Isovariance, the Gap Hypothesis, and bounded manifolds

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Abstract. If two closed oriented manifolds with smooth semifree group actions satisfy a condition called the Gap Hypothesis are equivariantly homotopy equivalent, then the equivalence between them can always be deformed to an isovariant homotopy equivalence. This paper gives and analog for compact bounded manifolds and equivariant homotopy equivalences that are isovariant on the boundary, showing that there are deformations to isovariant equivalences that are fixed on the boundary. The proof is homotopy theoretic like the author's approach to the existence result, but it also requires some additional machinery. A uniqueness theorem for isovariant deformations follows directly from the main result.

As indicated in our previous paper [Sc9], there are two basic notions of homotopy equivalence for spaces with group actions; namely, equivariant homotopy equivalences and the stronger concept of isovariant homotopy equivalence. In particular, the classifications of certain kinds of group actions on manifolds up to both types of equivalence have been extensively studied, and the main result of [Sc9] shows that these coincide if the manifolds in question satisfy a condition called the **Gap Hypothesis**.

There is a fairly extensive discussion of such matters in [Sc9], so we shall only summarize points that are important for our purposes. First, we shall restrict attention to group actions on manifolds that are smooth and **semifree** (the group acts freely off the fixed point set — note that this holds for *all* actions of a cyclic group $\mathbb{Z}_p$ of prime order p). Second, we shall assume the Gap Hypothesis condition, which implies that the dimensions of the ambient manifold X and the fixed point set F satisfy the inequality

$$\dim F + 1 \leq \tfrac{1}{2} \dim X \ .$$

The main results of [Sc9] (see Theorem 1.1) state that if M and N are closed (compact, unbounded, connected) smooth manifolds with semifree actions of a finite group G which satisfy the Gap Hypothesis, and $f : M \to N$ is a G-equivariant homotopy equivalence, then f is equivariantly homotopy to a G-isovariant homotopy equivalence. The main result of this paper is a relative analog of Theorem 1.1 in [Sc9] for manifolds with boundary.

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Theorem 1. *Let G be a finite group, let $(M, \partial M)$ and $(N, \partial N)$ be compact, bounded, smooth semifree G-manifolds that satisfy the Gap Hypothesis, and let $f : (M, \partial M) \to (N, \partial N)$ be an equivariant homotopy equivalence that restricts to an isovariant homotopy equivalence from ∂M to ∂N. Then f is equivariantly homotopic to an isovariant homotopy equivalence, and in fact one can choose the homotopy to be constant on ∂M.*

One important consequence of this result is the following uniqueness property for the isovariant homotopy equivalences given by this result and Theorem 1.1 from [Sc9]:

Theorem 2. *In the setting of the previous result, suppose that f and g are isovariant homotopy equivalences of pairs from $(M, \partial M)$ to $(N, \partial N)$ that are equivariantly homotopic, and assume further that the Gap Hypothesis holds for $M \times \mathbb{R}$ and $N \times \mathbb{R}$. Then there is an equivariant deformation of the equivariant homotopy between f and g to an isovariant homotopy, and this deformation is fixed on $M \times \{0, 1\}$; furthermore, if the original equivariant homotopy is isovariant on $\partial M \times [0, 1]$, then one can choose the equivariant deformation so that it is also fixed on the $\partial M \times [0, 1]$.*

Note that if $X \times \mathbb{R}$ satisfies the Gap Hypothesis then we have

$$\dim(F \times \mathbb{R}) + 1 = \dim F + 2 \leq \tfrac{1}{2} \dim(X \times \mathbb{R}) \leq \tfrac{1}{2}(\dim X + 1)$$

so that

$$\dim F + 1 \leq [\tfrac{1}{2}(\dim X + 1)] - 1 = \tfrac{1}{2}(\dim X - 1) < \tfrac{1}{2} \dim X$$

and hence X also satisfies the Gap Hypothesis. Therefore we obtain the following immediate consequence:

Corollary 3. *Suppose that M is a compact smooth semifree G-manifold where G is a finite group acting semifreely on M. Let $\mathcal{E}_G(M)$ be the group of all equivariant homotopy classes of equivariant homotopy self-equivalences of M, let $\mathcal{E}_G^{\mathbf{ISO}}(M)$ be the group of all isovariant homotopy classes of equivariant homotopy self-equivalences of $(M.\partial M)$, and let $j : \mathcal{E}_G^{\mathbf{ISO}}(M) \to \mathcal{E}_G(M)$ be the forgetful homomorphism. Then j is surjective if M satisfies the Gap Hypothesis and j is bijective if $M \times \mathbb{R}$ satisfies the Gap Hypothesis.*

In geometric topology, such relative theorems and uniqueness results are frequently straightforward corollaries of the corresponding absolute existence theorems, and the proofs are often included almost as afterthoughts to the latter. Since we did not include the relative and uniqueness results in [Sc9], we should explain why they do not follow immediately from the methods described in [Sc9]. A crucial step in the latter is to deform an equivariant homotopy equivalence so that it is isovariant near the fixed point set; in fact, one chooses the deformed map so that it satisfies a homotopy theoretic analog of transversality near the fixed point set that is called *normal straightening*. Suppose now that we are given an equivariant homotopy equivalence F of manifolds with boundary and that the restriction of F to the boundary determines an isovariant homotopy equivalence ∂F. The results of [DuS] imply that ∂F is isovariantly homotopic to a map that is normally straightened near the fixed point set, and the results of [Sc9] imply the map F is equivariantly homotopic to a map of pairs that is normally straightened near the fixed point set. Thus we obtain two deformations of ∂F to maps that are normally straightened near the fixed point set.

Question. *Can one choose these deformations so that they determine compatible normal straightenings near the fixed point set of the boundary?*

Here is one way of making the notion of compatibility more precise:

Sharpened question. *Are the restrictions of the maps to neighborhoods of the fixed point set isovariantly homotopic?*

If it is possible to answer the sharpened question affirmatively, then the balance of the proof of Theorem 1 will be a routine extension of the arguments in [Sc9]. However, the deformation to a normally straightened map in the latter involves the arbitrary choice of an equivariant fiber homotopy equivalence, so the method provides no way of ensuring a positive answer to the compatibility question. In fact it is very easy to construct examples for which the induced maps near the fixed point sets are not at all compatible (in particular, their restrictions to neighborhoods of the fixed point sets need not be isovariantly homotopic).

In order to overcome such difficulties we must introduce some means for guaranteeing compatibility. We shall do this using equivariant analogs of the normal invariants which arise in ordinary surgery theory (compare Rourke [Rk], p. 140, as well as Chapter 3 of Lück [L] and Browder's book [Br1]). As in the nonequivariant case, such maps can be defined and studied directly by homotopy theoretic methods without any need to discuss surgery problems as such. Using such invariants we shall describe *canonical choices* for normal straightenings with the desired compatibility properties. These invariants lie in equivariant analogs of some standard homotopy functors which arise in surgery theory (*cf.* [MaMi]). Although the constructions of these objects parallel the nonequivariant case, clear statements and proofs are difficult — and in some cases impossible — to find in the literature, so for the sake of completeness it will be necessary to spend some time describing the sets in which our equivariant normal invariants are defined.

In a subsequent paper [DoS] we shall use Theorem 1 in proving a generalization of the main result in [Sc9] (namely, Theorem 2.2) to some classes of examples just outside the Gap Hypothesis range satisfying $\dim M = 2\,\dim M^G + \varepsilon$, where ε is either 1 or 0. On the other hand, in yet another paper [Sc10] we shall construct examples to show that the main result in [Sc9] is false for some examples satisfying $\dim M = 2\,\dim M^G + 1$, and simple modifications of such examples will also show that Theorem 2 is also false for some examples satisfying $\dim M = 2\,\dim M^G + 2$.

