did, then by convexity the point Y in the preceding paragraph would also lie in the given half-plane, and similarly the point X would lie in this half-plane. Since $X \in L$ by assumption, this is impossible, and thus the three vertices cannot all lie on one side of L.

Next, we claim that at most one vertex lies on L. If, say, $A \in L$, then $L = AX$, and if either B or C were also on L we would have that $L = AB$ or AC, which in turn would imply that $X \in AB$ or AC, contradicting the condition that X lies in the interior of the triangle. The cases where $B \in L$ and $C \in L$ can be established by interchanging the roles of the three vertices in the preceding argument.

Suppose now that one vertex lies on L; we claim that the other two vertices must lie on opposite sides of L. Once again, it is enough to consider the case where $A \in L$, for the other cases will follow by interchanging the roles of the vertices. But if $A \in L$, then the Crossbar Theorem implies that L and (BC) have a point W in common (in fact (BC) and $(AX$ do), and therefore it follows that B and C lie on opposite sides of L. Furthermore, it follows that the line L meets the triangle in the distinct points A and W.

The only possibility left to consider is the case where no vertex lies on L; by the preceding discussion, we know that neither half-plane contains all three vertices, and thus two of the vertices are on one half-plane and one is on the other. As before, without loss of generality we may assume that A is on one side and B, C are on the other. But in this situation we know that the line L meets both (BC) and (AC). This completes the examination of all possible cases.■

C4. We shall follow the hint and solve for $\mathbf{x}$ in terms of $\mathbf{y}$. Since $\mathbf{y} = A\mathbf{x} + \mathbf{b}$ and A is invertible, it follows that $\mathbf{x} = A^{-1}(\mathbf{y} - \mathbf{b})$. If we substitute this into the defining equation for the plane, we see that

$$d \;=\; C\mathbf{x} \;=\; CA^{-1}(\mathbf{y} - \mathbf{b}) \quad \text{or equivalently} CA^{-1}\mathbf{y} \;=\; d \;+\; CA^{-1}\mathbf{b}$$

which shows that $\mathbf{y}$ is defined by an equation of the form $P\mathbf{y} = q$, where $P = CA^{-1}$ and $q = d + CA^{-1}\mathbf{b}$.■