Gap Hypotheses and equivariant homotopy equivalences

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Abstract In fundamental but unpublished results from the 1970s and 80s, S. Straus and W. Browder showed that two notions of homotopy equivalence for manifolds with smooth group actions — isovariant and equivariant — are equivalent under a condition known as a Gap Hypothesis. The proofs use deep results in geometric topology, mainly from C. T. C. Wall's theory of surgery on compact manifolds. No complete proof of this fundamental result has been published (or posted to the World Wide Web) during the intervening decades, and one purpose of this paper is to give a proof in the semifree case using the approach of Browder and Straus. We also obtain a simplified recognition principle for isovariant homotopy equivalences of closed manifolds with group actions satisfying a very weak Gap Hypothesis.

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1 Background

Although the notion of topological equivalence (or *homeomorphism*) is obviously central to most branches of topology, for more than a century mathematicians have also recognized the importance and usefulness of a weaker notion called *homotopy equivalence* (*e.g.*, see [64]). If we consider more structured objects given by a space with a continuous action of some topological group G, the natural analog of a topological equivalence is an *equivariant* topological equivalence φ satisfying $\varphi(g \cdot x) = g \cdot \varphi(x)$ for all x in the domain and all $g \in G$, but there are **two** distinct analogs of homotopy equivalence:

(1) *Equivariant homotopy equivalence*, for which the morphisms and homotopies are all equivariant.

(2) *Isovariant homotopy equivalence*, for which the morphisms and homotopies also satisfy the condition $g \cdot \varphi(x) = \varphi(x)$ if and only if $g \cdot x = x$ (in terms of isotropy subgroups [8], this means $G_{\varphi(x)} = G_x$ for all x).

If φ is an equivariant homeomorphism, then the condition in the second statement is automatically satisfied, so equivariant and isovariant topological equivalence are identical notions.

Example. If two spaces are isovariantly homotopy equivalent, then they are equivariantly homotopy equivalent. However, if we take the $\mathbb{Z}_2$-action Φ_- on the real line $\mathbb{R}$ sending x to $-x$, then the $\mathbb{Z}_2$-space $(\mathbb{R}, \Phi_-)$ is equivariantly but not isovariantly homotopy equivalent to the one point space $\{0\}$ with the trivial $\mathbb{Z}_2$-action.

During the 1960s, several major advances in geometric topology yielded a classification scheme for smooth or topological manifolds within a given homotopy type (compare [61]). This research and other considerations motivated analogous questions for smooth or continuous group actions on manifolds up to equivariant or isovariant homotopy equivalence. In particular, the following basic phenomena were discovered for manifolds with smooth group actions:

(1) Classifications up to some versions of isovariant homotopy equivalence can often be given using variants of established techniques in geometric topology (see [12], the commentary on the latter in [28], Section II.1 of [53] the final section of [23], and Weinberger's book [62]; additional references are also cited in the final section of this paper).

(2) The same hold for classifications up to some forms of equivariant homotopy equivalence PROVIDED the objects satisfy a type of condition called a *Gap Hypothesis*; for example, if $G = \mathbb{Z}_2$ and M is a smooth G-manifold, this states that the dimensions of M and its fixed point set M^G satisfy an inequality of the form $\dim M \geq 2 \dim M^G + \varepsilon$ for some integer ε close to zero. The importance and usefulness of the restriction became apparent in work of T. Petrie ([38] and [39]) as well as subsequent papers of Dovermann-Petrie [19], Dovermann-Rothenberg [20], and Lück-Madsen [35] (this list is not meant to be exhaustive).

Results of S. H. Straus in the 1970s [57] and independently obtained results of W. Browder in the 1980s [11] yield a strong connection between these two themes:

Browder-Straus Theorem. *Under suitable conditions (an appropriate versions of the Gap Hypothesis and other rairly simple restrictions), an equivariant* *homotopy equivalence of closed smooth G-manifolds is equivariantly homotopic to an isovariant homotopy equivalence, and two equivariantly homotopic isovariant homotopy equivalences are isovariantly homotopic.*

Unfortunately, a complete and correct proof of this decades-old theorem has not appeared in print, and the aim of this paper is to give a proof using the original surgery-theoretic methods of Straus and Browder. An earlier paper [54] attempted to give a more homotopy-theoretic proof, but the argument contains a mistake (specifically, the application of the Blakers-Massey Theorem is incorrect; this error occurs in the proof of Proposition 4.2 in [54]).

Overview of the paper.

Section 2 begins with background material on isovariant homotopy theory and continues with statements of the main results, ending with a brief discussion of some side issues. The main results can be viewed as variants of some results of C. T. C. Wall on splitting homotopy equivalences of manifolds (see Chapter 12A of [61]), and Section 3 is devoted to proving an extension of Wall's $\pi - \pi$ Theorem in Chapter 4 of [61] which will allow us to generalize the methods of Chapter 12A in [61]). In Section 4 we shall complete the proofs of the main results, and in Section 5 we shall discuss some known applications and related results. These include a new and simple recognition principle for isovariant homotopy equivalences of smooth G-manifolds satisfying a very weak version of the Gap Hypothesis. Finally, in Section 6 we shall discuss a variety of questions related to the results of this paper. In particular, we shall prove some extensions of the main results to a broad class of 4-dimensional semifree G-manifolds.

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2 The main results and related issues

Before stating the main result, we shall summarize some background information about isovariant homotopy theory.

Isovariance and almost isovariance

One important feature of equivariant homotopy theory is that we can modify many standard methods and results from ordinary homotopy theory and apply them effectively to spaces with group actions. Obviously we would like to do something similar for isovariant homotopy theory, but this requires a less direct approach based upon an intermediate concept of *almost isovariance*. We begin with some definitions.

Definition. Let X and Y be sets with actions of a topological group G, and let $f : X \to Y$ be a G-equivariant mapping. A point $p \in G$ is a *nonisovariant point* of f if the isotropy subgroup $G_{f(p)}$ properly contains G_x. Note that $G_x \subset G_{f(x)}$ for all $x \in X$ if f is equivariant, and $G_x \subset G_{f(x)}$ for all $x \in X$ if f is isovariant.

Definition. Let M and N be smooth G-manifolds, where G is a compact Lie group, and assume that the actions are semifree (the only isotropy subgroups are $\{1\}$ and G). Take D_M and D_N to be unions of closed tubular neighborhoods for the various fixed point set components in M^G and N^G, and denote their boundaries by S_M and S_N respectively . A continuous equivariant mapping $f : M \to N$ is *amost isovariant* with respect to D_M and S_M if the following hold:

1. (1) The sets of nonisovariant points for f is contained in the interior of D_M.
2. (2) The original mapping f is actually a map of triads from $(M; D_M, M - \operatorname{Int} D_M)$ to $(N; D_N, N - \operatorname{Int} D_N)$.

An arbitrary isovariant mapping $f : M \to N$ can always be deformed isovariantly to a map of triads as in the second condition (see Section 4 of [23]), so the only effective difference between isovariance and almost isovariance is that the latter includes maps which might not be isovariant near the fixed point set.

The main results of Section 4 in [23] imply that there is a 1–1 correspondence between isovariant homotopy classes of continuous mappings $M \to N$ and almost isovariant homotopy classes of continous equivariant mappings $M \to N$ which are almost isovariant with respect to D_M and D_N. Note that this is true for all possible fixed choices of D_M and D_N.

The concept of almost isovariance is useful for studying isovariant homotopy theory because the standard methods of homotopy theory extend in a straightforward manner to almost isovariant mappings (*cf.* [23] and [22]).

A definition of almost isovariance for more general actions is given in [23]. We have restricted attention to semifree actions because of their relatively simple orbit structure; furthermore, the results in the semifree case suffice to study several basic types of questions effectively.

Statement of the main results

We have already mentioned that our main results require a type of assumption known as a Gap Hypothesis. Here is a precise statement of what we need:

Definition. Let M be a smooth G-manifold, where G is a compact Lie group. We shall say that M satisfies the **Standard Gap Hypothesis** (for semifree actions) provided $\dim M - \dim M^G \geq 3$ and

$$2 \cdot \dim M^G \; + \; 1 \quad < \quad \dim M \; .$$

Observe that the second condition implies the first unless $\dim M^G = 0$ and $\dim M = 2$.

As indicated in the previous section, we are mainly interested in the following unpublished result, which is due to Straus [57] and Browder [11]. It implies a fairly strong, general and precise connection between almost isovariance and the Standard Gap Hypothesis.

Theorem 2.1 *(i) Let $f : M \to N$ be an equivariant homotopy equivalence of connected, compact, unbounded* (= closed) *and semifree smooth G-manifolds that satisfy the Standard Gap Hypothesis. Assume that M and N are at least 5-dimensional. Then f is equivariantly homotopic to an isovariant homotopy equivalence.*

(ii) Let f_0 and $f_1 : M \to N$ be a isovariant homotopy equivalences of connected, compact, unbounded (= closed) *and semifree smooth G-manifolds such that $M \times \mathbb{R}$ and $N \times \mathbb{R}$ satisfy the Standard Gap Hypothesis, and suppose that $H : M \times [0,1] \to N$ is an equivariant homotopy from f to g. Assume that M and N are at least 4-dimensional. Then H is equivariantly homotopic to an isovariant homotopy by a deformation which is constant on $M \times \{0,1\}$.*

Both conclusions in Theorem 2.1 are special cases of the following more general result on deforming equivariant homotopy equivalences to isovariant ones.

Theorem 2.2 *Let $f : (M, \partial M) \to (N, \partial N)$ be an equivariant homotopy equivalence of connected, oriented, compact, bounded smooth G-manifolds that satisfy the Standard Gap Hypothesis, and assume that the associated map $\partial f : \partial M \to \partial N$ is an isovariant homotopy equivalence. Assume that M and N*

are at least 5*-dimensional. Then f is equivariantly homotopic to an isovariant homotopy equivalence, and one can choose this deformation to be constant on the boundary.*

The first conclusion in Theorem 2.1 is the special case where $\partial M = \partial N = \varnothing$, and the second conclusion is the special case where $M = P \times [0,1]$ and $N = Q \times [0,1]$, where P and Q are smooth G-manifolds with $\partial P = \partial Q = \varnothing$.

Some further results

There are several directions in which one can consider extensions of the main theorems, and we shall summarize a few of them here. Additional details on some of these issues are given in Section 6.

1. *Equivariant homotopy equivalences coming from isovariant homotopy equivalences* . Given an equivariant homotopy equivalence φ of a closed smooth G-manifold with no assumptions about Gap Hypotheses, it is meaningful ask whether the equivariant homotopy class of φ can be represented by an isovariant homotopy equivalence. In general this is not possible, but in Theorem 5.2 we prove that an isovariant representative of φ will also be an isovariant homotopy equivalence if the groups act semifreely and satisfy a weak Gap Condition (namely, the fixed point sets have codimension ≥ 3). For the sake of simplicity we shall only state and prove the result for oriented manifolds such that all fixed point set components are also oriented.