Here is a summary of the paper. In Section 1 we shall discuss the equivariant homotopy functors in which the equivariant normal invariants are defined. Some of these ideas were previously considered in [Sc5] and [Sc4], but we shall need some further properties, and in particular we shall need a splitting theorem for such functors over spaces on which the group acts trivially. The second section contains the definitions of the equivariant normal invariants and some of their important formal properties; in particular, if an equivariant homotopy equivalence is isovariant, these yield a close relationship between the normal invariant to the behavior of the isovariant homotopy equivalence near the fixed point set. We shall use this relationship and ideas from [Sc9] to prove the required compatibility condition in Section 3. Finally, in Section 4 we shall prove the main theorems and include some additional remarks.

1. Equivariant fiber retraction structures

Unfortunately, the phrase "proper mapping" has two distinct meanings in geometric topology, and one of them plays a key role here. In this paper a **proper** map $f : (M, \partial M) \to (N, \partial N)$ of manifolds with boundary will always mean that f also maps interior of M to the interior of N, or equivalently that $f^{-1}(\partial N) = \partial M$. A simple application of the smooth Collar Neighborhood Theorem implies that if f is an arbitrary continuous map of bounded manifold pairs, then f is homotopic to a proper mapping in the sense described above; one can similarly use an equivariant version of the Collar Neighborhood Theorem to prove an equivariant analog if the bounded manifolds in question have smooth actions of compact Lie groups.

One crucial step in [Sc9] depends on the following result of K. Kawakubo [Ka]:

Homotopy invariance of stable equivariant normal bundles. *Let G be a compact Lie group, let M and N be compact oriented smooth G-manifolds, suppose that $f : (M, \partial M) \to (N, \partial N)$ is an equivariant homotopy equivalence of pairs, and let ν_M and ν_N denote the equivariant normal bundles for smooth proper equivariant embeddings of M and N in $\Omega \times \mathbb{R}_+$, where Ω is a finite dimensional orthogonal G-representation and $\mathbb{R}_+$ denotes the nonnegative reals with trivial G-action. Then the unit sphere bundles of ν_M and ν_N are stably fiber homotopically equivalent.*

Remark. Standard (equivariant smooth) embedding and isotopy theorems imply that the stable equivalence classes of the equivariant normal bundles ν_M and ν_N do not depend upon the choice of equivariant smooth embedding because all such embeddings are stably ambient isotopic.

In this paper we shall need a stronger version of this result. Specifically, the existence of a **canonical choice** of stable G-fiber homotopy equivalence between the equivariant normal bundles of M and N. Standard considerations involving direct sums of vector bundles imply the existence of such a stable equivalence is equivalent to the existence of a stable G-fiber homotopy trivialization for the unit sphere bundle of $\nu(N) \oplus \xi$, where $f^*\xi$ is stably inverse to $\nu(M)$ with respect to direct sum, and it is usually more convenient to work with this equivalent formulation.

The construction of such equivalences and trivializations is formally parallel to a standard well known construction in the nonequivariant case. In particular, a more general version of the latter is discussed in [Rk, p. 140], especially the discussion preceding diagram (3.8); this reference cites results of M. Spivak on Poincaré duality spaces [Spv], but since we are only interested in smooth manifolds one can replace this with a reference to Atiyah's earlier paper on Spanier-Whitehead duality and Thom complexes [At]. With this substitution, everything extends to manifolds with group actions if one uses K. Wirthmüller's equivariant version of Spanier-Whitehead duality [Wi] and the results of Kawakubo.

Standard results in algebraic topology yield a classifying space for equivalence classes of stable vector bundles with stable fiber homotopy trivializations, and we shall denote this space by F/O; this space is also frequently denoted by G/O (*cf.* [MaMi]), but it seems better to avoid conflicting uses for G in a paper about group actions. We shall need the equivariant analogs of this space and especially some basic properties with no counterparts in the nonequivariant case. Similar objects were considered in [Sc5, Section 2, page 261]. The most important conclusion for our purposes is Theorem 2.4, which will be especially important for the proofs of the main results.

Let G be a compact Lie group, and (X, x_0) be an invariantly pointed G-space; this means that $x_0 \in X$ is a fixed point of the action. Given a finite-dimensional orthogonal G-representation M, we shall say that an *M-pointed G-vector bundle over X with a G-equivariant fiber retraction* is a pair (ξ, ρ), where ξ is a pointed G-vector bundle over X (so we are given a G-isomorphism from M to the fiber over x_0) and $\rho : S(\xi) \to S(M)$ is a G-equivariant map such that the restriction of ρ to the fiber of x_0 yields the identity and the restriction of ρ to every fiber is a homotopy equivalence. As usual, $S(\xi)$ denotes the unit sphere bundle of ξ with respect to some riemannian metric. Two such objects (ξ_j, ρ_j) — where $j = 0$ and 1 — are equivalent if and only if there is a similar sort of object (η, σ) over $X \times [0, 1]$ and there are G-vector bundle isomorphisms $h_j : \xi_j \to \eta|X \times \{j\}$ such that $\rho_j = \sigma \circ h_j$. Not surprisingly, this yields an equivalence relation, and the equivalence classes form a based set that we shall call $F/O_{G,M}(X, x_0)$; the base point is given by the product bundle $X \times M$ with the usual projection map from $X \times S(M)$ to $S(M)$. The pullback construction makes the sets $F/O_{G,M}(X, x_0)$ into a G-homotopy functor on invariantly pointed G-spaces, and it is routine to check that the functor is equivariantly representable (*cf.* the discussion of a related construction in [Sc5, p. 261]). The Whitney sum determines well behaved natural transformations

$$F/O_{G,M}(X, x_0) \times F/O_{G,N}(X, x_0) \longrightarrow F/O_{G,M\oplus N}(X, x_0)$$

for each pair of representations M and N, and one can take direct limits in the usual way to form a corresponding stabilized functor $F/O_G(X, x_0)$, which is also a representable G-homotopy functor.

Given (X, x_0) and M as above, we can also define a set of M-based G-vector bundles $\mathrm{Vect}_{G,M}(X, x_0)$ consisting of a G-vector bundle and an identification of the fiber over x_0 with the representation M; as above, direct sum defines a natural transformation from $\mathrm{Vect}_{G,M} \times \mathrm{Vect}_{G,N}$ to $\mathrm{Vect}_{G,M\oplus N}$, and one can stabilize to obtain objects that can be called Vect_G; the existence of inverse bundles in equivariant K-theory (at least for reasonable choices of X including finite G-CW complexes) implies that the latter is isomorphic to the reduced equivariant real K-theory $KO_G(X, x_0)$. There are natural maps from the functors $F/O_{G,M}$ and F/O_G to their counterparts $\mathrm{Vect}_{G,M}$ and Vect_G, and these maps commute with direct sum constructions.

Likewise, given an element of $F/O_{G,M}(X, x_0)$, there is an underlying G-vector bundle, and if we stabilize we obtain an additive natural transformation from F/O_G to Vect_G. This natural transformation fits into long exact sequences which extend infinitely to the left as in Section 3 of [Sc5]. The "fiber functor" F_G for the natural transformation $F/O_G \to \mathrm{Vect}_G$ is equivariantly representable as follows: Given a G-representation M, let $\mathcal{E}_G(M)$ denote the space of equivariant self maps of the unit sphere $S(M)$ and let G act on this function space by conjugation. If N is another such representation then the join construction defines a continuous equivariant homomorphism from $\mathcal{E}_G(M)$ to $\mathcal{E}_G(M \oplus N)$, and if we take colimits over a suitable family of representations (for example, finite-dimensional subrepresentations of the countable direct sum $\oplus^\infty \mathbf{L}^2(G)$, where the summands are the usual L^2 group algebra), then we obtain a limit space $\mathcal{E}_G$ which represents the functor F_G.

