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PART I

INTRODUCTION TO ISOVARIANT HOMOTOPY THEORY

Background references and notation

Although we shall include a few remarks on the basic facts and concepts of differentiable transformation groups, a detailed account of the subject's foundations is beyond the scope of this article. The following references contain most if not all of the relevant background material:

(1) Chapters I–II and Sections VI.1–2 of Bredon, *Introduction to Compact Transformation Groups* (:= [Bre3]).

(2) Chapter I and Sections II.1–2 of tom Dieck, *Transformation Groups* (:= [tD2]).

(3) Chapters I–II and the Summary of Dovermann–Schultz, *Equivariant Surgery Theories and Their Periodicity Properties* (:= [DoS2]).

Most of the algebraic topology that we use can be found in the standard books by Spanier [Sp] and Milnor and Stasheff [MS].

We shall generally use standard notational conventions in transformation groups including M^G for the fixed point set of a group G acting on a space M, G_x for the isotropy subgroup of G at x, and $M_{(H)}$ for the set of all points whose isotropy subgroups are conjugate to H, and $|G|$ for the order of a finite group G. These (and many others) can be found in the references listed above.

As indicated by the title of this article, we shall deal mainly with smooth group actions. However, for purposes of comparison we shall occasionally comment on other families of group actions that lie between smooth and topological (*i.e.*, continuous) actions on manifolds. In particular, we shall discuss results for group actions that are *locally linear* (≡ locally smooth in the sense of Bredon [Bre3]), and if G is finite we shall also discuss actions that are *piecewise linear* (usually abbreviated to PL) or *PL locally linear* (≡ Gpl in [Ro, pp. 303–304]). Further information on these families can be found in the references cited, in [LaR], and in standard texts of PL topology such as Hudson [Hud] or Rourke and Sanderson [RSa]; the logical relations among the various types of actions are either obvious or implicit in work of S. Illman [Il3].

Finally, to simplify the exposition we adopt the following default hypothesis:

($\star$) *For all differentiable group actions, the group G that acts is assumed to be finite unless there are statements or contexts to indicate otherwise.*

1. Equivariant differential topology

To a great extent the foundations of the theory of transformation groups are developed by analogy. Specifically, given a fact or concept in some category of objects and morphisms $\mathcal{A}$, one attempts to describe something parallel in an equivariant category $\mathcal{A}_G$ with the following sorts of data:

(i) The objects of $\mathcal{A}_G$ are pairs consisting of an object X in $\mathcal{A}$ and a homomorphism $\Phi : G \to \mathrm{Aut}_{\mathcal{A}}(X)$ that may be required to satisfy additional technical restrictions (*e.g.*, that a naturally associated map $\varphi : G \times X \to X$ lie in $\mathcal{A}$).
(ii) The morphisms $f : (X, \Phi) \to (X', \Phi')$ satisfy the equivariance identity $\Phi'(g) \circ f = f \circ \Phi(g)$ for all $g \in G$.

In this article we are interested mainly in the category of smooth finite-dimensional manifolds. Two basic concepts in this category are the notions of *vector bundle* and *smooth embedding*. Each of these has a natural analog in the G-equivariant category; namely, a smooth G-vector bundle $E \downarrow M$ is taken to have a smooth action of G that sends the fiber over $x \in M$ linearly into the fiber over $g \cdot x$ for all $(g, x) \in G \times M$ (*cf.* [Bre3, pp. 303–304]), and an equivariant smooth embedding is merely a smooth embedding that is G-equivariant. In ordinary differential topology the Tubular Neighborhood Theorems relate smooth embeddings to smooth vector bundles in a fundamentally important manner, and one has a straightforward equivariant generalization with extremely far reaching consequences for the theory of differentiable transformation groups:

Theorem 1.1. (i) *Let M and N be smooth (finite dimensional) G-manifolds, and let $f : M \to N$ be a smooth equivariant embedding. Then there is an equivariant vector bundle $E \downarrow M$ and a smooth embedding $F : E \to N$ whose restriction to the zero section is essentially f.*