2. *An absolute analog of Theorem* 2.2. It is natural to ask if there is an parallel result for equivariant homotopy equivalences of manifolds with boundary if we do not assume that the boundary is sent to itself by an isovariant homotopy equivalence. We shall prove a result of this type when the boundary is sent to itself by an equivariant homotopy equivalence (Theorem 4.1) and give examples to show the need for the additional condition.

3. *Extensions of Theorem 2.1 to* 4*-manifolds.* We shall prove two results of this type in Section 6, one of which only requires that the group is not cyclic of order 2 (Theorem 6.2) and another of which holds if the fundamental groups of the manifolds are finite or abelian (Theorem 6.3). One could go further in this direction, proving analogs of Theorem 4.1 for bounded manifolds, but we shall not pursue this any further.

4. *Generalizations to nonsemifree smooth actions.* One reason for restricting attention to semifree actions is that it avoids complications regarding the placement of the fixed point sets for the various subgroups of G (we need only consider the fixed point set of G itself), and another is that the class of semifree actions is broad enough to include all actions of the cyclic group $\mathbb{Z}_p$ where p has prime order (in this case the only subgroups are $\{1\}$ and $\mathbb{Z}_p$ itself). Everything in this paper can be generalized to actions with *treelike isotropy structure* in the sense of [23]; this condition holds if the isotropy subgroups are normal and linearly ordered by inclusion, and therefore this class of group actions includes all actions of the cyclic p-groups $\mathbb{Z}_{p^r}$, where p is prime and $r \geq 1$. Specifically, one can generalize Theorems 2.1, 2.2, and 4.1 by combining the methods of this paper with an induction on the number of orbit types as in [23]. There is general agreement that one can generalize these results to fairly general actions of a finite group G on a manifold P which satisfy the following two conditions:

(1) Standard Gap Hypothesis. For each pair of isotropy subgroups $H \supsetneqq K$ in G and each pair of components $B \subset P^H$, $C \subset P^K$ such that $B \subsetneqq C$ we have $2 \cdot \dim B + 1 < \dim C$.

(2) If H is not maximal among the isotropy subgroups, then every component of P^H is at least 5-dimensional.

Further comments on such generalizations appear in Section 6.

5. *Generalizations to nonsmoothable actions.* Browder has noted that the methods of [11] (and this paper) can be extended to certain actions which are not smoothable, at least if one replaces isovariance by a suitable notion of almost isovariance. In particular, everything should go through for piecewise linear locally linear G-manifolds (see [43] for more information on the latter). Since the methods of this paper and [23] rely heavily on the existence of well-behaved neighborhoods for the fixed point sets of the isotropy subgroups, passage to more general classes of actions such as

(1) continuous locally linear actions (the locally smooth actions of [8]),

(2) homotopically stratified actions in the sense of [42] or [62]

is likely to be considerably less straightforward. Once again, further comments appear in Section 6.

6. *Optimality of the Gap Conditions in the main results.* Since the conclusions of the main results are fairly strong, and it is natural to expect that counterexamples exist for equivariant homotopy equivalences of closed smooth G-manifolds which do not satisfy some variant the Standard Gap Hypothesis. As noted after the statement of Theorem 5.1, in some cases one can weaken the condition in the Standard Gap Hypothesis, and in forthcoming work with K.H. Dovermann we shall give still other examples (some of these rely on results due to A. Bak and M. Morimoto in [3] and [4]). In contrast, there are also examples which suggest that positive results in such cases are extremely limited, and forthcoming work of S. Safii [46] will give counterexamples to the main results just outside the range of the Standard Gap Hypothesis.

3 Input from equivariant surgery theory

Let M and N be closed, smooth, semifree G-manifolds with fixed point sets M^G and N^G respectively, and let $E(M^G)$ and $E(N^G)$ denote closed equivariant tubular neighborhoods of the fixed point sets. If $f : M \to N$ is a G-isovariant homotopy equivalence, then the main results of [23] state that f is isovariantly homotopic to an isovariant homotopy equivalence of triads from $\big(M; E(M^G), M - \mathrm{Int}\, E(M^G)\big)$ to $\big(N; E(N^G), N - \mathrm{Int}\, E(N^G)\big)$. This splitting result for isovarian homotopy equivalences suggest a relationship between the following two questions:

(1) Given an equivariant homotopy equivalence $f : M \to N$ of closed smooth G-manifolds, is f equivariantly homotopic to an isovariant homotopy equivalence?

(2) Let CAT denote the smooth, piecewise linear or topological category, and suppose that W is a closed CAT-manifold which splits as $W = W_1 \cup W_2$, where W_1 and W_2 are compact bounded manifolds with $\partial W_1 = \partial W_2$. If V is a closed CAT-manifold and $f : V \to W$ is a simple homotopy equivalence, can f be deformed to a homotopy eqivalence of triads from $(V; V_1, V_2)$ to $(W; W_1, W_2)$ for a suitably chosen splitting $V = V_1 \cup V_2$?

Problems of this sort have been studied for nearly a century (*e.g.*, H. Kneser's connected sum conjecture for 3-manifolds [32]; see Chapter 7 of [26] for a proof). A thorough discussion of this topic is beyond the scope of the present article, so we shall limit ourselves to stating an important special case which appears in Chapter 12A of [61] (see Theorem 12.1 on pages 142–143; strictly speaking, the hypotheses here are slightly stronger than those in Wall's book).

Theorem 3.1 *Let* CAT *denote the smooth, piecewise linear or topological category, and suppose that* W *is a closed* CAT*-manifold of dimension* ≥ 6 *which splits as* $W = W_1 \cup W_2$*, where* W_1 *and* W_2 *are compact bounded manifolds with* $\partial W_1 = \partial W_2$ *and the latter is also connected. Assume that the induced map of fundamental groups from* $\pi_1(\partial W_1)$ *to* $\pi_1(W_1)$ *is an isomorphism. If* V *is a closed* CAT*-manifold and* $f : V \to W$ *is a simple homotopy equivalence,* *then f can be deformed to a homotopy eqivalence of triads from $(V; V_1, V_2)$ to $(W; W_1, W_2)$ for a suitably chosen splitting $V = V_1 \cup V_2$.*

In fact, the proofs of Theorems 2.1 and 2.2 in this paper are variants of the proof for Theorem 12.1 in [61], and the goal of this section is to formulate equivariant versions of Wall's $\pi - \pi$ Theorem ([61], Chapter 4) that are needed to extend Wall's proof of Theorem 3.1.

Background remarks. If W_1, W_2 and their common boundary are simply connected, then the conclusion of Theorem 3.1 is due to Browder (see [9], Section 1), with subsequent generalizations due to J. Wagoner [60] and D. Sullivan (unpublished). A few basic applications of the theorem are given in [6] and [9].

A class of equivariant surgery problems

We are primarily interested in equivariant degree 1 maps of compact, oriented, smooth semifree G-manifolds such that the induced map of fixed point sets is a homotopy equivalence. Let $f : M \to N$ be such a mapping. In any discussion of surgery theory some notion of bundle data is needed, but in this paper we shall only need a very weak version; namely, our bundle data will be a map of stable equivariant G-vector bundles $\mathbf{b} : E(\nu_M) \to E(\xi \downarrow N)$, where ν_M denotes the stable equivariant normal bundle of M in some linear G-action on a sufficiently large coordinate space $\mathbb{R}^k$, such that for each $x \in M$ the map sends the fiber over x to the fiber over $f(x)$ by a G_x-equivariant linear isomorphism (here G_x denotes the isotropy subgroup at x); the pair $(f, \mathbf{b})$ will be called a *degree* 1 *normal map*. As usual, if f is an equivariant homotopy equivalence then it defines an isomorphism f^* of reduced equivariant KO-groups $\widetilde{KO}_G(N) \cong \widetilde{KO}_G(M)$, and hence there is a G-vector bundle ξ, which is unique up to stable equivalence, for which one can define bundle data of the given form. If M and N have boundaries, possibly decomposed into smaller pieces, then the necessary bundle data shall include compatible bundle data for these subsets.

We can now use the preceding definitions and conventions to state the equivariant generalization of Wall's $\pi - \pi$ Theorem which is needed here. This result essentially goes back to [57].

Theorem 3.2 *Let W is a compact smooth G-manifold of dimension ≥ 6 such that ∂W splits as $\partial W = \partial_1 W \cup \partial_0 W$, where $\partial_1 W$ and $\partial_0 W$ are compact manifolds with $\partial\partial_0 W = \partial\partial_1 W$ and the former is also connected. Assume that the induced map of fundamental groups from $\pi_1(\partial_0 W)$ to $\pi_1(W)$ is an* *isomorphism. Furthermore, assume that W satisfies the* slightly weakened Gap Condition $2\dim W^G < \dim W$ *and G acts freely on $\partial_0 W$. If $(V, \partial V = \partial_0 V \cup \partial_1 V)$ satisfies the analogous conditions and*

$$(f, \mathbf{b}) : (V; \partial_1 V, \partial_0 V) \longrightarrow (W; \partial_1 W, \partial_0 W)$$

is a degree 1 normal map (in the sense described above) which is a homotopy equivalence on $\partial_1 V$, then $(f, \mathbf{b})$ is normally cobordant to a G-simple homotopy equivalence such that the cobordism is a product with $[0, 1]$ over $\partial_1 W$ and W^G.

Similar $\pi - \pi$ theorems for equivariant surgery are given in Section 3 of [35]; one difference between the results in [35] and Theorem 3.2 is that the fixed point reference data in the former map bijectively from $\partial_0 W$ to W while the corresponding data in the latter do not.

The basic idea of the proof is fairly easy to summarize. In the proof of Wall's result the normal cobordism is constructed in a series of elementary steps. For each of these steps, one starts out with a degree 1 normal map

$$(f', \mathbf{b}) : (V'; \partial_1 V, \partial_0 V') \longrightarrow (W; \partial_1 W, \partial_0 W)$$

and approximates certain maps of q-disks or spheres into V by smooth embeddings; in Wall's setting these can be constructed because $q \leq \frac{1}{2} \dim W$ or some closely related condition holds. In order to generalize this argument, it suffices to know that the images of these embeddings are disjoint from each other and from the fixed point set. If W satisfies the Gap Hypothesis, this can be achieved using standard transversality results such as Theorem 2.5 on page 78 of [27]. Thus the main issue in the proof of Theorem 3.2 is to explain more precisely how and why these things can be done.