Variants with orbit type restrictions

A fundamental difference between the preceding constructions and those of [Sc5] is that the latter involve *free* G-vector bundles in the sense of [Sc4]; *i.e.*, the group G acts freely off the zero section of the total space. The corresponding sets obtained using such vector bundles are denoted by $F/O_{G,\mathrm{free},M}(X, x_0)$, $F/O_{G,\mathrm{free}}(X, x_0)$, and $\mathrm{Vect}_{G,\mathrm{free}}(X, x_0)$. There are obvious natural maps

$$F/O_{G,\mathrm{free},M}(X, x_0) \longrightarrow F/O_{G,M}(X, x_0)$$

$$F/O_{G,\mathrm{free}}(X, x_0) \longrightarrow F/O_G(X, x_0)$$

$$\mathrm{Vect}_{G,\mathrm{free}}(X, x_0) \longrightarrow \mathrm{Vect}_G(X, x_0)$$

that are given by passage from free G-vector bundles to arbitrary G-vector bundles. By construction these maps all preserve direct sums. Similarly, one has functors $F_{G,\mathrm{free},M}$ and $F_{G,\mathrm{free}}$ defined using free G-representations rather than arbitrary ones. Furthermore, the natural homomorphism from $F/O_{G,\mathrm{free}}(X, x_0)$ to $\mathrm{Vect}_{G,\mathrm{free}}(X, x_0)$ fits into an exact sequence exactly as in the unrestricted case (see Section 3 of [Sc5]), and there is an exact commutative ladder diagram containing the two exact sequences.

We shall also need another variant of the constructions which lies between objects of the form $T_{G,\mathrm{free}}$ and T_G, where T is one of F, F/O or $\widetilde{KO}_G$ or their corresponding unstable versions. Specifically, one can define functors

$$F_{G,\mathrm{semifree},M}(X, x_0), \qquad F_{G,\mathrm{semifree}}(X, x_0)$$

$$F/O_{G,\mathrm{semifree},M}(X, x_0), \qquad F/O_{G,\mathrm{semifree}}(X, x_0)$$

$$\widetilde{KO}_{G,\mathrm{semifree}}(X, x_0)$$

such that all representations ane G-vector bundles are taken to have semifree actions. Everything discussed thus far (and more) will go through with only the obvious changes, and all of the natural maps

$$T_{G,\mathrm{free}} \longrightarrow T_G$$

described above factor canonically through the corresponding functors $T_{G,\mathrm{semifree}}$.

Inverse bundles. We have noted that $\mathrm{Vect}_G(X, x_0)$ is naturally isomorphic to $KO_G(X, x_0)$ if X is a reasonable space because G-vector bundles over such spaces have stable inverses. Clearly one can define abelian groups $KO_{G,\mathcal{A}}(X, x_0)$, where $\mathcal{A} =$ "free" or "semifree," and there are obvious natural homomorphisms from the abelian monoids $\mathrm{Vect}_G(X, x_0)$ to the abelian groups $KO_{G,\mathcal{A}}(X, x_0)$. These maps will be bijective if and only if $\mathrm{Vect}_G(X, x_0)$ is an abelian group. If G is finite cyclic and $\mathcal{A} =$ "free," then this is true by Proposition 1.4 in [Sc4]. More generally, it will be true for a group G if the following question has an affirmative answer:

Extension problem. *Suppose G is a compact Lie group that admits a free finite dimensional orthogonal representation, let H be a closed subgroup of G, and suppose that V is a free finite dimensional representation of H. Is there a free representation W of H such that $V \oplus W$ extends to a free representation of G?*

Certainly the answer to this question is false if $G = S^1$, and an example from Section 1 of [Sc4] shows that the monoid $\mathrm{Vect}_{G,\mathrm{free}}(X, x_0)$ need not have inverses in this case. It is considerably more difficult to determine whether there is an affirmative answer if G is finite; in particular, the answer is affirmative for many noncyclic examples, including generalized quaternionic groups. However, under suitable elementary conditions on (X, x_0) one can conclude that inverses always exist, and this turns out to suffice our purposes here. In particular, we have the following:

Lemma 1.0. *Suppose that (X, x_0) is an invariantly pointed finite G-equivariant CW complex and that G acts semifreely on X. Then inverse vector bundles exist in the abelian monoid* $\mathrm{Vect}_{G,\mathrm{semifree}}(X, x_0)$

Proof. In this case the answer to the extension problem is clearly yes for all isotropy subgroups of the action on X; it is a tautology for G, and it is also clearly true if $H = \{1\}$. These conditions are enough to carry out the construction of an inverse bundle described in Proposition 1.4 of [Sc4].■

Splitting principles for spaces with trivial actions

If the group G acts trivially on the space X, then basic results on equivariant K-theory imply that $KO_G(X)$ splits canonically into a direct sum of copies of the ordinary real, complex and quaternionic K-theories $KO(X)$, $KU(X) = K(X)$ and $KSp(X)$, where the sum is indexed by the list of irreducible representations and the type of K-theory depends upon whether the skew field for a representation is the real numbers, the complex numbers, or the quaternions (*cf.* [Se1, p. 134, lines 4–6]; in this passage Segal uses KR for the object now generally called KO). Corresponding decompositions for the reduced groups $\mathrm{Vect}_G(X, x_0)$ also follow immediately. As in [Sc4], similar results hold for $\mathrm{Vect}_{G,\mathrm{free}}$ and $\widetilde{KO}_{G,\mathrm{free}}$, the only difference being that the summation runs through all free irreducible G-representations; the corresponding results for $\mathrm{Vect}_{G,\mathrm{semifree}}$ and $\widetilde{KO}_{G,\mathrm{semifree}}$ also follow from such considerations, and the various decompositions all have the expected naturality properties. The following elementary relationship between $\mathrm{Vect}_{G,\mathrm{free}}$ and $\mathrm{Vect}_{G,\mathrm{semifree}}$ for trivial G-spaces is part of a pattern which is important for our purposes.

Proposition 1.1. *Suppose that (X, x_0) is a pointed G-space with a trivial action. Then there is a natural isomorphism from $\mathrm{Vect}_{G,\mathrm{semifree}}(X, x_0)$ to the direct sum*

$$\mathrm{Vect}_{G,\mathrm{free}}(X, x_0) \oplus \widetilde{KO}(X, x_0)$$

such that projection onto the second factor is given by taking the fixed point subbundle and injection into the first factor is the forgetful map from free to semifree G-vector bundles.

Proof. These follow directly from the direct sum decompositions of the groups

$$\mathrm{Vect}_{G,\mathrm{semifree}}(X, x_0), \qquad \mathrm{Vect}_{G,\mathrm{free}}(X, x_0)$$

into copies of nonequivariant real, complex and quaternionic K-theory; in the semifree case there is one extra summand of real K-theory arising from the unique irreducible semifree representation that is not free; namely, the trivial representation.■

We shall need corresponding and compatible splittings for our functors of the types F_G and F/O_G. These are less trivial to derive than the formulas for Vect, and the formula for F/O will be crucial to handling the main technical problem in this paper. The first of these follows directly from the methods of [Se2] and [BeS]. As in [BeS], if V is a free G-representation we define $F_G(V)$ to be the topological monoid of all G-equivariant self maps of the unit sphere $S(V)$ and take F_G to be the limit of the spaces $F_G(V)$ for some appropriate collection of free G-representations V; by the results of [BeS] the homotopy types of F_G and its classifying space BF_G do not depend upon the choice so long as the family has representations of arbitrarily large dimensions.