(ii) *Suppose that $f_j : E_j \to N$ (where $j = 0, 1$) satisfy* (i), *and suppose we are given invariant riemannian metrics on E_j with unit disk bundles $D(E_j)$. Then there is a metric preserving G-vector bundle isomorphism $\varphi : E_0 \to E_1$ covering the identity on M and a smooth equivariant ambient isotopy $H : N \times [0,1] \to N$ such that $H_1 \circ f_o = f_1 \circ \varphi$.*

This theorem follows from the results of [Bre3, Sections VI.2–3].

If the source manifold M in Theorem 1.1 is a homogeneous space G/H, then the theorem reduces to the *differentiable slice theorem* [Bre3, Cor. VI.2.4, p. 308]:

Corollary 1.2. *Let M be a smooth finite dimensional G-manifold without boundary, let $x \in M$ be given, and let H be the isotropy subgroup G_x. Then x has a G-invariant* *neighborhood that is G-diffeomorphic to an associated fiber space $G \times_H V \downarrow G/H$ for some finite dimensional G-representation V.*■

Default convention. Unless stated otherwise, all manifolds are henceforth assumed to be finite dimensional.

For our purposes it is important to have an analog of Theorem 1.1 for smooth G-manifolds with boundary. As in the nonequivariant category, the most important property of bounded smooth G-manifolds is the existence of *collar neighborhoods* for the boundary.

Proposition 1.3. *Let M be a smooth G-manifold with boundary ∂M. Then ∂M has a neighborhood that is equivariantly diffeomorphic to $\partial M \times [0,1)$ – with trivial action on the second coordinate – such that ∂M corresponds to $\partial M \times \{0\}$. If c_j are two smooth equivariant collar neighborhoods (where $j = 0,1$), then $c_0|\partial M \times [0,\frac{1}{2}]$ and $c_1|\partial M \times [0,\frac{1}{2}]$ are ambient isotopic.*

This result follows from the same sort of argument that appears in M. Hirsch's differential topology textbook [Hi, Thm. 4.6.1, pp. 113–114].■

As noted in [Hi, Ch. 4, Sec. 6], the analog of the Tubular Neighborhood Theorem for bounded manifolds requires a suitably refined notion of embedding for manifolds with boundary. Specifically, one needs to work with *neat embeddings* of $(M, \partial M)$ in $(N, \partial N)$ such that boundary goes to boundary and interior to interior (*i.e.*, a proper embedding of manifolds with boundary) and some neighborhood of ∂M in M meets ∂N orthogonally along the boundary (say with respect to some collar neighborhoods). We then have the following equivariant analog of the results for bounded manifolds in [Hi, Thms. 4.6.3 and 4.6.5, pp. 114–116].

Theorem 1.4. *(i) Let $(M, \partial M)$ and $(N, \partial N)$ be smooth (finite dimensional) G-manifolds with boundary, and let $f : (M, \partial M) \to (N, \partial N)$ be a smooth equivariant embedding satisfying the neatness condition described above (with respect to some invariant collar neighborhoods). Then there is an equivariant vector bundle $E \downarrow M$ and a neat smooth embedding $F : E \to N$ whose restriction to the zero section is essentially f.*

(ii) Suppose that $f_j : E_j \to N$ (where $j = 0,1$) satisfy (i), and suppose we are given invariant riemannian metrics on E_j with unit disk bundles $D(E_j)$. Then there is a metric preserving G-vector bundle isomorphism $\varphi : E_0 \to E_1$ covering the identity on M and a smooth equivariant ambient isotopy $H : N \times [0,1] \to N$ such that $H_1 \circ f_o = f_1 \circ \varphi$.■