Algebraic considerations. Wall's proof of Theorem 3.1 requires an algebraic lemma which gives a condition for certain homology groups to be projective or stably free modules (specifically, Lemma 2.3 on page 26 of [61]); the latter applies to chain complexes over group rings which are free in each dimension, finitely generated in each dimension, and zero in all but finitely many dimensions. We shall need a slight generalization of this result when the chain groups are direct sums of permutation modules $\mathbb{Z}[\Gamma/H_\alpha]$ which are free abelian groups on the cosets in Γ/H_α, where Γ is a fixed group and the subgroups H_α are variable. Our standard examples of chain complexes with such chain groups come from lifting a G-action on a finite G-CW complex X to the universal covering of X; such liftings exist provided X^G is nonempty (compare Theorem I.9.2 in[8] and Theorem 1.13 in [16]).

In order to state our generalization of Lemma 2.3 in [61] we need some notational conventions. Let G be a finite group, let X be a connected finite G-CW complex, let $A \subset X$ be a G-subcomplex, and let X' denote the universal covering of X. If A^G is nonempty (so that X^G is also nonempty), then as in the preceding paragraph the fundamental group Γ of X/G is isomorphic to a semidirect product of $\pi_1(X, x_0)$ and G, where $x_0 \in A^G$ and G acts on $\pi_1(X, x_0)$ via the action of G on (X, x_0). Furthermore, the action of G on X lifts to a Γ-action on the universal covering space X', and similarly the G-equivariant cell complex structure on X lifts to a Γ-equivariant cell complex structure on X'.

The singular set of X, denoted by $\mathrm{Sing}\,(X)$ is the union of all equivariant cells of the form $G/H \times e_\alpha$ such that H is a nontrivial subgroup, and similarly for A. By the definition of equivariant CW complexes we know that $\mathrm{Sing}\,(X)$ and $\mathrm{Sing}\,(A)$ are equivariant subcomplexes of X and A respectively.

Finally, given an arbitrary subset $Y \subset X$, its inverse image in the universal covering will be denoted by Y'. By construction, if Y is an equivariant subcomplex of X then Y' is an equivariant subcomplex of X'.

Lemma 3.3 *Suppose that we are given the data of the preceding paragraph, and assume further that the induced map of singular sets from* $\mathrm{Sing}\,(A)$ *to* $\mathrm{Sing}\,(X)$ *is an equivariant homotopy equivalence. Then there is a* $\mathbb{Z}[\Gamma]$ *chain complex* B_* *such that each chain group is a finitely generated free* $\mathbb{Z}[\Gamma]$*-module which is trivial in all but finitely many dimenions, such that for every* $\mathbb{Z}[\Gamma]$*-module* M *we have isomorphisms*

$$H_*(X', A'; M) \cong H^*(B; M) \ , \qquad H^*(X', A'; M) \cong H_*(B; M) \ .$$

Since the proof of Wall's Lemma 2.6 goes through for chain complexes satisfying the conditions in the lemma, it follows that the homology cohomology groups of C_*, and hence also the corresponding homology and cohomology groups of (X, A), automatically satisfy the projectivity and stable freeness conclusions in Wall's lemma. Important special cases of this result are mentioned briefly in the proof of Theorem 1.2 in [36].

Proof. Let $C_*(X', A')$ denote the Γ-equivariant cellular chain complex of the pair (X', A') By construction the inverse images of the singular sets define a chain subcomplex $S_* = C_*(\,\mathrm{Sing}\,(X)', \mathrm{Sing}\,(A)'\,)$, and the associated quotient complex B_* satisfies the finiteness and freeness conditions in the lemma. Since the induced map of singular sets from $\mathrm{Sing}\,(A)$ to $\mathrm{Sing}\,(X)$ is an equivariant homotopy equivalence, it follows that the homology groups of $S_* \otimes_{\mathbb{Z}[\Gamma]} M$ are trivial for for every $\mathbb{Z}[\Gamma]$-module M, and therefore by the long exact homology and cohomology sequences of the pairs

$$S_* \otimes_{\mathbb{Z}[\Gamma]} M \subset C_*(X', A') \otimes_{\mathbb{Z}[\Gamma]} M$$

we have canonical isomorphisms from the homology and cohomology groups of (X', A') with coefficients in M to the corresponding groups for the chain complex B_*. □

Proof of Theorem 3.2. Since the argument is a modification of the proof in Chapter 3 of [61], we shall concentrate on the points at which changes to the latter are needed; as in Wall's book, the even- and odd-dimensional cases must be treated separately.

Even-dimensional case. Assume that $\dim W = 2k$, where $k \geq 3$. The first step in [61] is to perform surgery on embedded spheres and disks to make a normal map k-connected over W and $(k-1)$-connected over $\partial_0 W$. The relevant spheres and disks have dimension $\leq k-1$ in the first case and $\leq k-2$ in the second. Since the group action on ∂W_0 is free, there is no problem performing surgery over the boundary, and the Gap Hypothesis translates to the inequality $\dim W^k \leq k - 2$, and hence general position implies that one can choose the invariant submanifolds in W up to isotopy so that they are disjoint from W^G. Once this is done, the next step in [61] involves the middle dimension, in which case we obtain a finite collection of neat embeddings (in the sense of [27]) of (D^{2k}, S^{2k-1}) into $(V, \partial_0 V)$. In the equivariant case we want embeddings of $G \times (D^{2k}, S^{2k-1})$ into $(V, \partial_0 V)$ which are disjoint from W^G. We can use the Gap Hypothesis once more to obtain this sharper conclusion. If U denotes an invariant, neat, closed tubular neighborhood for this configuration of disks, then as in [61] we obtain a simple homotopy equivalence if we remove the interior of U.

Odd-dimensional case. Assume that $\dim W = 2k + 1$, where $k \geq 3$. As in the preceding case we can convert the original normal map to one that is k-connected over both W and $\partial_0 W$, so we might as well assume that the original normal map

$$(f, \mathbf{b}) : (V; \partial_1 V, \partial_0 V) \longrightarrow (W; \partial_1 W, \partial_0 W)$$

has this property.

Following Lemma 2.2 in [61], let $K_i(V, V_0)$ denote the kernel of the algebraic map $H_i(V'V_0') \to H_i(W', W_0')$, where A' denotes the universal covering of A; note that in our example the fundamental groups of V, W, V_0 and W_0 are canonically isomorphic. Similarly, let $K_i(V)$ and $K_i(V_0)$ denote the respective kernels of the algebraic maps $H_i(V') \to H_i(W')$ and $H_i(V_0') \to H_i(W_0')$. The same considerations as in the preceding paragraph then imply that our map is normally cobordant to one for which $K_k(V, V_0)$ and $K_{k+1}(V)$ are both zero (the second follows from the first by the duality statement in Lemma 2.2 of [61]); as before, the Gap Hypothesis implies that one can perform the necessary surgery-theoretic constructions away from the fixed point set. Once again, this means we might as well assume that the original normal map also satisfies the additional algebraic conditions.

The arguments and Wall and the earlier discussions in this section now imply that the group $K_{k+1}(V, V_0)$ can be assumed to be a finitely generated free $\mathbb{Z}[\Gamma]$-module. Take a set of mappings $g_i : (D^{k+1}, S^k) \to (V, V_0)$ which represent the free generators of the given module. As in [61] we can take these maps to be smooth immersions with trivial normal bundles, and the next step in Wall's program is to show that the associated maps from $G \times S^k$ to V_0 are regularly homotopic to pairwise disjoint smooth embeddings. The construction of these embeddings involves some auxiliary embeddings of 2-disks in V, and the Gap Hypothesis implies that we can always take these 2-disks to be disjoint from W^G. Therefore, as in [61] we can attach equivariant handles of the form $G \times D^{k+1} \times D^k$ along the constructed embeddings of $G \times S^k \times D^k$ in V_0. If we do so, we obtain a new normal map $(X, X_0) \to (W, W_0)$; define $K_i(X, X_0)$, $K_i(X)$ and $K_i(X_0)$ to be the analogs of $K_i(V, X_0)$, $K_i(V)$ and $K_i(V_0)$. The arguments in [61] then generalize directly to show that $K_{k+1}(X, X_0) \cong K_{k+1}(V, V_0)$ and $K_{k+1}(X) \cong K_{k+1}(X, X_0)$ (note that $K_{k+1}(X_0) \cong K_{k-1}(X_0) = 0$ also holds in our setting). In fact, as in [61] we can go further and conclude that $K_k(X_0) = 0$ and hence that $X_0 \to W_0$ is a simple homotopy equivalence.

The final geometric step in Chapter 4 of [61] involves surgery on generators from $K_k(X)$, and once again the Gap Hypothesis shows that we can choose the generators to be disjoint from the fixed point set X^G. These equivariant surgeries yield a new normal map $(Y, X_0) \to (W, W_0)$, and the algebraic formalism in [61] now shows that the map $Y \to W$ is a simple homotopy equivalence, which in turn implies that $(Y, X_0) \to (W, W_0)$ is a simple homotopy equivalence of pairs and hence completes the proof of the theorem. □

4 Proofs of the main results

It will be convenient to begin with some notational conventions and elementary observations in order to simplify the main discussion and the proofs.

Summary of notational conventions

Let P be a closed smooth G-manifold, where G is a finite group. By local linearity of the action we know that the fixed point set P^G is a union of connected smooth submanifolds; as before, denote these connected components by P_α. For each α let $D(P_\alpha)$ denote a closed tubular neighborhood. By construction these sets are total spaces of closed unit disk bundles over the manifolds P_α, so let $S(P_\alpha)$ and denote the associated unit sphere bundles; it follows that

$$\partial D(P_\alpha) \quad = \quad S(P_\alpha) \ .$$

Suppose now that M and N are smooth semifree G-manifolds and $f : M \to N$ is an equivariant homotopy equivalence. Then the associated map f^G of fixed point sets defines a $1-1$ correspondence between the components of M^G and N^G, and we shall use the following terminology to discuss the fixed point data attached to these G-manifolds:

(1) $\{N_\alpha\}$ denotes the set of components of N^G where we may as well assume that α runs through the elements of $\pi_0(N^G)$.

(2) If for each α we let

$$M_\alpha \quad = \quad f^{-1}[N_\alpha] \cap M^G$$

then f_α is the continuous map from M_α to N_α determined by f.

(3) If the equivariant normal bundles of M_α and N_α in M and N are ξ_α and ω_α respectively, and let $S(\nu)$ and $D(\nu)$ generically represent the unit sphere and disk bundle of the vector bundle ν (with the associated group action since ν is a G-vector bundle).

We shall also use some notational conventions we have previously developed and mentioned.

Consequences of Theorem 2.2

Before proving Theorem 2.2, we shall explain how it implies several other central results.