Proposition 1.2. *If (X, x_0) as above is a trivial G-space then there is a natural isomorphism $F_{G,\mathrm{semifree}}(X, x_0) \cong [X, F] \times [X, F_G]$ such that the following diagram commutes:*

$$\begin{array}{ccccc}
F_{G,\mathrm{free}}(X, x_0) & \longrightarrow & F_{G,\mathrm{semifree}}(X, x_0) & \longrightarrow & F_{\{1\}}(X, x_0) \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
[X, F_G] & \longrightarrow & [X, F_G] \times [X, F] & \longrightarrow & [X, F]
\end{array}$$

The horizontal arrows on the left are given by the inclusion of the family of free representations into the family of semifree representations and the horizontal arrows on the right are given by passage to fixed point sets.

This follows directly from the methods of [BeS] and [Se2].■

The topological monoid $F_G \times F$ has a classifying space of the form $BF_G \times BF$, and by results of S. Waner [Wa] the homotopy functor $[X, BF_G \times BF]$ classifies stabilized equivariant spherical fibrations over X such that the equivariant homotopy type of the fiber is a sphere with a semifree orthogonal action. In analogy with our constructions for vector bundles, we can define unstable and stable functors on trivial G-spaces

$$\mathrm{Sph}_{G,(\mathrm{semi})\mathrm{free},M} \qquad \mathrm{Sph}_{G,(\mathrm{semi})\mathrm{free}} \qquad \mathrm{Sph}_G$$

and there is an additive structure given by fiberwise joins which is compatible with direct sums for vector bundles. In particular, there are obvious additive maps from $\mathrm{Vect}_{G,\mathcal{A}}$ to $\mathrm{Sph}_{G,\mathcal{A}}$, and one can extend the previously described infinite exact sequences relating $F_{G,\mathcal{A}}$, $F/O_{G,\mathcal{A}}$, and $\mathrm{Vect}_{G,\mathcal{A}}$ one step to the right so that they include $\mathrm{Sph}_{G,\mathcal{A}}$.

Claim. *If $\mathcal{A}$ is either* "free" *or* "semifree"*, then all objects in such exact sequences are abelian groups with respect to direct sum or fiberwise join.*

Sketch of proof. We know that everything in sight has a commutative monoid structure, and we know that $\mathrm{Vect}_{G,\mathcal{A}}$ has a group structure by the decomposition of an element into a sum of ordinary vector bundles over various division rings. We also know that $\mathrm{Sph}_{G,\mathcal{A}}$ is represented by BF_G if $\mathcal{A} =$ "free" and by $BF_G \times BF$ if $\mathcal{A} =$ "semifree." The fiberwise join construction defines compatible binary operations on these spaces, and by the results of [BeS2] they have the homotopy-everything properties needed to show that the operations come from infinite loop space structures. Thus in the exact sequences of functors

$$\cdots \to F_{G,\mathcal{A}} \longrightarrow F/O_{G,\mathcal{A}} \longrightarrow \mathrm{Vect}_{G,\mathcal{A}} \longrightarrow \mathrm{Sph}_{G,\mathcal{A}}$$

all terms except possibly $F/O_{G,\mathcal{A}}$ are abelian group valued. An elementary diagram chase then shows that this remaining term is also abelian group valued.■

We are finally ready to state and prove a result that plays a crucial role in our paper.

Proposition 1.3. *If (X, x_0) as usual is a trivial G-space then there is a natural abelian group isomorphism $F/O_{G,\mathrm{semifree}}(X, x_0) \cong [X, F/O] \oplus F/O_{G,\mathrm{free}}(X, x_0)$ such that the projection onto the first factor is given by taking fixed point sets and the injection from the second factor is given by inclusion of the family of free representations in the family of semifree representations. These splittings are compatible with similar splittings in which F/O is replaced by F or* Vect *or* Sph.

Proof. First of all, note that the composite

$$F/O_{G,\mathrm{free}}(X, x_0) \longrightarrow F/O_{G,\mathrm{semifree}}(X, x_0) \longrightarrow [X, F/O]$$

is obviously trivial because it takes a free G-vector bundle to the subbundle determined by its zero section. Next observe that the projection onto $[X, F/O]$ has a one-sided inverse given by inclusion of the family of trivial representations into the family of semifree representations, and this map is clearly additive and compatible with similar maps for F or Vect or Sph; for each of the latter, one has direct sum decompositions of the type described in the proposition. Using all these facts, one can prove the required direct sum decomposition for $F/O_{G,\mathrm{semifree}}(X, x_0)$ by a straightforward but slightly tedious diagram chase.■

Unreduced analogs of the stabilized functors

Although the unstable functors $T_{G,\mathcal{A},M}$ depend upon the base point — or more correctly, on the connected component of the latter in the fixed point set — the stabilized versions do not really require choices of base points. To see this, given (X, x_0) consider the invariantly pointed G-space (X_+, ∞) where $X_+ = X \sqcup \{\infty\}$ and G acts trivially on the new added base point. We shall **define** the corresponding absolute group $T_{G,\mathcal{A}}(X)$ to be $T_{G,\mathcal{A}}(X_+, \infty)$. There is a unique "initial object" map from (X_+, ∞) to (X, x_0) sending X to itself by the identity and sending ∞ to x_0, and this induces a canonical natural homomorphism θ from $T_{G,\mathcal{A}}(X, x_0)$ to $T_{G,\mathcal{A}}(X_+, \infty)$.

Claim. *The map θ is an isomorphism.*

Proof of Claim. The inclusion map of $\{\infty, x_0\}$ in X_+ is an equivariant retract, for an explicit one sided inverse is given by the map that is the identity on ∞ and sends all points of X to x_0. If we combine this with the long exact homotopy sequence of the equivariant cofibration $\{\infty, x_0\} \subset X_+ \to X$, we see that θ is split injective. In order to prove that θ is surjective, it suffices to check that

$$F/O_{G,\mathcal{A}}\big(\{\infty, x_0\},\, \infty\big) \;=\; \{0\}$$

which reduces to showing that (at least stably) every equivariant homotopy self-equivalence of a sphere with an orthogonal action of a finite group is equivariantly homotopic to an orthogonal mapping. As noted in [Sc9], this is well known and in particular follows from the Equivariant Hopf Theorem (*cf.* tom Dieck [tD], Thm. 8.4.1, pp. 213–214).■

2. Equivariant normal invariants and their properties

We have already mentioned that a homotopy equivalence of compact (possibly) bounded smooth manifolds $f : (M, \partial M) \to (N \partial N)$ has an associated *smooth normal invariant* which lies in the group $[N, F/O]$. The purpose of this section is to define similar objects for equivariant homotopy equivalences. Most if not all of this has been known for some time but has not appeared in print explicitly. Since we shall need some basic properties of these generalizations to prove our main results, it is necessary to include formal statements here. Our main objectives are formulas involving equivariant homotopy equivalences that are isovariant, either everywhere or only on the boundary (see Proposition 2.3 and Theorem 2.4 below).

General remark. *None of the results in this section assume the Gap Hypothesis.*

Following standard terminology as in [At], if X is a space and α is a vector bundle over X then X^α will denote the Thom space of α, with a canonical invariant base point given by the "point at infinity." Similarly, if X is a manifold with boundary then $(X, \partial X)^\alpha$ will denote the quotient obtained by collapsing the portion of bundle over ∂X to a point.

