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PART II

ISOVARIANT HOMOTOPY, CLASSIFICATION PROBLEMS, AND GENERAL POSITION

Background material

As in Part I we assume basic concepts in algebraic topology and transformation groups in Bredon, tom Dieck, Dovermann-Schultz, Milnor-Stasheff, and Spanier. Beyond this, we shall frequently mention the concept of *simple homotopy equivalence* as presented in Milnor's article [Mln2] or M. Cohen's book [Co], and we shall also use data from the *Sullivan-Wall surgery exact sequence*. The standard reference for the latter is Section 10 of Wall's book, *Surgery on Compact Manifolds* (:= [Wl]). Most of the material of immediate interest in this article is summarized in [Brw2], and in particular the Sullivan-Wall sequence is presented in [Brw2, p. 29] with a minimum of technical diversions. However, we shall use notation that differs slightly from that of [Wl] and [Brw2], mainly because we shall also need to consider variants of the structure sets described in those references.

Both [Wl] and [Brw2] deal with structure sets $\mathcal{S}^{Diff}(X)$ of simple homotopy structures on a simple Poincaré complex X and with certain algebraically defined surgery obstruction groups $L_k(\pi)$. As noted in Section 17 of [Wl] one can define analogous homotopy structure sets, surgery obstruction groups, and exact sequences for homotopy structures on a Poincaré complex; for this theory, equivalent structures are h-cobordant rather than diffeomorphic. To distinguish between the two structure set theories in the smooth category we shall denote the simple homotopy objects by $\mathcal{S}^{s,Diff}$ and $L^s_k(\pi, w)$, and we shall denote the ordinary homotopy objects by $\mathcal{S}^{h,Diff}$ and $L^h_k(\pi, w)$; here w refers to the homomorphism $\pi \to \mathbb{Z}_2$ defined by the first Stiefel-Whitney class. We shall also use somewhat different notation for the bordism classes of degree one normal maps that are called $N^{Diff}(X)$ in [Brw2]. Standard results in algebraic and geometric topology imply that $N^{Diff}(X)$ is isomorphic to the set of homotopy classes $[X, F/O]$, where F/O is the space classifying stable fiber homotopy trivializations of stable vector bundles of X; this space is considered in [Brw3, Sec. II.4] where it is called G/O (we use F/O rather than G/O because the two names are essentially used interchangeably in the literature and G will frequently denote a finite group in the discussion that follows). A full discussion of the isomorphism $N^{Diff}(X) \approx [X, F/O]$ appears in [Brw3, Thm. II.4.4, pp. 46–49]. This description of $N^{Diff}(X)$ is useful because there is an exact sequence of abelian groups

$$\cdots \to \widetilde{KO}(\Sigma X) \to \{\Sigma X, S^0\} \to [X, F/O] \to \widetilde{KO}(X) \to \widetilde{KSph}(X)$$

where $\widetilde{KO}$ denotes reduced real K-theory, $\{-, S^0\}$ denotes stable cohomotopy, and $\widetilde{KSph}$ denotes the analog of reduced K-theory for stable spherical fiber spaces.

Finally, we shall also note the existence of *relative structure sets* $\mathcal{S}^{c,Diff}(X, \partial X)$ for Poincaré complexes with formal boundaries; here $c = s$ or h. The basic idea is to take simple homotopy (resp. homotopy) equivalences $(M, \partial M) \to (X, \partial X)$ such that the map of boundaries is a diffeomorphism. There is an extension of the Sullivan-Wall exact sequence to such objects modulo some adjustments; the Wall groups are simply those for the fundamental group and first Stiefel-Whitney class of X, but the normal bordism set $N^{Diff}(X, \partial X)$ in this case is equivalent to $[X \cup \mathrm{Cone}(\partial X), F/O]$.

Recent work of M. Dawson [Daw] includes an independent proof of the main results in Section 1 and applications to smooth variants of the Cappell-Weinberger *replacement theorems* (*e.g.*, see Theorem 5.1 below for a statement of one such result in the locally linear topological or PL categories). Additional remarks on this work appear in Section 5.