The equivariant Tubular Neighborhood Theorems imply that certain invariant subsets of a smooth G-manifold are also smoothly embedded submanifolds or unions of submanifolds (where the dimension may vary from one component to the next). For example, this holds for the fixed point set M^G and the constant orbit type set $M_{(H)}$. Since M is the pairwise disjoint union of these subsets $M_{(H)}$, where (H) runs through the conjugacy classes of subgroups of G, and the induced G-action on $M_{(H)}$ is completely determined by fiber bundle considerations, we therefore have a decomposition of M into G-invariant pieces that can be studied effectively by standard topological methods. In order to understand things more thoroughly it is necessary to determine how these pieces fit together; a fairly complete account of this can be found in work of M. Davis [Dav]. As indicated in [DoS2, Sec. II.4], the process can be viewed as a special case of the Thom-Mather theory of smoothly stratified sets.

Convention on abuse of language. If we are given an invariant closed subset $N \subset M$ of a smooth G-manifold M such that N is a disjoint union of smoothly embedded submanifolds N_j, and the maps $\varphi_j : E_j \to N_j$ define pairwise disjoint tubular (*i.e.*, vector bundle) neighborhoods as in the preceding results, we shall often say that the disjoint union

$$\coprod \varphi_j : \coprod E_j \to \coprod N_j \ (\approx N)$$

is an invariant tubular neighborhood of N in M; if α is the disjoint union of the associated vector bundles α_j (whose fibers may have different dimensions!), then $D(\alpha)$ and $S(\alpha)$ will denote the unions of the associated disk and sphere bundles $\coprod D(\alpha_j)$ and $\coprod S(\alpha_j)$ respectively.

2. Equivariant homotopy theory

One of the main themes in algebraic topology is the use of cohomology groups to analyze the homotopy classes of maps from one space to another. The classical approach to this general question involves *obstruction theory* (*cf.* [Wh1]; for historical background see [EiL] and [Wh2]). Although several new and powerful techniques for studying homotopy classes have emerged over the past four decades, in many cases the obstruction-theoretic approach is still the most useful or illuminating. The applicability of obstruction theory to equivariant homotopy was already understood by the mid nineteen fifties (*cf.* [Dg]), especially when the group action is FREE (*i.e.*, all isotropy subgroups are trivial). Systematic investigations of equivariant homotopy theory began in the nineteen sixties. A historical summary appears in the first part of [Sc10, Sec. 1]; we shall merely sketch the basic mathematical points here.

In principle, classical obstruction theory yields an algebraic setting for describing homotopy classes of maps provided one has the following data:

- (i) Cellular decompositions for the source spaces.
- (ii) Suitably defined cohomology groups for the source spaces, in general with twisted coefficients, that can be computed from small cochain complexes reflecting the cell structure. The coefficients are determined by the homotopy type of the target space and some information involving fundamental groups.

In [Bre2] G. Bredon described general and usable equivariant analogs of (i) and (ii). The objects in (i) were forerunners of *$G - CW$ complexes* (*e.g.*, see [tD2, Sec. II.1]), and the cohomology functors in (ii) evolved into the so-called *Bredon (equivariant) cohomology groups*; the latter were first defined by Bredon [Bre2], and a close relation of these to ordinary singular cohomology follows from an alternative definition due to Th. Bröcker [Brö] (in this connection also see [Il1]). Equivariant coefficient systems in

Bredon cohomology are more complicated than ordinary coefficient groups and require families of abelian groups indexed by the components of each fixed point set Y^H (where H is a subgroup of G), but one still has small cochain complexes for computing the Bredon cohomology groups ${}_{BR}H^*(X, \mathcal{A})$.

The following results reflect the strong analogies between ordinary and equivariant homotopy theory.