As in the nonequivariant case, one can define normal invariants more generally for certain mappings of degree ± 1 with some extra structure that always exists if one has a homotopy equivalence. In fact there are several equivariant generalizations of such *normal maps* (or surgery problems) in the literature, so we begin by describing the concept needed in this paper. As before, ν_X denotes the equivariant normal bundle of some suitably chosen proper equivariant smooth embedding of the compact smooth G-manifold X.

Definition. Let G be a compact Lie group, and let $(N, \partial N)$ be a compact smooth G-manifold with boundary. A **weakly structured G-normal map** is a triple $(f, \mathbf{b}, \xi)$ consisting of (1) a proper G-mapping $f : (M, \partial M) \to (N, \partial N)$ of smooth G manifolds with boundary such that the dimensions of M and N are equal, (2) a G-vector bundle ξ over N such that the dimensions of the fibers of ξ and ν_M are equal, (3) a map $\mathbf{b} : E(\nu_M) \to E(\xi)$ of total spaces such that for each $x \in M$ the map $\mathbf{b}$ sens the fiber $E_x(M)$ to the fiber $E_{f(x)}(\xi)$ by a linear isomorphism.

Two such objects $(f : M \to N, \mathbf{b}, \xi)$ and $(h : P \to N, \mathbf{d}, \omega)$ are said to be *stably concordant* if there is an equivariant diffeomorphism $\varphi : (P, \partial P) \to (M, \partial M)$, a pair of linear G-representations V and W, and an equivariant map $\Phi : E(\nu_P \oplus V) \to E(\nu_M \oplus W)$ such that (i) for each $y \in P$ the map Φ sends the fiber $E_y(\nu_P \oplus V)$ to $E_{f(y)}(\nu_M \oplus W)$ by a linear isomorphism, (ii) we have $\mathbf{d} = \mathbf{b} \circ \Phi$.

Once again, this construction depends upon choosing equivariant embeddings (in complete analogy to the nonequivariant case), but as in the nonequivariant case there are canonical bijections between stable concordance classes for arbitrary pairs of equivariant embeddings for the sorts of reasons mentioned at the beginning of Section 1.

Before proceeding, we need to verify that equivariant homotopy equivalences determine weakly structured equivariant normal maps as defined above.

Proposition 2.0. *Let $(M, \partial M)$ and $(N, \partial N)$ be compact (possibly) bounded smooth G-manifolds, and let $f : (M, \partial M) \to (N, \partial N)$ be an equivariant homotopy equivalence. Take L to be an equivariant homotopy inverse to f, and let $\xi = h^*\nu_M$. Then there is a mapping $\mathbf{b}$ such that the triple $(f, \mathbf{b}, \xi)$ is a weakly structured G-normal map.*

Sketch of proof. (Compare Rourke [Rk]; see also Sections 3.3 and 5.3 of Lück [L].) We only need to check that such a mapping can be defined, but this follows because $h \circ f$ is equivariantly homotopic to the identity and the equivariant homotopy invariance of pullbacks implies that $f^*\xi = f^*h^*\nu_M$ is equivariantly isomorphic to ν_M.$\blacksquare$

Our definition of normal map does not place any condition on the degrees of either the globally defined maps or their restrictions to components of fixed point sets. In order to proceed we shall require a fairly strong restriction on these degrees.

Defintition. Let G be a compact Lie group, and let M and N be compact smooth G-manifolds of the same dimension. Suppose that $f : (M, \partial M) \to (N, \partial N)$ is an equivariant continuous mapping, and assume that for each isotropy subgroup H the mapping of fixed point sets f^H defines a bijection from the arc components of M^H to the arc components of N^H and also from the arc components of ∂M^H to the arc components of ∂N^H. Assume further that all the components described above are orientable. We shall say that f has *total equivariant degree* ± 1 if for each subgroup H and each component of M^H or ∂M^H the map induced by f^H has degree ± 1.

Basic Example. If f is an equivariant homotopy equivalence (of pairs), then f automatically has total equivariant degree ± 1.

In most situations, it suffices to know the degree condition for a reasonable family $\mathcal{A}$ of subgroups. For example, if the action is effective and all the isotropy subgroups are normal (as in the semifree case), then one can take $\mathcal{A}$ to be the famil of isotropy subgroups.

If we are given a normal map $(f, \mathbf{b}, \xi)$ of degree ± 1 in the nonequivariant case, the next step is to define its normal invariant in the group of homotopy classes $[N, F/O]$. For weakly structured equivariant normal maps of total degree ± 1 we can carry out a similar construction to obtain a class in $F/O_G(N)$.

Construction 2.1. Let $f : (M, \partial M) \to (N \partial N)$ be a weakly structured equivariant normal map of compact smooth G-manifolds as above, with total equivariant degree ± 1. The *G-equivariant normal invariant* of f is the element of the group $F/O_G(N)$ represented by the object given as follows:

(1) Let $\mathbf{b}^\bullet : M^\nu \to N^\xi$ be the associated map of Thom spaces given by taking one point compactifications, and let $\mathbf{b}_1 : (M, \partial M)^\nu \to (N, \partial N)^\xi$ be the map of quotients obtained by collapsing the restrictions ∂M^ν and ∂N^ξ to the base points. Let $c_M : S^{V \oplus \mathbb{R}} \to (M, \partial M)^\nu$ be given by the Pontryagin-Thom map which collapses the complement of the interior of a neat tubular neighborhood of $M \subset V \times [0, \infty)$ to a point.

(2) Let τ_N denote the equivariant tangent bundle of N, and let α be a G-vector bundle such that $\alpha \oplus \xi \oplus \tau_N$ is isomorphic to $N \times W$ for some G-representation W; the existence of stable inverses implies that such bundles exist and are unique up to stable equivalence. Let $\delta : N^\alpha \wedge (N, \partial N)^\xi \to S^{V \oplus W \oplus \mathbb{R}}$ be a standard G-equivariant Spanier-Whitehead duality map, and let η be the map $\delta \circ (\, \mathbf{id}[N^\alpha] \wedge \langle \mathbf{b}_1 \circ c_M \rangle \,)$. By construction, η is an equivariant map from $N^{\alpha \oplus V \oplus \mathbb{R}} \cong N^\alpha \wedge S^{V \oplus \mathbb{R}}$ to $S^{V \oplus W \oplus \mathbb{R}}$ such that the restriction to a suspended fiber compactification $S^{V \oplus W \oplus \mathbb{R}} \cong S^W \wedge S^{V \oplus \mathbb{R}}$ is homotopic to the identity.

(3) Let γ be the map from the sphere bundle $S(\alpha \oplus W \oplus \mathbb{R}^2)$ to $N^\alpha \wedge V \oplus \mathbb{R}$ given by collapsing the a canonical cross section, and let φ be the composite of γ and η. Then φ determines an equivariant fiber homotopy trivialization of the sphere bundle.

Remarks on the construction. 1. This is an equivariant version of the construction cited in [Rk, p. 140]; the latter works with the category of piecewise linear manifolds, but everything goes through for smooth manifolds if one replaces F/PL by F/O. Conversely, the discussion in Sections 3.3–3.5 and 5.3 of Lück [L] deals with the smooth category and describes the adaptations for piecewise linear and topological manifolds in Section 5.4. — There is also a dual approach to normal invariants in the nonequivariant case which is described in M. A. Armstrong's paper [Ar], once again stated in the PL category with a straightforward adaptation to the smooth category. It is also possible to carry out the dual approach equivariantly; at one crucial step in the nonequivariant case one needs a result stating that the total spaces of the equivariant normal bundles of homotopy equivalent manifolds are stably equivalent, and the equivariant analog of this result is given by work of Kawakubo [K2] (related results are also due to S. Kwasik [Kw]).