Proof that Theorem 2.2 implies Theorem 2.1. The first conclusion is the special case of Theorem 2.1 for which the boundaries are empty, and the second conclusion may be derived as follows: Let $h : M \times [0,1] \to N$ be an equivariant homotopy between two isovariant homotopy equivalences, and define $H : M \times [0,1] \to N \times [0,1]$ so that its projections onto N and $[0,1]$ are

h and the usual coordinate projection respectively. By construction, H determines an equivariant homotopy equivalence of pairs from $(M\times[0,1], M\times\{0,1\})$ to $(N\times[0,1], N\times\{0,1\})$ which is isovariant on the boundary, and therefore H is equivariantly homotopic as a map of pairs to an isovariant homotopy equivalence K, and we can in fact find a homotopy which is fixed on $M\times[0,1]$. The composite of K followed by coordinate projection onto N will then define the desired isovariant homotopy from f to g. □

We shall continue by stating an absolute version of Theorem 2.2:

Theorem 4.1 *(i) Let $f:(M,\partial M)\to(N,\partial N)$ be an equivariant homotopy equivalence of connected, compact smooth semifree G-manifolds with boundary that satisfy the Standard Gap Hypothesis. Assume that* either *M and N are at least 6-dimensional* or *M and N are 5-dimensional and the action of G preserves orientations. Then f is equivariantly homotopic, as a map of pairs, to an isovariant homotopy equivalence.*

(ii) Let f_0 and $f_1:(M,\partial M)\to(N,\partial N)$ be isovariant homotopy equivalences of connected, compact, compact smooth semifree G-manifolds with boundary such that $M\times\mathbb{R}$ and $N\times\mathbb{R}$ satisfy the Standard Gap Hypothesis, and suppose that

$$H:(M\times[0,1], \partial M\times[0,1]) \quad\longrightarrow\quad (N,\partial N)$$

is an equivariant homotopy of pairs from f to g. Assume that M and N are at least 5-dimensional. Then H is equivariantly homotopic, as a map of pairs, to an isovariant homotopy equivalenceby a deformation which is constant on $M\times\{0,1\}$.

Examples. It is easy to see that *the first part of Theorem 4.1 does* **not** *generalize to equivariant homotopy equivalances which do not come from homotopy equivalences of pairs*, and here are some explicit counterexamples. Let G be the cyclic group $\mathbb{Z}_n$ for some integer $n\geq 2$, let G act trivially on $\mathbb{R}$, and take the standard action by complex multiplication on $\mathbb{C}$. For each positive integer k the unit disks in $\mathbb{R}^k\times\mathbb{C}^{2k}$ and $\mathbb{R}^k\times\mathbb{C}^{3k}$ are equivariantly contractible, and hence these unit disks are equivariantly homotopy equivalent. However, if they were *isovariantly* homotopy equivalent then the complements of their fixed point sets would also be isovariantly homotopy equivalent. Since these complements are nonequivariantly homotopy equivalent to S^{4k-1} and S^{6k-1} respectively, it follows that the unit disks in $\mathbb{R}^k\times\mathbb{C}^{2k}$ and $\mathbb{R}^k\times\mathbb{C}^{3k}$ cannot be isovariantly (or even equivariantly) homotopy equivalent.

One can also construct counterexamples involving bounded manifolds where the domain and codomain have the same dimension. Given an integer $m \geq 2$, consider the linear $\mathbb{Z}_m$-action on $D^2 \subset \mathbb{C}$ sending (z, v) to the product $z\,v$. If $k \equiv 1 \mod m$ and $k > 1$, then the map $\Psi_k(v) = v^k$ again defines a $\mathbb{Z}_m$-isovariant map from D^2 to itself which is a $\mathbb{Z}_m$-equivariant homotopy equivalence, but is not an isovariant homotopy equivalence because Ψ_k is not a homotopy equivalence on the complements of the fixed points sets. One can easily modify this construction to obtain many other examples; for example, one can take products with linear disks (and round corners equivariantly).

Proof that Theorem 2.2 implies Theorem 4.1. We shall begin by verifying that the equivariant homotopy equivalence of pairs is equivariantly homotopy equivalent, as a map of pairs, to an isovariant homotopy equivalence. Since the associated map of boundaries $\partial f : \partial M \to \partial N$ satisfies the hypotheses of Theorem 2.2, it follows that ∂f is equivariantly homotopy equivalent to an isovariant homotopy equivalence, and by the Equivariant Homotopy Extension Property the original map of pairs is equivariantly homotopic to an equivariant homotopy equivalence of pairs which is isovariant on the boundary. Another application of Theorem 2.2 now implies the latter map is equivariantly homotopic to an isovariant homotopy equivalence such that the homotopy leaves the boundary fixed.

The proof of the second statement follows similarly. Let H be an equivariant homotopy from $(M \times [0,1], \partial M \times [0,1])$ to $(N, \partial N)$ for which the top and bottom maps f_0 and f_1 are isovariant homotopy equivalences, and let

$$K : (M \times [0,1], \partial M \times [0,1]) \quad \longrightarrow \quad (N \times [0,1], \partial N \times [0,1])$$

be the map whose projection onto N and $[0,1]$ are given by H and the usual coordinate projection onto $[0,1]$. Let $K^\bullet$ denote the associated map from $\partial M \times [0,1]$ to $\partial N \times [0,1]$. Another application of Theorem 2.2 shows that we can equivariantly deform $K^\bullet$ to an isovariant homotopy equivalence $L^\bullet$, and by the Equivariant Homotopy Extension Property we can extend this homotopy to obtain a homotopic map of pairs L whose restriction to $\partial M \times [0,1]$ is $L^\bullet$ and whose restriction to $M \times \{0,1\}$ is given by the isovariant mappings f_0 and f_1. By construction this map is isovariant on all of $\partial(M \times [0,1])$, and therefore one final application of Theorem 2.2 shows that L is equivariantly homotopic to an isovariant homtopy equivalence such that the homotopy is fixed on the boundary. This deformation yields the desired isovariant homotopy equivalence of pairs. □

Proof of Theorem 2.2 in the unbounded case. Following [61] we begin with the case in which M and N have no boundaries because all the main issues are already present for such examples but many objects are simpler to describe.

Assume we are given an equivariant homotopy equivalence $f : M \to N$, where M and N are smooth semifree G-manifolds satisfying the hypotheses of the theorem and f is an equivariant homotopy equivalence. As in [61] we can use an equivariant homotopy inverse for f to find bundle data $\mathbf{b}$ for f in the previously described sense, and hence we have an equivariant surgery problem $(f, \mathbf{b}) : M \to N$.

The fixed point set N^G splits into a disjoint union of finitely many components F_α; choose pairwise disjoint closed equivariant tubular neighborhoods E_α for the respective components, set E equal to the union of the neighborhoods E_α, and let

$$C \quad = \quad N \ - \ \cup_\alpha \ \mathrm{Int}\,(E) \ .$$

By construction, we have $\partial C = \partial E$. Since G acts freely on this invariant submanifold, standard transversality theorems imply that f is equivariantly homotopic to a smooth equivariant map which is transverse to $\partial C = \partial E$. Without loss of generality, we may as well assume that f also has these properties.

Assuming the conditions in the previous sentence, for each component F_α of N^G let $f_\alpha : (E'_\alpha, \partial E'_\alpha) \to (E_\alpha, \partial E_\alpha)$ denote the restriction of f to the inverse image of E_α, and let $f_0 : (C', \partial C') \to (C, \partial C)$ denote the restriction of f to the inverse image of C. By equivariance M^G is contained in the interior of E', so that G acts freely on C' and likewise for $\partial C' = \partial E'$. For each α the Gap Hypothesis implies that $\dim(N^G \cap E_\alpha) \leq \dim(E_\alpha) - 3$, and from this we caonclude that the inclusions $\partial E_\alpha \subset E_\alpha$ induce isomorphisms in fundamental groups. Therefore Theorem 3.2 implies that for each α there is a normal cobordism

$$(\psi_\alpha, \mathbf{c}_\alpha) : (Q_\alpha, P_\alpha) \quad \longrightarrow \quad (E_\alpha, \partial E_\alpha)$$

from f_α to a simple equivariant homotopy equivalence. Set Q and P equal to $\amalg_\alpha \, Q_\alpha$ and $\amalg_\alpha \, P_\alpha$ respectively. It follows that we can assemble the equivariant surgery problems $(\psi_\alpha, \mathbf{c}_\alpha)$ into a single object of the form $(\psi, \mathbf{c}) : (Q, P) \to (E, \partial E)$.

Proceeding further as in [61] (see the last paragraph on page 142), we may now construct a normal cobordism by attaching a copy of $M \times [0, 1]$ to Q along $E' \times \{1\}$ with corners suitably rounded at $\partial E' \times \{1\}$ (compare the drawing at the top of [61], p. 143). For each α let L_α denote the top of the cobordism

Q_α, and as before let $L = \amalg_\alpha L_\alpha$. We then have an equivariant normal map of triads

$$\big(W; M \times \{0\} \cup L, C' \times \{1\} \cup P\big) \to \ (N \times [0,1]; N \times \{0\} \cup E \times \{1\}, C \times \{1\})\ .$$

By construction, this map is an equivariant simple homotopy equivalence on the first pieces of the respective boundaries. We claim it induces an isomorphism of fundamental groups on the second piece; specifically, we are claiming that the inclusion of $C \times \{1\}$ in $N \times [0,1]$ induces an isomorphism of fundamental groups. The proof of this assertion is more direcct that its counterpart in [61], and it proceeds as follows: Since the inclusion of $N \times \{1\}$ in $N \times [0,1]$ is a homotopy equivalence, it is enough to show that the inclusion of $C \times \{1\}$ in $N \times \{1\}$ induces an isomorphism of fundamental groups, or equivalently that the same statement is true for the inclusion $C \subset N$. By construction, C is equivariantly homotopy equivalent to $N = N^G$, and since $\dim N^G \leq \dim N - 3$ by the Gap Hypothesis it follows that $N - N^G \subset N$ induces an isomorphism of fundamental groups. Combining these observations, we see that the inclusion of $C \times \{1\}$ in $N \times [0,1]$ induces an isomorphism of fundamental groups, which is what we wanted to prove. As noted above, this implies the second conclusion in Theorem 2.1. □

Proof of Theorem 2.2 in the bounded case. We shall use the setting of the preceding argument, but in order to take the boundaries of M and N into account we add the convention that if X is a compact bounded manifold and $A \subset X$, then $A^\bullet$ will denote the intersection $A \cap \partial X$.

Note that the fixed point components $F_\alpha \subset N^G$ satisfy $\partial F_\alpha = F_\alpha^\bullet$, and in this case the boundaries of the closed tubular neighborhoods split as $\partial E_\alpha = E_\alpha^\bullet \cap S_\alpha$, where $E_\alpha^\bullet$ is a closed tubular neighborhood of ∂F_α in $\partial N = N^\bullet$, and S_α is the unit sphere bundle for the disk bundle $E_\alpha \downarrow F_\alpha$. As usual, we assume that the tubular neighborhoods are neat with respect to a collar neighborhood for $\partial N \subset N$ in the sense of [27].