Property 2.0. (G-homotopy extension property) *Let G be a compact Lie group, let X be a $G-CW$ complex, let $A \subset X$ be a subcomplex, let $f : X \to Y$ be a continuous equivariant map into some G-space Y, and let $h_t : A \to Y$ be an equivariant homotopy such that $h_0 = f|A$. Then h_t extends to an equivariant homotopy $H_t : X \to Y$ such that $H_0 = f$.*■

Property 2.1. (Extension theorem) *Let G, A, X, Y be as above, and let $h : A \to Y$ be continuous and equivariant. Then f extends to a continuous equivariant map from A to Y if a sequence of obstructions valued in the i-dimensional Bredon cohomology groups of (X, A), with coefficients in the $(i-1)$-dimensional homotopy groups of fixed set components of Y, is trivial.*■

Property 2.2. (Classification theorem) *Let G, X, Y be as above, and let $h_j : X \to Y$ be continuous and equivariant for $j = 0, 1$. Then h_0 and h_1 are equivariantly homotopic if a sequence of obstructions valued in the i-dimensional Bredon cohomology groups of X, with coefficients in the i-dimensional homotopy groups of fixed set components of Y, is trivial.*■

Note the difference in coefficient groups between 2.1 and 2.2.

Property 2.3. (Barratt-Federer spectral sequence) *Let G, X, Y be as above, and also assume X is finite-dimensional. Modulo some mildly exceptional behavior in dimensions 1 and 0 there is a spectral sequence such that*

$$E^2_{i,j} = {}_{BR}H^{-i}_G(X; {}_G\pi_j(Y))$$

where ${}_G\pi_j(Y)$ is a coefficient system derived from the homotopy groups of components of fixed point sets Y^H (where H runs through subgroups of G) and $E^\infty_{i,j}$ gives a series for $\pi_{i+j}(F_G(X, Y))$, where $F_G(X, Y)$ is the space of G-equivariant continuous maps from X to Y with the compact open topology.■

Precise statements of 2.1–2.3 appear in [DuS, Sec. 1]. Unfortunately, the preceding results are computationally less useful than their nonequivariant counterparts because Bredon cohomology groups are far more difficult to compute than ordinary singular cohomology. One the other hand, results from [Sc2] yield alternatives to 2.1–2.3 that involve ordinary cohomology groups; not surprisingly, there is a price to pay for this – roughly speaking, each Bredon cohomology group is replaced by a finite list of ordinary cohomology groups. In particular, the analog of Property 2.3 given by [Sc2] is the Barratt-Federer/Fáry spectral sequence in [DuS, Thm. 1.5].

One can in fact extend nearly all the basic concepts and results in algebraic topology to the category of $G - CW$ complexes, including an equivariant version of the Whitehead theorem for recognizing equivariant homotopy equivalences (see [Bre2]), Postnikov decompositions, and localization at a subring of the rationals [MMT]. This can be interpreted as a special case of a more general observation (see [DF, p. 131]; in this connection also see [May] and [DuS, statement (1.8)]). There is also a corresponding analog of equivariant stable homotopy theory; references for the latter include tom Dieck's book on the Burnside ring [tD1], an exhaustive account by G. Lewis, J. P. May, and M. Steinberger [LMS], and a more recent survey article by G. Carlsson [Car].

Application to smooth G-manifolds

Of course, if we wish to apply the preceding machinery to compact smooth G-manifolds it is necessary to find appropriate interpretations of the latter as $G - CW$ complexes. There are two ways of doing this. Results of A. Wassermann [Wa] yield a version of Morse Theory for G-invariant smooth functions on smooth G-manifolds, and from this it is elementary to show that smooth G-manifolds have the G-homotopy type of finite-dimensional $G - CW$ complexes; in fact, for compact smooth G-manifolds one can choose the $G - CW$ complexes to be finite (also see [Ko] for a self-contained account of these results). On the other hand, for many purposes it is more useful to invoke a stronger result due to S. Illman [Il3]: *Every smooth G-manifold (G finite) has a G-equivariant triangulation that is smooth in the sense of J. H. C. Whitehead* (see [Mun] for the corresponding result in ordinary differential topology).