2. Clearly the construction involves numerous choices, so at some point it is necessary to check that the constructed element of $F/O_G(N)$ does not depend upon these choices, at least if one stabilizes by taking direct sums with a product bundle. This proceeds exactly as in the nonequivariant case; for example, if we choose another neat equivariant embedding, then the original and new embeddings will determine stably homotopic objects that become equal in $F/O_G(N)$. The independence of choices at other points also follow by straightforward adaptations from the nonequivariant case.

We have described the construction of normal invariants in detail because we actually need a refinement of the general concept for semifree actions.

Proposition 2.2. *In the preceding discussion, suppose that G acts semifreely on M and N. Then the normal invariant has a canonical lifting to $F/O_{G,\mathrm{semifree}}(N)$.*

In order to verify this we need to notice a few things. First, one can embed a compact smooth semifree G-manifold smoothly and neatly in $V \times [0, \infty)$, where V has a semifree G-action. Second, since $\mathrm{Vect}_{G,\mathrm{semifree}}$ has inverse bundles, we can choose α and W to have semifree actions. Third, the verification that the construction is independent of choices goes through essentially unchanged.■

We can now state generalizations for some basic properties of nonequivariant normal invariants.

Proposition 2.3. *The equivariant normal invariants described above (and their refinements for semifree actions) have the following properties:*

(*i*) *The invariant is trivial for an equivariant diffeomorphism, and equivariantly homotopic equivariant homotopy equivalences have the same normal invariants.*

(*ii*) *If $\mathbf{q}(f)$ is the normal invariant of f and $\partial f : \partial M \to \partial N$ is the induced map of boundaries, then $\mathbf{q}(\partial f)$ is the image of $\mathbf{q}(f)$ under the restriction map from $F/O_{G,\mathcal{A}}(N)$*

to $F/O_{G,\mathcal{A}}(\partial N)$; here $\mathcal{A}$ is either the family of all representations or the family of all semifree representations.

(iii) If we are given two equivariant homotopy equivalences $f : (M, \partial M) \to (N, \partial N)$ and $g : (N, \partial N) \to (P, \partial P)$ then the normal invariant of the composite is given by the formula $\mathbf{q}(g \circ f) = \mathbf{q}(g) + [g^*]^{-1}\mathbf{q}(f)$.

(iv) Suppose f splits into a map manifold triads $(M; M_0, M_1) \to (N; N_0, N_1)$ and that f_0 is the induced map on $(M_0, \partial M_0)$. Let j_0 be the inclusion of N_0 in N. Then we have $\mathbf{q}(f_0) = j_0^*\mathbf{q}(f)$.

(v) Suppose that ξ is a G-vector bundle over the compact smooth G-manifold N and

$$f : \big(D(\xi), S(\xi) \big) \longrightarrow N \times \big(D(W),\, S(W) \big)$$

is an equivariant fiber homotopy trivialization. If β denotes the class determined by ξ and f in $F/O_{G,\mathcal{A}}(N)$, then the normal invariant of f is equal to β.

In the literature one sometimes sees a formula like the last one with a minus sign; this is because some papers use a definition of normal invariant that is inverse to the one given here and in [Rk].

Specialization to isovariant homotopy equivalences. Suppose now that the map $f : (M, \partial M) \to (N, \partial N)$ is **isovariant** map and that f is normally straightened in the sense of [DuS] and [Sc9]; since every isovariant homotopy equivalence is isovariantly homotopic to one with this property by [DuS. p. 31], there will be no loss of generality if we make this assumption. The latter implies know that f splits into a map of triads $(M; M_0, M_1) \to (N; N_0, N_1)$ where M_0 and N_0 are closed tubular neighborhoods of the fixed point set; actually, since we are making no assumptions on the dimensions of the various components of the fixed point set, it might be preferable to view this as a union of tubular neighborhoods over these components. As in [Sc9], let $\{N_\alpha\}$ denote the set of components of N^G; the associated map f^G of fixed point sets defines a $1-1$ correspondence between the components of M^G and N^G, and for each α we take

$$M_\alpha \;=\; f^{-1}(N_\alpha) \cap M^G \;.$$

As in [Sc9], $f_\alpha : M_\alpha \to N_\alpha$ will denote the partial map of fixed point sets determined by f, and the equivariant normal bundles of M_α and N_α in M and N will be called ξ_α and ω_α respectively. Finally, $S(\nu)$ will again generically represent the unit sphere bundle of a vector bundle ν.

Since we are assuming f is normally straightened, it follows that its restriction $f_{0,1}$ to $M_0 \cap M_1$ splits into pieces corresponding to the components of the fixed point sets of M and N. Suppose now that the associated map f_α defines an ordinary homotopy equivalence from $(M_\alpha, \partial M_\alpha)$ to $(N_\alpha, \partial N_\alpha)$, and let L_α denote a homotopy inverse to f_α. Given an index variable α corresponding to such a pair of components, take

$$h_\alpha : \big(D(M_\alpha), S(M_\alpha) \big) \longrightarrow \big(D(M_\alpha), S(M_\alpha) \big)$$

to be the associated map of pairs defined by $f_{0,1}$. By [Sc9] we know that the mappings h_α are equivariant fiber homotopy equivalences of pairs covering the mappings f_α. Each h_α determines a class $\lambda(h_\alpha) \in F/O_{G,\mathrm{free}}(N_\alpha)$ by a simple stabilization trick. Specifically, purely formal considerations imply that h_α determines a canonical equivariant fiber homotopy equivalence h'_α from $S(L^*\xi_\alpha)$ to $S(\omega_\alpha)$; note that both are free G-vector bundles over N_α. Therefore, if μ_α represents an inverse G-vector bundle to ω_α then the fiberwise join $h'_\alpha \oplus \mathrm{id}[\mu_\alpha]$ defines an equivariant fiber homotopy trivialization from $S(L^*_\alpha\xi_\alpha \oplus \mu_\alpha)$ to $S(\omega_\alpha \oplus \mu_\alpha) \cong N_\alpha \times S(W_\alpha)$, where W_α is a free G-representation such that $\omega_\alpha \oplus \mu_\alpha \cong N_\alpha \times W_\alpha$. Standard considerations imply that the stabilized class of $\lambda(h_\alpha)$ in $F/O_{G,\mathrm{free}}(N_\alpha)$ does not depend upon the choice of (a stable) inverse bundle, and in fact the class itself depends only on the isovariant homotopy class of the original mapping f.

The following basic relationship between $\lambda(h_\alpha)$ and the normal invariant of f is absolutely indispensable for our purposes.

Theorem 2.4. *In the setting above, let $j_\alpha : N_\alpha \to N$ denote the inclusion mapping. Then under the canonical isomorphism*

$$F/O_{G,\mathrm{semifree}}(N_\alpha) \cong F/O_{G,\mathrm{free}}(N_\alpha) \times [N_\alpha, F/O]$$

the restricted normal invariant $j^*_\alpha \mathbf{q}(f)$ corresponds to $(\lambda(h_\alpha), \mathbf{q}(f_\alpha))$.