General considerations about equivariant homotopy equivalences show that the isovariant homotopy equivalence $\partial f : \partial M \to \partial N$ sends each component of M^G to a unique component of N^G, and in fact we have induced homotopy equivalences of pairs

$$f_\alpha : (F'_\alpha, \partial F'_\alpha) \ \ \longrightarrow \ \ (F_\alpha, \partial F_\alpha)$$

where F'_α is the unique component of M^G which maps to N^G. Furthermore, the results of [23] imply that we can isovariantly deform f as a map of pairs so that ∂f sends the submanifolds

$$S'_\alpha\ ,\ \ E'_\alpha\ ,\ \ C' \quad \subset \quad M\ , \qquad S'^{\bullet}_\alpha\ ,\ \ E'^{\bullet}_\alpha\ ,\ \ C'^{\bullet} \quad \subset \quad M^\bullet = \partial M$$

to their counterparts

$$S_\alpha \ , \ \ E_\alpha \ , \ \ C \quad \subset \quad N \ , \qquad S^\bullet_\alpha \ , \ \ E^\bullet_\alpha \ , \ \ C^\bullet \quad \subset \quad N^\bullet = \partial N$$

and the deformed maps on M and ∂M are smoothly transverse to each of the given submanifolds of N or ∂N. For the remainder of the proof we shall assume that f and its induced boundary map ∂f satisfy these additional conditions.

As in the unbounded case we can apply our version of the $\pi - \pi$ Theorem to obtain normal cobordisms (Q_α, P_α) from the mappings f_α to equivariant homotopy equivalences of pairs, but there is one major difference: We now have normal cobordisms of triads

$$\Phi_\alpha : (Q_\alpha; G_\alpha, R_\alpha) \quad \longrightarrow \quad (E_\alpha; E^\bullet_\alpha, S_\alpha)$$

such that the restricted maps $G_\alpha \to E^\bullet_\alpha$ are equivalent to composites

$$E'^{\bullet}_\alpha \times [0,1] \quad \longrightarrow \quad E'^{\bullet}_\alpha \quad \longrightarrow \quad E^\bullet_\alpha$$

where the left hand map is projection onto the first factor and the right hand map is ∂f_α.

Following the conventions in the proof for the unbounded case, we shall write $S = \amalg\, S_\alpha$, $R = \amalg\, R_\alpha$, $G = \amalg\, G_\alpha$ and so on. With this notation, we can form a normal map of triads from

$$(W; M \times \{0\} \cup \partial M \times [0,1] \cup L \cup G_\alpha, C' \times \{1\} \times R)$$

to the standard triad

$$(N \times [0,1]; \partial N \times [0,1] \cup N \times \{0\} \cup E \times \{1\}, C \times \{1\})$$

by gluing the maps Φ_α to $f \times \mathrm{id}_{[0.1]}$ via the identification of $E = \partial_0 Q$ with $E \times \{1\} \subset E \times [0,1]$.

A straightforward application of the Seifert-van Kampen Theorem implies that the inclusion of $C \times \{1\}$ in $N \times [0,1]$ induces an isomorphism of fundamental groups and therefore we may use the same argument in the unbounded case to show that the normal map of triads is normally covbordant to an equivariant simple homotopy equivalence Λ such that the isovariant cobordism is a product with an interval over each of the tubular neighborhood maps $G_\alpha \to E_\alpha$. If the domain triad for this equivariant simple homotopy equivalence is given by

$$(W^*; M \times \{0\} \cup \partial M \times [0,1] \cup L \cup G_\alpha, C^* \times \{1\} \times R)$$

then the s-cobordism Theorem implies that $(W^*, M \times \{0\})$ is equivariantly diffeomorphic to $(M \times [0,1], M \times \{0\})$, and it follows that the restriction of Λ to $W \times \{1\}$ is an almost isovariant homotopy equivalence. We can now apply the results of [23] to conclude that Λ is almost isovariantly homotopic to an isovariant homotopy equivalence, where for each $t \in [0,1]$ the homotopy Θ_t is given on the boundaries by the original boundary map ∂f; this homotopy satifies all the conditions in the conclusion of the theorem. □

Remark There is a slight difference between the Gap Conditions in Theorems 3.2 and 2.2; specifically, in the first result we have $\dim M - 2\dim M^G > 0$ while in the second result we have $\dim M - 2\dim M^G > 1$. The reason for this apparent disparity is that the proof of Theorem 2.2 applies Theorem 3.2 to a $(\dim M + 1)$-dimensional cobordism whose boundary contains M (and its fixed point set has dimension equal to $\dim M^G + 1$).

5 Some applications

In [57] Straus derived an interesting application of the Theorem 2.1 to cyclic reduced products of manifolds, and a result from [54] gives a criterion for recognizing certain isovariant homotopy equivalence with fewer conditions than the general statements of Section 4 in [23]. For the sake of completeness we are including proofs of both results.

Homotopy invariance of deleted reduced products

Given a topological space X and an integer n, its n-fold cyclic reduced product is defined to be the quotient of the product space X^n (*i.e.*, n copies of X) modulo the action of $\mathbb{Z}_n$ on the latter by permuting coordinates, and the deleted cyclic reduced product is the subset of the latter obtained by removing the image of the diagonal $\Delta(X^n)$ consisting of those points whose coordinates are all equal. In his thesis [57] Straus used his version of Theorem 2.1 to obtain the following homotopy invariance property for such spaces:

Theorem 5.1 *Let M and N be closed smooth manifolds of dimension ≥ 2, let p be an odd prime, and suppose that M and N are homotopy equivalent. Let $\mathbb{Z}_p$ act smoothly on the p-fold self products $\prod^p M$ and $\prod^p N$ (where $\prod^p X = X \times \cdots \times X$, with p factors) by cyclically permuting the coordinates, and let $\mathbf{D}^p(M)$, $\mathbf{D}^p(N)$ be the invariant subsets sets given by removing the diagonals from $\prod^p M$ and $\prod^p N$. Then the deleted reduced cyclic products $\mathbf{D}^p(M)/\mathbb{Z}_p$ and $\mathbf{D}^p(N)/\mathbb{Z}_p$ are homotopy equivalent.*

As also noted in [57], this result does not extend to compact bounded manifolds, and in fact closed unit disks of different dimensions yield simple but systematic counterexamples. The results of [52] imply that Theorem 5 extends to simply connected manifolds if $p = 2$, but results of R. Longoni and P. Salvatore [34] imply that the result does not extend to 3-dimensional lens spaces when $p = 2$. Further results on the relationship between $\mathbf{D}^2(M)/\mathbb{Z}_2$ and $\mathbf{D}^2(N)/\mathbb{Z}_2$ for homotopy equivalent manifolds appear in a paper by P. Löffler and R. J. Milgram [33].

The argument below follows Straus' approach in [57] very closely; it also appears in [54].

Sketch of proof. We shall first prove the result when $\dim M = \dim N \geq 3$ and then dispose of the remaining cases afterwards. If $f : M \to N$ is a homotopy equivalence then $\prod^p f : \prod^p M \to \prod^p N$ is an equivariant homotopy equivalence of closed smooth $\mathbb{Z}_p$-manifolds. All actions of $\mathbb{Z}_p$ are semifree if p is prime, so this condition holds automatically. Furthermore, the fixed point sets of the action on $\prod^p M$ and $\prod^p N$ are the respective diagonals $\Delta\left(\prod^p M\right) \cong M$ and $\Delta\left(\prod^p N\right)^n \cong N$, and since

$$\begin{array}{ccccc} \dim \Delta^p\left(\prod^p X\right) & = & \dim X & = & \left(\dim \prod^p X\right)/p \quad \leq \\ \frac{1}{3}\dim \prod^p X & < & \frac{1}{2}\dim \prod^p X \; - \; 1 & & \end{array}$$

if $X = M$ or N is at least 3-dimensional and p is odd, then the Standard Gap Hypothesis also holds. Therefore Theorem 2.1 implies that $\prod^p f$ is equivariantly homotopic to an isovariant homotopy equivalence, and the latter in turn yields an equivariant homotopy equivalence from $\mathbf{D}^p(M)$ to $\mathbf{D}^p(N)$. The induced map of orbit spaces is the desired homotopy equivalence from $\mathbf{D}^p(M)/\mathbb{Z}_p$ to $\mathbf{D}^p(N)/\mathbb{Z}_p$.

Suppose now that $\dim M = \dim N \leq 2$. In these cases homotopy equivalent manifolds are homeomorphic, so we can take the homotopy equivalence $f : M \to N$ to be a homeomorphism. It follows immediately that $\prod^p f$ is a homeomorphism and as such is automatically isovariant. One can now complete the proof as in the last two sentences of the preceding paragraph. □

Recognizing isovariant homotopy equivalences

If X and Y are smooth manifolds with semifree differentiable actions of a finite group G, then the Isovariant Whitehead Theorem in [23] (Theorem 4.10 on p. 35) implies that an isovariant map $f : X \to Y$ is an isovariant homotopy equivalence if and only if it induces homotopy equivalences on subsets of X and Y as follows:

(1) f maps X to Y by a homotopy equivalence.

(2) The induced map of fixed points sets $f^G : X^G \to Y^G$ is a homotopy equivalence.

(3) The induced map of fixed point set complements $X - X^G \to Y - Y^G$ is a homotopy equivalence.

(4) For each component F_α of X^G, the map f induces a fiber homotopy equivalence from the sphere bundle S_α for a tubular neighborhood of F_α to the corresponding sphere bundle S'_α of the component $F'_\alpha \subset Y^G$ which corresponds to F_α under f.

Some of the methods and results in [23] yield stronger results in a few different directions. For example, Corollaries 4.11 and 4.12 in [23] (see pp. 35–37) imply that if X and Y are smooth G-manifolds, then the last condition is redundant, at least if one imposes some orientability hypotheses (see Remark 1 below for further discussion). Other results along these lines appeared in [54]. In fact, one can strengthen all of these results as follows:

Theorem 5.2 *Let $f : M \to N$ be an equivariant homotopy equivalence of connected, compact, unbounded* (= closed) *and oriented smooth G-manifolds such that* (1) *G acts semifreely,* (2) *all components of the fixed point sets M^G and N^G are orientable,* (3) *the weak gap condition* $\dim M - \dim M^G = \dim N - \dim N^G \geq 3$ *is satisfied. If f is isovariant, then f is an isovariant homotopy equivalence.*

Remarks.