3. Isovariant homotopy theory

Recall that a finite group action on a reasonable (say paracompact Hausdorff) space X is *free* if the isotropy subgroups at all points are trivial. Basic results in topology state that a free action of a finite group G on such a space is determined by its orbit space X/G and a homotopy class of maps from X/G to a universal base space $BG \simeq K(G, 1)$. In [Pa] R. S. Palais extended this to a classification theory for G-spaces that are not necessarily free; this involves a special class of maps that Palais called ISOVARIANT. Specifically, a G-equivariant map $f : X \to Y$ is said to be isovariant if for all $x \in X$ the isotropy subgroups satisfy $G_x = G_{f(x)}$; an equivariant map f automatically satisfies $G_x \subset G_{f(x)}$, but it is easy to see that equality often does not hold. During the nineteen sixties and seventies isovariant maps were also discussed in connection with various topological problems, and questions about isovariant homotopy arose naturally; much of this is discussed at various points of [DuS]. The usefulness of isovariant homotopy for classifying G-manifolds became explicit in the work of Browder and Quinn [BQ] on stratified surgery theory. Various *ad hoc* techniques for dealing with isovariant homotopy theory gradually emerged, and by the mid nineteen eighties it was clear that one could analyze isovariant homotopy effectively by the basic techniques of algebraic topology. The key idea is expressed in [DuS] as follows:

Isovariant homotopy for smooth G-manifolds is essentially equivalent to equivariant homotopy theory for suitable diagrams of smooth G-manifolds.

We shall explain this statement in several steps, beginning with a discussion of diagram categories. If $\mathbb{D}$ is a small category and $\mathcal{S}$ is some category of topological spaces, then a *$\mathbb{D}$-diagram with values in $\mathcal{S}$* is merely a covariant functor $\mathbb{D} \to \mathcal{S}$ and a *morphism of $\mathbb{D}$-diagrams* is a natural transformation of functors. Results of W. Dwyer and D. M. Kan [DK] show that one can extend much of classical homotopy theory and obstruction theory to categories of $\mathbb{D}$-diagrams with values in the category of CW complexes, and subsequent generalizations of E. Dror Farjoun [DF] yield corresponding results for suitable categories of $\mathbb{D}$-diagrams of G-CW complexes. More precisely, we need to choose a small category associated to some finite partially ordered set $\mathbb{P}$ that is closed under taking greatest lower bounds, and we also must restrict attention to $\mathbb{P}$-diagrams satisfying a few simple admissibility conditions (see [DuS, conditions $(i) - (iii)$ preceding (1.6)]. The basic results of equivariant algebraic topology extend directly to such equivariant diagrams and are summarized in [DuS, (1.6)–(1.10)].

Before discussing the types of diagrams we need, a technical remark is in order. Strictly speaking, the results of [DuS] only deal with a restricted class of group actions, but this class includes all actions of the cyclic groups $\mathbb{Z}_{p^r}$, where p is a prime, and all fixed point free actions of $\mathbb{Z}_{pq}$, where p and q are distinct primes. Extensions to more general actions will appear in a sequel to [DuS]. We shall first discuss the types of diagrams needed in an important special case and then outline an inductive extension to the general case.

The semifree case

The conclusions of [DuS] apply directly to actions that are *semifree* (= free off the fixed point set), so we shall describe the appropriate diagrams of G-manifolds in this important special case. If M is a compact smooth G-manifold, let M^G denote its fixed point set as usual, let α_M be the equivariant normal bundle of M^G in M, let $D(\alpha_M)$ and $S(\alpha_M)$ be the associated unit disk and sphere bundles respectively, and let $M \overset{*}{-} M^G$ denote the closure of $M - D(\alpha_M)$. Then the relevant diagram of spaces, which is denoted by $B(\mathbf{QF}_M)$ in [DuS], is the following partially ordered set:

$$\begin{tikzcd}
S(\alpha_M) \arrow[r] \arrow[d] & D(\alpha_M) \arrow[d] & M^G \arrow[l] \\
M \overset{*}{-} M^G \arrow[r] & M &
\end{tikzcd}$$