Proof. The map h_α factors as a composite

$$D(\xi_\alpha) \longrightarrow D\left([f^*_\alpha]^{-1}\xi_\alpha\right) \longrightarrow D(\omega_\alpha)$$

where for each $x \in M_\alpha$ the first map sends the fiber over x to the fiber over $f_\alpha(x)$ by a linear isomorphism and the second map is fiber preserving over N_α and determines a fiber homotopy equivalence of bundle pairs. By the formula for normal invariants of composites, it suffices to prove the formula in the theorem for each of the factors. The first of these follows directly from the construction of normal invariants, and the second follows from the final part of Proposition 2.3.■

The preceding observations also lead directly to the following basic fact about equivariant homotopy equivalences that are isovariant on the boundary.

Theorem 2.5. *Let M, N, M_α, N_α, ξ_α and ω_α be as above. Assume $f : (M, \partial M) \to (N, \partial N)$ is an equivariant map of total degree ± 1 that is isovariant on the boundary and determines a (nonequivariant) homotopy equivalence of fixed point sets. Assume that the map on the boundary is normally straightened so that we have equivariant maps of pairs of bundle pairs*

$$h_\alpha : \left(D(\xi_\alpha|\partial M_\alpha),\, S(\xi_\alpha|\partial M_\alpha) \right) \longrightarrow \left(D(\omega_\alpha|\partial N_\alpha),\, S(\omega_\alpha|\partial N_\alpha) \right)$$

covering $f_\alpha|\partial M_\alpha$ that are (equivariant) homotopy equivalences on each fiber. Then there are free G-representations W_α and equivariant homotopy equivalences of pairs

$$K_\alpha : \left(D(\xi_\alpha \oplus W_\alpha),\, S(\xi_\alpha \oplus W_\alpha) \right) \longrightarrow \left(D(\omega_\alpha \oplus W_\alpha),\, S(\omega_\alpha \oplus W_\alpha) \right)$$

covering f_α that are (equivariant) homotopy equivalences on each fiber such that the restriction of K_α to the inverse image of the boundary is the fiberwise join of h_α and the identity on W_α.

Sketch of proof. In fact, one can construct K_α by taking fiberwise joins of the normal invariants over the components N_α with the identities on the bundles ω_α. By Theorem 2.4 the restrictions to the boundaries are just the stabilizations of the original maps h_α.■

3. Proofs of main results

If the Gap Hypothesis holds, then the conclusion of Theorem 2.5 can be strengthened.

Theorem 3.1. *Let M, N, M_α, N_α, ξ_α and ω_α be as above. Assume $f : (M, \partial M) \to (N, \partial N)$ is an equivariant map of total degree ± 1 that is isovariant on the boundary and determines a (nonequivariant) homotopy equivalence of fixed point sets. Assume that the map on the boundary is normally straightened so that we have equivariant maps of pairs of bundle pairs*

$$h_\alpha : \big(\, D(\xi_\alpha|\partial M_\alpha),\, S(\xi_\alpha|\partial M_\alpha)\,\big) \longrightarrow \big(\, D(\omega_\alpha|\partial N_\alpha),\, S(\omega_\alpha|\partial N_\alpha)\,\big)$$

covering $f_\alpha|\partial M_\alpha$ that are (equivariant) homotopy equivalences on each fiber. Suppose further that M and N satisfy the Gap Hypothesis. Then there equivariant homotopy equivalence of pairs

$$k_\alpha : \big(\, D(\xi_\alpha),\, S(\xi_\alpha)\,\big) \longrightarrow \big(\, D(\omega_\alpha),\, S(\omega_\alpha)\,\big)$$

covering f_α that are (equivariant) homotopy equivalences on each fiber such that the restriction of k_α to the inverse image of the boundary is equal to h_α,

Proof. We need to show that if the Gap Hypothesis holds, the the map in Theorem 2.5 can be desuspended to a map of pairs

$$\big(\, D(\xi_\alpha),\, S(\xi_\alpha)\,\big) \longrightarrow \big(\, D(\omega_\alpha),\, S(\omega_\alpha)\,\big)$$

and this can be done using a homotopy that is fixed on the boundary of M_α. The obstructions to doing this lie in cohomology groups of the pair $(N_\alpha,\, \partial N_\alpha)$ with coefficients in the relative homotopy groups of $(F_G, F_G(V_\alpha)\,)$, where V_α is the normal representation at a fixed point in N_α. As in the proof of Proposition 5.1 in [Sc9], the dimensions of M_α and N_α are less than or equal to the connectivity of the pair, and therefore the obstructions vanish because they lie in cohomology groups that are trivial.■

The preceding result allows us to state and prove the relative normal straightening property that we need in order to prove a relative version of the main result in [Sc9], but before doing so it will be useful to describe an elementary construction we shall need.

Definition. Suppose that G is a compact Lie group and X is a compact smooth G-manifold with boundary. Let γ be a G-vector bundle over X, $E(\gamma)$ denote the associated total space with its usual structure as a smooth G-manifold and let $D(\gamma)$ and $S(\gamma)$ be the associated unit disk and sphere bundles. Suppose we are given a smooth equivariant collar neighborhood of the boundary $c : \partial X \times [0.2) \to X$ and let

$$\widetilde{c} : E(\gamma|\partial X) \longrightarrow E(\gamma)$$

be a corresponding collar neighborhood covering c that maps fibers to fibers orthogonally. The *standard tapered pinching map* $\rho(\gamma, \widetilde{c})$ associated to these data is the map defined by

$$\rho\widetilde{c}(y,t) \;=\; \widetilde{c}\big(\,(1-t)\cdot y\,\big)$$

if $t \leq 1$ and by $\rho(w) = 0 \cdot w$ otherwise. This yields a well defined continuous equivariant map that is the identity on the boundary and collapses everything outside the open collar

$$\widetilde{c}\big(\,\partial X \times [0,1)\,\big)$$

to the zero section. Frequently we shall write $\rho(\gamma)$ if there is no ambiguity about the equivariant collar neighborhoods we are working with, and in some cases we may abbreviate this further to ρ.

Proposition 3.2. *The map $\rho(\gamma, \widetilde{c})$ is homotopic to the identity by a fiber preserving homotopy that is fixed on the boundary.*

Proof. It suffices to take the straight line homotopy given by

$$t\,y \;+\; (1-t)\,\rho(\gamma, \widetilde{c})$$

which is the identity on the boundary because the same is true for ρ.■

We can now prove the result on relative normal straightenings fairly directly.

Theorem 3.3. *Let M, N, M_α, N_α, ξ_α and ω_α be as above. Assume $f : (M, \partial M) \to (N, \partial N)$ is an equivariant map of total degree ± 1 that is isovariant on the boundary and determines a (nonequivariant) homotopy equivalence of fixed point sets. Assume that the map on the boundary is normally straightened. Suppose further that M and N satisfy the Gap Hypothesis. Then f is equivariantly homotopic to an equivariant map F that is normally straightened and agrees with f on the boundary. In fact, one can construct an equivariant homotopy between f and F that is the constant homotopy on the boundary.*

Proof. Standard considerations imply that we might as well assume f is well behaved with respect to closed equivariant collar neighborhoods for ∂M and ∂N. Specifically, if c_M and c_M are the collar neighborhood embeddings, then we have $f \circ c_M(x,t) = c_N(\,f(x),\,t)$ for all $x \in \partial M$ and $t \in [0,1]$. As usual, we also assume these collars are neat with respect to all fixed point set components. For each component M_α of M^G let T_α be a neat equivariant tubular neighborhood of M_α in M, and let T'_α be defined similarly for N_α in N^G. By uniform continuity, for sufficiently small choices of T_α we have $f(T_\alpha) \subset T'_\alpha$, so we shall also assume this for each α. All of this can be done so that the normal straightening on the boundary is left unchanged (at least up to a change of scale for the unit disk and sphere bundles).