1. *Orientability of fixed point set components.* As on page 36 of [23], the orientability condition (2) is added in order to avoid complications involving cohomology with local coefficients. The orientability condition is always satisfied if G has odd order (because all nontrivial real G-representations come from complex representations and hence the equivariant normal bundles of fixed point set components are always orientable). One can use considerations involving oriented double coverings to prove an extension of Theorem 5.2 for examples where M^G and N^G have nonorientable components.

2. *The codimension ≥ 3 hypothesis.* The conclusion of the theorem does not necessarily hold if $\dim M - \dim M^G = \dim N - \dim N^G = 2$. Specifically, for each $n \geq 4$ consider the infinite family of smooth $\mathbb{Z}_m$-actions on S^n with knotted $(n-2)$-spheres as fixed point sets in [58]. Let V be the tangent space at a fixed point with the associated linear $\mathbb{Z}_m$-action. By Alexander Duality the complement of the fixed point set has the homology groups of S^1, and one can use the methods of Section 2 in [48] to construct a degree 1 isovariant map from one of these actions to the linear sphere $S(V \oplus \mathbb{R})$ such that the map of fixed point sets is a homeomorphism. This map is an equivariant homotopy equivalence, but it cannot be an isovariant homotopy equivalence because the complement of the fixed point set in the linear action is homotopy equivalent to S^1 but the complements of the fixed point sets in the exotic actions are not (specifically, by Section II of [58] the Alexander polynomials of the examples are nontrivial).

Proof of Theorem 5.2. Following the notation at the beginning of Section 4, denote the corresponding components of M^G and N^G by M_α and N_α, let $E_\alpha(M)$ and $E_\alpha(N)$ denote pairwise disjoint invariant closed tubular neighborhoods of these components, and set E_M and E_N equal to the unions of these closed tubular neighborhoods. Next, let $S_\alpha(M) = \partial E_\alpha(M)$ and $S_\alpha(N) = \partial E_\alpha(N)$ be the boundaries of the respective components, let S_M and S_N be the unions, and finally let C_M and C_N be the closures of the complements of E_M and E_N respectively. By construction S_M and S_N define G-invariant splittings of $M = E_M \cup C_M$ and $N = E_N \cup C_N$ respectively.

We claim that the pairs (M, C_M) and (N, C_N) are 2-connected. This follows because (1) C_M and C_N are deformation retracts of $M - M^G$ and $N - N^G$ respectively, (2) the pairs $(M, M - M^G)$ and $(N, N - N^G)$ are 2-connected because $\dim M - \dim M^G = \dim N - \dim N^G \geq 3$.

Let M' and N' denote the universal coverings of M and N respectively, and let $f' : M' \to N'$ be a lifting of the equivariant homotopy equivalence f. Furthermore, let E'_M, C'_M and S'_M be the inverse images of E_M, C_M and S_M with respect to the universal covering map $M' \to M$, and define E'_N, C'_N and S'_N in terms of the submanifolds E_N, C_N and S_N and the universal covering map $N' \to N$. Finally, let Λ denote the group ring $\mathbb{Z}[\pi_1(M)] \cong \mathbb{Z}[\pi_1(N)]$.

By Theorem 4.5 on page 30 of [23], the isovariant map f is isovariantly homotopic to a map of triads $(M; E_M, C_M) \to (N; E_N, C_N)$, and therefore the induced map f' of universal coverings also splits into a map of triads

$$(M'; E'_M, C'_M) \quad \longrightarrow \quad (N'; E'_N, C'_N) \ .$$

Since the homotopy equivalence f has degree ± 1, the discussion of degree 1 maps in Chapter 2 of [61] applies, and accordingly the homology mappings

$$H_j(E'_M; \Lambda) \quad \longrightarrow \quad H_j(E'_N; \Lambda)$$
$$H_j(S'_M; \Lambda) \quad \longrightarrow \quad H_j(S'_N; \Lambda)$$

$$H_j(C'_M;\Lambda) \quad \longrightarrow \quad H_j(C'_N;\Lambda)$$

are split surjections, Let $K_j(E'_M)$, $K_j(S'_M)$ and $K_j(C'_M)$ denote their respective kernels. Since f is a homotopy equivalence, we can use the reasoning at the top of page 93 in [13] to conclude that

$$K_j(S'_M) \quad \cong \quad K_j(E'_M) \ \oplus \ K_j(C'_M) \ .$$

By construction we know that the map of pairs $(E_M, S_M) \to (E_N, S_N)$ splits into maps of connected components $(E_\alpha(M), S_\alpha(M)) \to (E_\alpha(N), S_\alpha(N))$, and since f is an equivariant homotopy equivalence it follows that the underlying maps of spaces $E_\alpha(M) \to E_\alpha(N)$ are homotopy equivalences. As in the proof of Corollary 4.12 on page 37 of [23], an argument involving normal degrees shows that the maps $S_\alpha(M) \to S_\alpha(N)$ are homotopy equivalences, which in turn implies that $K_j(S_\alpha(M)')$ vanish for all j. Therefore the direct sum decomposition of the preceding paragraph implies that $K_j(C'_M) = 0$ for all j.

By Corollary 4.12 of [23], it suffices to prove that the map $C_M \to C_N$ is a homotopy equivalence. At the beginning of the proof we noted that the pairs (M, C_M) and (N, C_N) are 2-connected, and since M and N are connected the same is true for C_M and C_N. The next step is to verify that the map $C_M \to C_N$ induces an isomorphism of fundamental groups. To see this consider the following commutative diagram:

$$\begin{array}{ccc} \pi_1(C_M) & \longrightarrow & \pi_1(C_N) \\ \downarrow & & \downarrow \\ \pi_1(M) & \longrightarrow & \pi_1(N) \end{array}$$

The vertical morphisms are bijective by the 2-connectivity condition, and the bottom morphism is bijective since $M \to N$ is a homotopy equivalence, so the top diagram must also be an isomorphism by a diagram chase. These isomorphisms of fundamental groups imply that C'_M and C'_N are the universal covering spaces of C_M and C_N respectively, so that the map $C'_M \to C'_N$ is a lifting of $C_M \to C_N$ to universal coverings. By the preceding paragraph we know that the map $C'_M \to C'_N$ induces isomorphisms in homology, and therefore it follows that $C_M \to C_N$ is a homotopy equivalence, which is what we wanted to prove. □

6 Final remarks

It is natural to ask about possible extensions of the main results to more general settings, and in this section we shall summarize a few results and specific questions.

Equivariant mappings of degree 1

Results of K. H. Dovermann [18] show that certain equivariant degree 1 mappings of smooth G-manifolds are normally cobordant to isovariant maps (in the sense of equivariant surgery theory). In particular, Dovermann's results apply if a degree 1 map $f : M \to N$ satisfies the following conditions:

(1) The actions on M and N are semifree.

(2) The map of fixed point sets $f^G : M^G \to N^G$ is a homotopy equivalence.

(3) The dimensions of M and N are sufficiently large.

(4) One can define suitable bundle data similar to that of Section 3.

In contrast, the following results of Browder (implicit in [11]) yield equivariant degree 1 maps of such that (1) the underling G-manifolds satisfy the Gap Hypothesis, (2) the maps are not equivariantly homotopic to isovariant maps.

Examples 6.1 Let k and q be distinct positive integers such that q is even and G has a free q-dimensional linear representation (*i.e.*, the group acts freely except at the zero vector). Let $N = S^k \times S^q$ with trivial action on the first coordinate and the one point compactification of the free linear representation on the second, let M_0 be the disjoint union of N and two copies of the space $G \times S^k \times S^q$ (where G acts by translation on itself and trivially on the other two coordinates), and define an equivariant map $f_0 : M_0 \to N$ by taking the identity on M, the unique equivariant extension of the identity map on $S^k \times S^q$ over one copy of $G \times S^k \times S^q$, and the unique equivariant extension of an orientation reversing self diffeomorphism of $S^k \times S^q$ over the other copy. By construction this map has degree one, and one can attach 1-handles equivariantly to M_0 away from the fixed point set to obtain an equivariant cobordism of maps from f_0 to a map f on a connected 1-manifold M that is nonequivariantly diffeomorphic to a connected sum of $2 \cdot |G| + 1$ copies of $S^k \times S^q$. Since the fixed point sets of M and N are k-dimensional and the manifolds themselves are $(k+q)$-dimensional, it follows that the Standard Gap Hypothesis holds if we impose the stronger restriction $q \geq k + 2$. By construction the map f determines a homotopy equivalence of fixed point sets and is isovariant on a neighborhood of the fixed point set.

Assertion It is not possible to deform f equivariantly so that the set of non-isovariant points lies in a tubular neighborhood of the fixed point set. In particular, it is also not possible to deform f equivariantly to an isovariant map.

Proof. To prove the assertion, assume that one has a map h equivariantly homotopic to f with the stated property, and let U be a tubular neighborhood of M^G that contains the set of nonisovariant points. Let X be a submanifold of the form $\{g\} \times \{v\} \times S^q$ in M that arises from one of the copies of $G \times S^k \times S^q$ in M_0. Although X and U may have points in common, by the uniqueness of tubular neighborhoods we can always isotop X into a submanifold X' that is disjoint from U. By the hypotheses on h we know that $h(X')$ is disjoint from $N^G = S^k \times S^0$, and therefore $h(X')$ is contained in

$$N - N^G \quad \cong \quad S^k \times S^{q-1} \times \mathbb{R}$$

so that the image of the generator of $H_q(X') = \mathbb{Z}$ maps trivially into $H_q(N)$. However, h is supposed to be homotopic to a map which is nontrivial on the latter by construction, so we have a contradiction, and therefore it is not possible to find an isovariant map h that is equivariantly homotopic to f. □

A refinement of the preceding argument shows that if Y is a subset of M such that the image of $H_q(Y)$ in $H_q(M)$ is equal to the image of $H_q(X)$, then Y must contain some nonisovariant points of any equivariant map that is equivariantly homotopic to f.

Remark By construction, Browder's examples are normally cobordant to the identity; an explicit normal cobordism from the identity to f_0 is given by

$$W \quad = \quad N \times [0,1] \ \amalg \ G \times S^k \times S^q \times [0,1]$$

where $\partial_- W = N \times \{0\}$ and $\partial_+ W$ is the remaining $2|G|+1$ components of the boundary, and one can obtain a normal cobordism to f by adding 1-handles equivariantly along the top part of the boundary. More generally, results of K. H. Dovermann [18] imply that one can always construct equivariant normal cobordisms to equivariant normal maps if the Gap Hypothesis holds and the map is already an equivariant homotopy equivalence on the singular set as in Browder's examples.

However, it is also possible to construct examples like Browder's that are not cobordant to highly connected maps. It suffices to let $k \equiv 0(4)$ and replace $G \times S^k \times S^q$ by $G \times S(\gamma)$, where the latter is the sphere bundle of a fiber homotopically trivial vector bundle γ over S^k with nontrivial rational Pontryagin classes; one must also replace the equivariant folding map from $G \times S^k \times S^q$ to N by its composite with the identity on G times a fiber homotopy equivalence from $S(\gamma)$ to $S^k \times S^q$. Characteristic number arguments imply the map obtained in this fashion is not cobordant to a k-connected map. Of course, a degree 1 map of this type does not have the bundle data required for a normal map in the sense of equivariant surgery theory.