In the preceding special case the main result of [DuS, Sec. 4] can be stated as follows:

Theorem 3.1. (compare [DuS, Thm. 4.5]) *Let G be a finite group, let X and Y be compact smooth semifree G-manifolds, and let $B(\mathbf{QF}_X)$ and $B(\mathbf{QF}_Y)$ be the diagrams* *described above. Then there is a canonical isomorphism:*

$$\begin{bmatrix} G - \textit{equivariant} \\ \textit{homotopy classes of} \\ \textit{continuous equivariant} \\ \textit{diagram morphisms} \\ B(\mathbf{QF}_X) \to B(\mathbf{QF}_Y) \end{bmatrix} \cong \begin{bmatrix} G - \textit{isovariant} \\ \textit{homotopy classes of} \\ \textit{continuous isovariant} \\ \textit{maps of spaces} \\ X \longrightarrow Y \end{bmatrix}$$

■

The proof of this result is a fairly straightforward application of standard compactness considerations and the covering homotopy property for fiber bundles; details appear in [DuS, Sects. 2–4].

Results of Dwyer and Kan [DK] and E. Dror Farjoun [DF] show that one can "do algebraic topology" in the diagram category corresponding to the left hand side of the correspondence displayed in the theorem. In particular, the associated obstruction theories in the diagram categories (*cf.* [DuS, Sec. 1]) are essentially obstruction theories for isovariant homotopy theory. Furthermore, as noted in [DuS, (1.8)–(1.10)] one also has Postnikov decompositions, localization at subrings of the rationals, and spectral sequences for the homotopy groups of isovariant function spaces analogous to those of J. Møller and the author [Mø, Sc2]. At the end of [DuS, Sec. 4] these ideas are applied to prove isovariant analogs of the Whitehead Theorems for recognizing (ordinary or equivariant) homotopy equivalences. Here is one special case:

Theorem 3.2. *Let G be a finite group, let X and Y be compact,* UNBOUNDED, *semifree smooth G-manifolds, let $f : X \to Y$ be an isovariant map, and also assume that all fixed point sets X^H and Y^H are orientable. Then f is an isovariant homotopy equivalence if and only if for each isotropy subgroup H the map f induces homotopy equivalences from X^H to Y^H and from $X^H - \mathrm{Sing}(X^H)$ to $Y^H - \mathrm{Sing}(Y^H)$.*

Notation and remark. The *singular set* of M^H, denoted by $\mathrm{Sing}(M^H)$, is the set of all points in M^H whose isotropy subgroups PROPERLY contain H. By definition an isovariant map $X \to Y$ automatically takes $X^H - \mathrm{Sing}(X^H)$ to $Y^H - \mathrm{Sing}(Y^H)$.■

Related isovariant analogs of the Whitehead Theorem also appear in [DuS, Thm. 4.10 and Cor. 4.11]).

Actions with more general isotropy structures

The preceding results extend to arbitrary compact differentiable G-manifolds (where G is finite as usual). In principle the argument combines the techniques used in the semifree case with an inductive framework that we shall describe below; details will appear in a sequel to [DuS].

Definition. If G is a compact Lie group acting with finitely many orbit types (*e.g.*, if G is finite), then the *isotropy depth* of the action is the largest nonnegative integer d such that one has a sequence of isotropy subgroups

$$H_0 \subsetneqq H_1 \subsetneqq \cdots \subsetneqq H_d.$$

In particular, an action with only one isotropy type (*e.g.*, a free action) has isotropy depth zero, and a semifree action has isotropy depth one; the results of [DuS] include extensions of 3.1 and 3.2 to actions with isotropy depth one. Aside from semifree actions, the class with isotropy depth one also includes fixed point free, effective actions of $\mathbb{Z}_{pq}$, where p and q are distinct primes (since the group in question contains exactly two nontrivial proper subgroups, and neither contains the other).