Let h_α be the normal straightening on the boundary, let k_α be the extension given by Theorem 3.1, and let $\widetilde{k_\alpha}$ be obtained by pushing k_α over to a map from T_α to T'_α. Let σ and σ' be the relative pinching maps for the equivariant normal bundles $\xi_\alpha \downarrow M_\alpha$ and $\omega_\alpha \downarrow N_\alpha$ pushed over to T_α and T'_α respectively. It follows that the maps on T_α defined by $\sigma' \circ k_\alpha$ and $f_\alpha \circ \sigma$ are equal. Since the first is equivariantly homotopic to $\widetilde{k_\alpha}$ and the second is equivariantly homotopic to $f|T_\alpha$ with the boundary held fixed, it follows that the latter two maps are equivariantly homotopic with the boundary held fixed. Therefore, by the Equivariant Homotopy Extension Property we have an equivariant homotopy from f to some equivariant map f' such that $f'|T_\alpha = \widetilde{k_\alpha}$ and the homotopy is fixed on the boundary. The map f' is normally straightened, and therefore we have constructed the desired relative equivariant deformation of f.■

Proof of Theorem 1. In principle, all we need to check is (i) that the approach in Section 5 of [Sc9] goes through for bounded manifolds and equivariant homotopy equivalences that are isovariant on the boundary, (ii) in carrying this out we can keep the homotopy fixed on the boundary.

The first step in Proposition 5.1 of [Sc9] is to make the map isovariant near the fixed point set so that nothing on the boundary is disturbed. This is established in Theorem 3.3 above. Next, we need to show that the set of nonisovariant points can be equivariantly engulfed into tubular neighborhoods of the fixed point set components. This can be done exactly as in Proposition 5.2 of [Sc9], and since the map is already isovariant on the boundary, the set of nonisovariant points lies entirely in the interior of M and the equivariant isotopies which engulf the nonisovariant points can be chosen to be fixed near the boundary. We then need to show that since the nonisovariant points are suitably engulfed the equivariant deformation of our original map can be deformed further into an almost isovariant mapping. This can be done exactly as in part (iii) of Theorem 2.1 in [Sc9]. Finally, we need to show that the almost isovariant mapping can be suitably deformed into an isovariant mapping and that the latter is an isovariant homotopy equivalence. The first of these follows from [DuS], and the second follows by combining the same sorts of duality considerations appearing in the proof of Proposition 5.3 in [Sc9] with the Isovariant Whitehead Theorem from Section 4 of [DuS].■

Proof of Theorem 2. It is convenient to start with the unbounded case. Let $H : M \times [0,1] \to N$ be an equivariant homotopy between the isovariant homotopy equivalences f and g; standard elementary considerations imply that we may assume H is fixed for t close to 0 or 1. Consider the map

$$H_\#(x,t) \;=\; \big(H(x,t),\, t \big)\ .$$

We claim that $H_\#$ is an equivariant homotopy equivalence that is isovariant on the boundary. The second part follows immediately from the isovariance of f and g, and the first follows because a straightforward elementary argument shows that if k is a homotopy inverse to f then $k \times \mathrm{id}_{[0,1]}$ is a homotopy inverse to $H_\#$ (as maps of triads, in fact). Therefore Theorem 1 implies that $H_\#$ is equivariantly homotopic, leaving the boundary fixed, to an isovariant homotopy equivalence, say $K_\#$. If we let K denote the projection of $K_\#$ onto the N factor, we obtain the desired isovariant homotopy from f to g.

Suppose now that M is bounded and that the original homotopy is isovariant on the boundary. Then we may form $H_\#$ exactly as before, and it will define an equivariant homotopy equivalence on $M \times [0,1]$ that is isovariant on

$$\partial\Big(M \times [0,1]\Big) \;=\; M \times \{0,1\} \;\cup\; \partial M \times [0,1] \;.$$

There are standard questions about smoothness at the "corner set" $\partial M \times \{0,1\}$, but since we already have a map that is isovariant near the entire topological boundary we need not be concerned with such issues here (because everything near the boundary is left unchanged). We can now use the proof of Theorem 1 to conclude that $H_\#$ is equivariantly homotopic, leaving the boundary fixed, to an isovariant homotopy equivalence, and the projection onto N will be the desired isovariant homotopy.

Finally suppose M is bounded but we do not know if the map on the boundary is isovariant. Then we can apply the unbounded case from [Sc9] to deform the map from ∂M to ∂N into an isovariant homotopy equivalence, and by the Relative Equivariant Homotopy Extension Property we may deform the original map, leaving $M \times \{0,1\}$ fixed, to a map which is also isovariant on $\partial M \times [0,1]$. Therefore we have obtained an equivariant homotopy equivalence that is homotopic to $H_\#$ leaving $\partial M \times [0,1]$ fixed and is isovariant on the entire boundary. By the preceding paragraph the latter is equivariantly homotopic, leaving the entire boundary fixed, to an isovariant homotopy equivalence. Combining these homotopies, we have an equivariant homotopy from $H_\#$ to an isovariant homotopy equivalence leaving $\partial M \times [0,1]$ fixed. If we take the projection of this combined homotopy onto N, we obtain the desired equivariant homotopy from the original equivariant homotopy to an isovariant one.■

If we combine the preceding results with the proof of Theorem 5.4 of [Sc9], we obtain the following generalization of the latter to manifolds with boundary.

Theorem 3.4. *Let $f : (M, \partial M) \to (N, \partial N)$ be a continuous isovariant mapping of oriented compact semifree smooth G-manifolds that satisfy the Gap Hypothesis. Then f is an isovariant homotopy equivalence of pairs if and only if f is an ordinary homotopy equivalence of pairs (forgetting the group action), and the associated map of fixed point set pairs $f^G : (M^G, \partial M^G) \to (N^G, \partial N^G)$ is also a homotopy equivalence of pairs.*

Sketch of proof. The proof requires a generalization of one fact from [Sc9] to manifolds with boundary. Namely, an isovariant map is an isovariant homotopy equivalence if it is an equivariant homotopy equivalence (see Proposition 4.3 in [Sc9]); this follows from duality considerations and the Isovariant Whitehead Theorem in [DuS] (compare the assertions at the end of the proof of Theorem 1 above). Once we know this the argument proving Theorem 5.4 in [Sc9] goes through unchanged.■

4. Remarks on the proof(s)

The proofs of our relative results (Theorems 1 and 2) require considerably more work than their absolute counterparts in [Sc9]. On the other hand, in the surgery-theoretic approach of Straus [St] and Browder [Br2] the proofs of the relative results are basically straightforward analogs of the absolute cases. A few comments on the reasons for this disparity might yield additional insight into the relationship between the two approaches.

First of all, as noted in the introduction to [Sc9], the indirect approach through surgery theory requires a great deal more technical input and reflects the powerful nature of surgery-theoretic methods. Our direct approach requires far less machinery, and it is not surprising that such arguments may require more work.

Also, one advantage of the direct approach is that it leads to concepts and results of independent interest. For example, Theorem 2.5 yields an invariant which generates potential obstructions to isovariance when the Gap Hypothesis fails. In a subsequent paper we shall give examples realizing nontrivial obstructions of this type.

Finally, it is important to observe that the proofs of the results in this paper and [Sc9] are **not** easy applications of the obstructions to isovariance in [DuS]. In particular, if one looks at the cohomological obstruction groups in that paper, it soon becomes clear that there are many obstructions that are potentially nontrivial. Sorting through these possibilities requires some work, and the latter would require arguments and constructions like those of [Sc9] and this paper.

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