Exceptional low-dimensional cases

The dimension hypotheses in Theorem 2.1 arise because similar conditions are needed to apply surgery theory. Since the Gap Hypothesis only applies to manifolds of dimension ≥ 3, the only cases not covered by the main result are $\dim M = \dim N = 3, 4$. However, the second part of Theorem 2.1 also applies to 4-manifolds, and in fact we can prove that the first part of Theorem 2.1 is also true for certain group actions on 4-manifolds.

Theorem 6.2 *Let $f : M \to N$ be an equivariant homotopy equivalence of connected, compact, unbounded* ($=$ closed) *and semifree smooth G-manifolds that satisfy the Standard Gap Hypothesis. Assume that M and N are 4-dimensional, and if G is isomorhic to $\mathbb{Z}_2$ also assume that* $\dim M^G = \dim N^G = 0$. *Then f is equivariantly homotopic to an isovariant homotopy equivalence.*

Notes. If G is not cyclic of order 2 and the actions have nonempty fixed point sets, then $\dim M^G = \dim N^G = 0$ automatically holds because all free representations of G are even-dimensional. On the other hand, if $G \cong \mathbb{Z}_2$ then semifreeness and the Gap Hypothesis only imply the weaker condition $\dim M^G = \dim N^G \leq 1$. The hypothesis in the theorem is satisfied if M and N are oriented and $G \cong \mathbb{Z}_2$ acts orientation-preservingly.

Proof. We shall assume that M^G and N^G are nonempty because equivariant maps between free G-manifolds are automatically isovariant. Since the given actions on the 4-manifolds satisfy the Gap Hypothesis, we know that $4 = \dim M^G > 1 + 2 \dim M^G$, so that $\dim M^G \leq 1$. As indicated in the preceding paragraph, the hypotheses of the theorem imply that $\dim M^G = \dim N^G = 0$ in all cases. If we now take trivial G-action on S^1, we see that both of the product G-manifolds $M \times S^1$ and $N \times S^1$ satisfy the Gap Hypothesis.

It follows that the equivariant homotopy equivalence $f \times \mathrm{id}(S^1) : M \times S^1 \to N \times S^1$ satisfies the hypotheses in the first part of Theorem 2.1. Therefore we can now apply the first half of Theorem 2.1 to conclude that $f \times \mathrm{id}(S^1)$ is equivariantly homotopic to an isovariant homotopy equivalence f'. Let H denote an equivariant homotopy relating these two maps.

Standard results on lifting maps to covering spaces now imply that H lifts equivariantly to the infinite cyclic coverings $X \times \mathbb{R} \to X \times S^1$, where X is either $M \times [0,1]$ or N. Therefore the lifting of H defines an equivariant homotopy from $f \times \mathbb{R}$ to some isovariant homotopy equivalence F' (with G acting trivially on $\mathbb{R}$). If $Y = M$ or N then the injections $\varphi_Y : Y \to Y \times \mathbb{R}$ (with $\varphi_Y(y) = (y, 0)$) and coordinate projections $\rho_Y : P \times \mathbb{R} \to Y$ are isovariant homotopy equivalences which are mutually inverse in the isovariant homotopy category, and hence the composite $\rho_N \circ H \circ \varphi_M$ defines an equivariant homotopy from f to the isovariant homotopy equivalence $\rho_N \circ F' \circ \varphi_M$ which is isovariantly homotopic to f. □

Although it seems likely that a similar conclusion holds when $\dim M^G = \dim N^G = 1$, it is not clear how one might try to verify this in complete generality. However, if the fundamental groups of M and N are small in the sense of [24] (*e.g.*, finite or abelian), then we have the following extension of Theorems 2.1 and 6.2:

Theorem 6.3 *Let $f : M \to N$ be an equivariant homotopy equivalence of connected, compact, unbounded* (= closed) *and semifree smooth G-manifolds that satisfy the Standard Gap Hypothesis. Assume that M and N are 4-dimensional and that $\pi_1(M) \cong \pi_1(N)$ is small in the sense of* [24]. *Then f is equivariantly homotopic to an isovariant homotopy equivalence.*

The differences between this and the preceding result are that the fundamental groups have been restricted but the conclusion now covers the cases where $\dim M^G = \dim N^G = 1$.

Proof. It is only necessary to replace the input from smooth surgery theory in the proof of Theorem 2.1 with its counterparts in the 4-dimensional topological surgery of [24] for manifolds with small fundamental groups. In particular, one must use the latter to prove an analog of our equivariant $\pi - \pi$ Theorem (*i.e.*, Theorem 3.2) for certain 5-dimensional topological G-manifolds with small fundamental groups (specifically, this covers group actions that are semifree and tame in the sense of Chapter 14B in [61]). □

In the remaining case where $\dim M = \dim N = 3$, the class of semifree group actions satisfying the Gap Hypothesis is far more restricted than it is in higher dimensions. Specifically, *if a smooth semifree, nonfree G-action on a 3-manifold satisfies the Standard Gap Hypothesis, then the fixed point set must be 0-dimensional and the group G must be isomorphic to $\mathbb{Z}_2$.* It is natural to ask whether currently known techniques from 3-dimensional topology (including the equivariant geometrization theorems) can be applied to verify the main result of this paper for such examples.

Generalizations to nonsemifree actions.

The reasons for our restriction to semifree actions are discussed in Remark 4 at the end of Section 2. If we are given an arbitrary smooth action of a finite group G on a manifold M, it is necessary to work with all the fixed point sets M^H, where H runs through the isotropy subgroups of G, and with invariant tubular neighborhoods of the components of these fixed point sets. The notion of *smooth stratification* is particularly well-suited for managing such data (*e.g.*, see [12], Section 2.5 of [21], or the more general approach in Chapter 4 of [40]); however, it is also possible to study the underlying issues without introducing stratifications explicitly (see [15], [19], or [20], for example). In any case, the basic approach to working with nonsemifree action is by induction on the strata, assuming results have been established for all strata strictly less than a given one and adapting the methods of this paper to show that, under this hypothesis, the results will be true for all strata up to and including the given one; for semifree actions, one knows that the desired results are true on the strata of the fixed point set because an equivariant map of trivial G-spaces is automatically isovariant, and for this reason one can view our arguments as a special case of an inductive step.

Strictly speaking, there are **two** inductive processes here, one of which is an extension of the diagram-theoretic setting in [23] to nonsemifree actions, and the other of which is an extension of the surgery-theoretic methods in the present article. As noted before, it seems very likely that both of these programs can be carried out by a well-informed researcher who has the patience, determination and motivation to do so. Of course, the urgency of completing this work may depend upon finding applications of such generalizations.

Analogous results for more general actions

A few general comments about this issue are stated in Remark 5 at the end of Section 2; for the sake of simplicity we shall only consider semifree actions here. As noted in the remark, one fundamental difficulty involves the structure of small neighborhoods for the components of the fixed point set. An improved understanding of such local structure from the viewpoint of controlled topology as in [41] and [62] would be extremely useful for studying possible extensions of our main results to group actions that are not necessarily smooth but are still somehow well behaved topologically.

Any study of such questions for locally linear actions seems likely to have some interaction with the theory of isovariant homotopy structure sets for locally linear G-manifolds due to S. Weinberger [62]. A general analysis of isovariant homotopy for continuous locally linear actions will require new techniques because the sets $\mathrm{Sing}(X^H)$ do not necessarily have equivariant mapping cylinder neighborhoods in this case (as before, see [42]); it seems quite likely that some sort of sheaf-theoretic machinery will be required.

Isovariance and the Gap Hypothesis

Questions about the role of the Gap Hypothesis in transformation groups have been around for some time (*cf.* [51]). Such questions rarely have clear cut answers, but the main results of this paper provide further evidence that the usefulness of the Gap Hypothesis is closely related to

(1) the strong implications of isovariance for analyzing existence and classifications questions for group actions,

(2) some very close relationships between isovariant homotopy and equivariant homotopy when the Gap Hypothesis holds.

As noted at the beginning of this paper, examples of these phenomena have been well known for some time. Isovariant techniques play a central role in several classification results for group actions when the Standard Gap Hypothesis fails; in many cases where such machinery is not used explicitly, the work can readily be interpreted in these terms. One fundamentally important breakthrough in this area was due to Browder and Quinn [12] (see also the commentary on the latter in [28]), and a more general discussion of the situation in the smooth category — which also extends earlier work on the smooth classification of topologically linear actions on spheres — appears in Section II.1 of [53] (see also the final section of [23]). In the piecewise linear and topological categories, there is a distinct body of results which is largely based on techniques from controlled topology (*e.g.,* Weinberger's book [62], the survey articles by Hughes-Weinberger [28] and Cappell-Weinberger [14], and the doctoral dissertation of A. Beshears [7]). A full historical account of the topic is beyond the scope of this paper, but some additional references include the work of A. Assadi with Browder [1] and P. Vogel [2], the monograph by L. Jones on symmetries of disks [31], and the material on symmetries of aspherical manifolds in Weinberger's paper on higher Atiyah-Singer invariants [63].

In certain respects the main results of this paper imply that equivariant and isovariant homotopy equivalence of closed G-manifolds are equivalent notions provided the Gap Hypothesis holds. One possible conclusion from these results is that the study of classification questions outside the range of the usual

Gap Conditions should be largely based on isovariant rather than equivariant homotopy equivalence. In particular, our main results and the conclusions of [23] suggest a two step approach to analyzing smooth G-manifolds within a given equivariant homotopy type if the Gap Hypothesis does not necessarily hold; namely, the first step is to study the obstructions to isovariance for an equivariant homotopy equivalence and the second step is to study one of the versions of the isovariant surgery theories in [53] or [62]. It would be interesting to see how well this succeeds in some test cases.

References

[1] **A. Assadi and W. Browder**, *On the existence and classification of extensions of actions on submanifolds of disks and spheres*, Trans. Amer. Math. Soc. **291** (1985), 487–502.

[2] **A. Assadi and P. Vogel**, *Actions of finite groups on compact manifolds*, Topology **26** (1987), 239–263.

[3] **A. Bak and M. Morimoto**, *Equivariant surgery and applications*, Topology Hawaii (Conf. Proc., Honolulu, HI, 1990), pp. 13–25, World Sci. Publishing, River Edge, NJ, 1992.

[4] **A. Bak and M. Morimoto**, *Equivariant surgery with middle-dimensional singular sets. I*, Forum Math. **8** (1996), 267–302.