One of the main steps in extending Theorem 3.1 is the description of an analog to $B(\mathbf{QF}_M)$ for actions of isotropy depth greater than one. Suppose we are given a compact smooth G-manifold M; we begin by choosing a G-invariant metric on M and using it to construct a system of invariant tubular neighborhoods of the fixed point sets M^H where H runs through the isotropy subgroups of the action. If we are given an action of isotropy depth zero, then the finiteness of G implies that M splits into a disjoint union of codimension zero submanifolds M^K where K runs through the conjugates of H, and the *isovariance diagram* $\mathbf{IsD}(M)$ is given by the partially ordered set whose members are the sets M^K. One then assumes that $\mathbf{IsD}(Y)$ has been defined for smooth actions of isotropy depth less than d such that $\mathbf{IsD}(Y)$ satisfies some appropriate conditions (*e.g.*, it contains all fixed point sets Y^H where H runs through the isotropy subgroups of the action).

Consider next a smooth action on M with isotropy depth equal to d, and let $F_d \subset M$ be the set of points whose isotropy subgroups are maximal. It follows that F_d is an invariant union of smooth submanifolds (neatly embedded if $\partial M \neq \varnothing$). Let $S\langle d\rangle$ and $D\langle d\rangle$ be the unit sphere and disk bundles for an invariant tubular neighborhood of F_d (with the abuse of language convention at the end of Section 1), and let $C\langle d\rangle := M \stackrel{*}{-} F_d$ be the closure of the complement of $D\langle d\rangle$. The diagram $\mathbf{IsD}(M)$ of G-invariant subspaces of M is then constructed in pieces as follows: Since $S\langle d\rangle$ and $C\langle d\rangle$ have isotropy depth $\leq d-1$ we can take $\mathbf{IsD}(M; M \stackrel{*}{-} F_d)$ to be the union of $\mathbf{IsD}(S\langle d\rangle)$ and $\mathbf{IsD}(C\langle d\rangle)$. The set F_d is a pairwise disjoint union of the subsets M^H where H runs through the maximal isotropy subgroups of the action, so define $\mathbf{IsD}(M; F_d)$ to be the family of all such sets $M_{(K)}$. Next, define $\mathbf{IsD}(M; D\langle d\rangle)$ to be the family of all subsets $D\langle d\rangle^H$; the partially ordered set $\mathbf{IsD}(M)$ is then defined to be a family of subsets generated from $\mathbf{IsD}(M; M \stackrel{*}{-} F_d)$ and $\mathbf{IsD}(M; D\langle d\rangle)$ by adjoining suitable unions $P \cup Q$ where $P \in \mathbf{IsD}(M; M \stackrel{*}{-} F_d)$ and $Q \in \mathbf{IsD}(C\langle d\rangle)$.

With these definitions of isovariance diagrams Theorem 3.1 generalizes to a similar canonical isomorphism:

$$\begin{bmatrix} G-\text{equivariant} \\ \text{homotopy classes of} \\ \text{continuous equivariant} \\ \text{diagram morphisms} \\ \mathbf{IsD}(M) \to \mathbf{IsD}(N) \end{bmatrix} \cong \begin{bmatrix} G-\text{isovariant} \\ \text{homotopy classes of} \\ \text{continuous isovariant} \\ \text{maps of spaces} \\ M \longrightarrow N \end{bmatrix}$$

where M and N are compact smooth G-manifolds. Roughly speaking, this can be done inductively by first adjusting the map on $D\langle d\rangle$ and then using an inductive hypothesis to adjust the map on $M \stackrel{*}{-} F_d$ leaving $S\langle d\rangle$ fixed.

4. Isovariance versus equivariance

One basic question in equivariant homotopy theory is to determine whether a continuous map between two G-spaces is homotopic to an equivariant map (*cf.* [LW], [MP], and [La]). In this section we are interested in a corresponding question for isovariant homotopy theory; namely, whether a continuous equivariant map is equivariantly homotopic to an isovariant map. Several special cases have been studied in by geometric methods. One example will be discussed in Section II.2; other results include Illman's work on isovariance and equivariant general position [Il4] and results with applications to embedding manifolds in the metastable range (see Haefliger [Hae, Prop. 2, p. 245/06] and Harris [Har, Prop. 13, p. 24]). Each approach seems to yield insights not apparent from the others.