[5] **J. C. Becker and R. E. Schultz**, *Equivariant function spaces and stable homotopy theory*, Comment. Math. Helv. **49** (1974), 1–34.

[6] **R. Benlian and J. Wagoner.** *Type d'homotopie et réduction structurale des fibrés vectoriels.* C. R. Acad. Sci. (Paris) Sér. A–B **265** (1967), A207–A209.

[7] **A. Beshears**, *G-isovariant structure sets and stratified structure sets*, Ph. D. Dissertation, Vanderbilt University, 1997.
(Available from ProQuest Digital Dissertations, Order No. 9724937).

[8] **G. E. Bredon**, Introduction to Compact Transformation Groups (Pure and Applied Mathematics, Vol. 46). Academic Press, New York–London, 1972.

[9] **W. Browder**, *Embedding* 1*-connected manifolds*, Bull. Amer. Math. Soc. **72** (1966), 225-231.

[10] **W. Browder**, *Surgery and the theory of differentiable transformation groups*, Proceedings of the Conference on Transformation Groups (Tulane, 1967), pp. 3–46, Springer-Verlag, Berlin-Heidelberg-New York, 1968.

[11] **W. Browder**, *Isovariant homotopy equivalence*, Abstracts Amer. Math. Soc. **8** (1987), 237–238.

[12] **W. Browder and F. Quinn**, *A surgery theory for G-manifolds and stratified sets*, Manifolds—Tokyo, 1973 (Conf. Proc., Univ. of Tokyo, 1973), pp. 27–36, University of Tokyo Press, Tokyo, 1975.

[13] **S. Cappell**, *A splitting theorem for manifolds*, Invent. Math. **33** (1976), 69–170.

[14] **S. Cappell and S. Weinberger**, *Surgery theoretic methods in group actions*, Surveys on surgery theory, Vol. 2, Ann. of Math. Studies No. 149, pp. 285–317, Princeton Univ. Press, Princeton, NJ, 2001.

[15] **M. Davis**, *Smooth G-manifolds as collections of fiber bundles*, Pac. J. Math. **77** (1978), 315–363.

[16] **H. Donnelly and R. Schultz**, *Compact group actions and maps into aspherical manifolds*. Topology **21** (1982), 443–455.

[17] **K. H. Dovermann**, *$\mathbb{Z}_2$ surgery theory*, Michigan Math. J. **28** (1981), 267–287.

[18] **K. H. Dovermann**, *Almost isovariant normal maps*, Amer. J. Math. **111** (1989), 851–904.

[19] **K. H. Dovermann and T. Petrie**, *G-Surgery II*, Memoirs Amer. Math. Soc. **37** (1982), No. 260.

[20] **K. H. Dovermann and M. Rothenberg**, *Equivariant Surgery and Classification of Finite Group Actions on Manifolds*, Memoirs Amer. Math. Soc. **71** (1988), No. 379.

[21] **K. H. Dovermann and R. Schultz**, *Equivariant Surgery Theories and their Periodicity Properties*, [Springer] Lecture Notes in Mathematics, Vol. 1443, Springer-Verlag, Berlin-Heidelberg-New York (*etc.*), 1990.

[22] **E. Dror Farjoun**, *Homotopy and homology of diagrams of spaces*, Algebraic Topology (Workshop Proceedings, Seattle, 1985), pp.93–134, [Springer] Lecture Notes in Mathematics Vol. 1286, Springer-Verlag, Berlin-Heidelberg-New York (*etc.*), 1987.

[23] **G. Dula and R. Schultz**, *Diagram cohomology and isovariant homotopy theory*, Memoirs Amer. Math. Soc. **110** (1994), No. 527.

[24] **M. H. Freedman and F. S. Quinn**, *Topology of 4-manifolds*, Princeton Mathematical Series Vol. 39. Princeton University Press, Princeton, NJ, 1990.

[25] **J.-C. Hausmann and P. Vogel**, *Geometry on Poincaré spaces*, Princeton Mathematical Notes No. 41, Princeton University Press, Princeton, NJ, 1993.

[26] **J. Hempel**, *3-Manifolds*, Ann. of Math. Studies, No. 86. Princeton University Press, Princeton, NJ, 1976.

[27] **M. W. Hirsch**, "Differential topology. Corrected reprint of the 1976 original." Graduate Texts in Mathematics No. 33. Springer-Verlag, Berlin-*etc.*, 1994.

[28] **B. Hughes and S. Weinberger**, *Surgery and stratified spaces*, Surveys on surgery theory, Vol. 2, Ann. of Math. Studies No. 149, pp. 319–352, Princeton Univ. Press, Princeton, NJ, 2001.

[29] **S. Illman** , *Equivariant algebraic topology*, Ph. D. Dissertation, Princeton University, 1972.
(Available from ProQuest Digital Dissertations, Order No. 7229793).

[30] **S. Illman**, *Smooth equivariant triangulations of G-manifolds for G a finite group*, Math. Ann. **233** (1978), 199–220.

[31] **L. Jones**, *Combinatorial symmetries of the m-dimensional ball*, Mem. Amer. Math. Soc. **62** (1986), No. 352.

[32] **H. Kneser**, *Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten*, Jahresber. der Deutsch. Math. Verein. **38** (1929), 248–260.

[33] **P. Löffler and R. J. Milgram**, *The structure of deleted symmetric products*, Braids (Santa Cruz, CA, 1986), Contemp. Math. **78** (1988), 415–424.

[34] **R. Longoni and P. Salvatore**, *Configuration spaces are not homotopy invariant*, Topology **44** (2005), 375–380.

[35] **W. Lück and I. Madsen**, *Equivariant L-theory I*, Math. Zeitschrift **203** (1990), 503–526.

[36] **W. Lück and I. Madsen**, *Equivariant L-theory II*, Math. Zeitschrift **204** (1990), 253–268.

[37] **R. S. Palais**, *The classification of G-spaces*, Memoirs Amer. Math. Soc. No. 36 (1960).

[38] **T. Petrie**, *Pseudoequivalences of G-manifolds*, Proc. AMS Sympos. Pure Math **32** Pt. 1 (1978), 169–210.

[39] **T. Petrie**, *One fixed point actions on spheres I–II*, Adv. Math. **46** (1982), 3–14 & 15–70.

[40] **M. J. Pflaum**, *Analytic and geometric study of stratified spaces*, [Springer] Lecture Notes in Mathematics, Vol. 1768, Springer-Verlag, Berlin-Heidelberg-New York (*etc.*), 2001.

[41] **F. S. Quinn**, *Ends of maps II*, Invent. Math. **68** (1982), 353–424.

[42] **F. S. Quinn**, *Homotopically stratified sets*, J. Amer. Math. Soc. **1** (1987), 441–499.

[43] **T. St. J. Redmond**, *PL equivariant characteristic classes and G-signature*, Ph. D. Dissertation, Princeton University, 1984 (summarized in Dissertation Abstracts International **45** (1984/1985), 1496B–1497B).
(Available from ProQuest Digital Dissertations, Order No. AAT 8417439).

[44] **F. Rhodes**, *On the fundamental group of a transformation group*, Proc. London Math. Soc. (3) **16** (1966), 635–650.

[45] **M. Rothenberg**, *Review of the book, "Equivariant Surgery Theories and Their Periodicity Properties," by K. H. Dovermann and R. Schultz*, Bull. Amer. Math, Soc. (2) **28** (1993), 375–381.

[46] **S. Safii**, Ph. D. Thesis, University of California at Riverside, in preparation.

[47] **R. Schultz** , *Homotopy decomposition of equivariant function spaces I*, Math. Zeitschrift **131** (1973), 49–75.

[48] **R. Schultz.** *Differentiable group actions on homotopy spheres. I. Differential structure and the knot invariant*, Invent. Math. **31** (1975), 105–128.

[49] **R. Schultz**, *Spherelike G-manifolds with exotic equivariant tangent bundles*, Studies in Algebraic Topology (Adv. in Math. Supplementary Studies Vol. 5), pp. 1–39, Academic Press, New York, 1979.

[50] **R. Schultz**, *Differentiable group actions on homotopy spheres. II. Ultrasemifree actions*, Trans. Amer. Math. Soc. **268** (1981), 255–297.

[51] **R. Schultz (**ed.**)**, *Problems submitted to the AMS summer research conference on group actions*, Group actions on manifolds (Boulder, CO, 1983), Contemp. Math. **36**, pp. 513–568, Amer. Math. Soc., Providence, RI, 1985.

[52] **R. Schultz**, *A splitting theorem for manifolds with involution and two applications*, J. London Math. Soc. (2) **39** (1989), 183–192.

[53] **R. Schultz**, *Isovariant homotopy theory and differentiable group actions*, Algebra and topology 1992 (Taejŏn, Korea), Proc. KAIST Math. Workshops, Vol. 7, pp. 81–148, Korea Adv. Inst. Sci. Tech., Taejŏn, Korea, 1992.

[54] **R. Schultz.** *Isovariant mappings of degree* 1 *and the Gap Hypothesis.* Alg. Geom. Top. (electronic) **6** (2006), 739–762.

[55] **R. Schultz.** *Isovariantly exotic symmetries of products of spheres.* Abstracts Amer. Math. Soc. **28** (2007).
(Available online at http://math.ucr.edu/∼res/miscpapers/exoticisovariant.pdf).

[56] **R. Schultz.** *Equivariant function spaces and isovariant homotopy equivalences.* (to appear)

[57] **S. H. Straus**, *Equivariant codimension one surgery*, Ph. D. Dissertation, University of California, Berkeley, 1972.

[58] **D. W. Sumners**, *Smooth $\mathbb{Z}_p$-actions on spheres which leave knots pointwise fixed*, Trans. Amer. Math. Soc. **205** (1975), 193–203.

[59] **W. Thurston**, *The topology and geometry of* 3*-manifolds*, lecture notes, Princeton University, 1980.
Edited online electronic version: http://library.msri.org/book/gt3m

[60] **J. B. Wagoner**, *Smooth and piecewise linear surgery*, Bull. Amer. Math. Soc. **73** (1967), 72–77.

[61] **C. T. C. Wall**, *Surgery on Compact Manifolds*, (Second edition. Edited and with a foreword by A. A. Ranicki), Mathematical Surveys and Monographs, Vol. 69, American Mathematical Society, Providence, RI, 1999.

[62] **S. Weinberger**, *The topological classification of stratified spaces*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1994.

[63] **S. Weinberger**, *Higher ρ-invariants*, Tel Aviv Topology Conference: Rothenberg Festschrift (1998), Contemp. Math. **231**, pp. 315–320, Amer. Math. Soc., Providence, RI, 1999.

[64] **G. Whitehead**, Elements of Homotopy Theory. Graduate Texts in Mathematics, Vol. 61. Springer-Verlag, New York-Berlin, 1978.