Default hypothesis: To keep the notation relatively uncomplicated we shall only consider semifree actions on connected manifolds whose fixed point sets are connected.

Similar results also hold for more general actions, but the terminology quickly becomes far more complicated.

We begin with an elementary observation.

Proposition 4.1. *Let M and N be as above, let α_M and α_N be the equivariant normal bundles of the fixed point sets, let $D(-)$ denote an associated unit disk bundle, and let $f : M \to N$ be an equivariant map. Then f is equivariantly homotopic to a map h such that $h(D(\alpha_M)) \subset D(\alpha_N)$. Furthermore, if h_0 and h_1 are two equivariant maps with this property and Φ_t is an equivariant homotopy from h_0 to h_1, then there is an equivariant homotopy Ψ_t from h_0 to h_1 such that $\Psi_t(D(\alpha_M)) \subset D(\alpha_N)$ for all $t \in [0, 1]$.*

This is essentially a special case of [DuS, Prop. 5.1].■

Theorem 3.1 and Proposition 4.1 combine to yield the following lifting condition for finding an isovariant map in an equivariant homotopy class:

Theorem 4.2. *Let M and N be as above, and let $f : M \to N$ be a continuous equivariant map satisfying the conditions of Proposition 4.1. Then f is equivariantly homotopic to an isovariant map if the associated maps $S(\alpha_M) \to D(\alpha_N)$ and $M \mathbin{\underline{*}} M^G \to N$ lift – equivariantly and compatibly – to $S(\alpha_N)$ and $N \mathbin{\underline{*}} N^G$ respectively.*

This is a special case of [DuS, Thm. 5.3].■

One can obtain cohomological isovariance obstructions by combining Theorem 4.2 with obstruction theory in several different ways. Typical results along this line are given in [DuS, Thms. 5.4–5.5]. The remarks at the end of [DuS, Sec. 5] discuss variants and special cases of these theorems.

Comparative computations

Another way of studying the difference between equivariant and isovariant homotopy is to compare the homotopy groups of an equivariant function space $F_G(X, Y)$ with those of the corresponding isovariant function space $IF_G(X, Y)$, say if X and Y satisfy the basic assumptions of [DuS]. For example, one could use the appropriate Barratt-Federer spectral sequences from [Sc2] and [DuS] in each case, and as a first step it might be worthwhile to take tensor products with the rationals and obtain information about rational homotopy groups.

The final section of [DuS] provides a comparison along these lines when $G = \mathbb{Z}_p$ and $X = Y = S(V)$, where p and q are distinct odd primes and V is a linear G-representation such that V contains at least two free irreducible summands. If $F_G(V)$ is the space of equivariant self maps of $S(V)$ then the Barratt-Federer spectral sequence of [Sc2] shows that $\pi_k(F_G(V)) \otimes \mathbb{Q} = 0$ for all but finitely many k. On the other hand, if $IF_G(V)$ is the space of isovariant self maps of $S(V)$ then the results of [DuS, Sec. 6] show that each rational homotopy group $\pi_k(IF_G(V)) \otimes \mathbb{Q}$ is finite dimensional, but the sequence of dimensions d_k satisfies $\mathrm{limsup}_{k\to\infty} d_k/k^n = \infty$ for every positive integer n. There is an explanation for this difference in terms of the Barratt-Federer spectral sequences: In the equivariant setting one has cohomology groups with coefficients in the homotopy groups of spheres, and rationally these vanish in all but at most two dimensions. On the other hand, in the isovariant setting one has cohomology groups with coefficients in the homotopy groups of wedges of spheres, and the ranks of these groups tend to grow exponentially as the dimension increases.