1. Isovariant homotopy structures

The main ideas of surgery theory began to emerge in the nineteen fifties, and they became well established with the work of M. Kervaire and J. Milnor [KM] on classifying smooth manifolds that are homotopy equivalent to spheres (*i.e.*, homotopy spheres). Subsequent work of W. Browder and S. P. Novikov yielded far reaching extensions of [KM] to existence and classification questions for simply connected manifolds of a fixed homotopy type, and still further work of Wall extended the theory to manifolds with arbitrary fundamental groups [WL].

It soon became clear that surgery theory also yielded valuable information on existence and classification questions for group actions on manifolds (*cf.* [Brw1]). In particular, many striking applications to free differentiable group actions on spheres were made during the nineteen sixties (*e.g.*, see [HH], [Hs], [LdM]). Systematic efforts to study nonfree actions also began in the nineteen sixties with work of Browder and Petrie [BP] and Rothenberg and Sondow [RSo] on classifying smooth G-actions that are semifree and homotopically linear – in other words, both M and M^G are closed manifolds that are homotopy equivalent to spheres (see also [Brw1] and [Sc3]). Actions of this type can be viewed as smooth G-manifolds that are equivariantly homotopy equivalent to a linear G sphere $S(V)$ given by the unit sphere in some orthogonal, semifree representation of G on a finite dimensional real inner product space V. In analogy with the Browder-Novikov-Wall extension of [KM] to arbitrary closed manifolds, it is natural to search for an extension of the Browder-Petrie and Rothenberg-Sondow work to more general G-manifolds.

A major step in this direction was due to W. Browder, who presented his ideas in a lecture at a conference in 1971 (see p. *vii* in the book containing [MnY1]). This work was later extended by F. Quinn and published jointly in [BQ]. The basic idea was to consider manifolds that are *isovariantly* homotopy equivalent to a given model such that the isovariant equivalence satisfies a *transverse linearity* condition. For the sake of simplicity we shall only describe this for semifree actions. In such cases the isovariant homotopy equivalence $f : M \to N$ is supposed to be a map of triads from $(M; M \overset{*}{-} M^G, D(\alpha_M))$ to $(N; N \overset{*}{-} N^G, D(\alpha_N))$, where α_Y refers to the (equivariant) normal bundle of Y^G in Y as in Part I, and the induced map from $D(\alpha_M)$ to $D(\alpha_N)$ is assumed to be (orthogonally) linear and fiber preserving. If one specializes this theory to G-manifolds modeled by linear G-spheres, one obtains a theory that maps naturally into the Browder-Petrie and Rothenberg-Sondow theories and includes many infinite families of examples from (both of) the latter.

One of the most important properties of the Browder-Quinn setting is the existence of a surgery exact sequence that is formally parallel to the Sullivan-Wall sequence (*cf.* [BQ, Thm. 2.2, p. 29]):

$$\cdots \to L^{c,BQ}_{n+1}(X) \to \mathcal{S}^{c,BQ}_G(X) \to [X/G, F/O] \to L^{c,BQ}_n(X)$$

In this sequence X is a closed smooth G-manifolds, the symbol c denotes either s (for equivariant simple homotopy; *cf.* Illman [Il2] or Rothenberg [Ro]) or h (for ordinary equivariant homotopy), and the groups $L^{c,BQ}_*(X)$ are the *Browder-Quinn surgery obstruction groups* as defined and studied in [BQ] and [DoS2, Sec. 2]. Although these groups are written in terms of X, they are in fact determined by weaker data that is summarized in the *geometric reference* R_X of W. Lück and I. Madsen [LüMa, Definitions (2.3) and (3.1), pp. 512 and 516]. As noted in [BQ] and [DoS2], certain natural exact couples determine spectral sequences converging to the groups $L^{c,BQ}_*(X)$ such that the initial terms are ordinary Wall groups L^c_*, and therefore one can view the terms in the Browder-Quinn surgery sequence as computable up to determination of the homotopy groups of F/O and the appropriate Wall groups. In analogy with the Sullivan-Wall sequence, there are also relative versions of the Browder-Quinn sequence involving structure sets $\mathcal{S}^{c,BQ}(X, \partial X)$ represented by transverse linear G-homotopy equivalences $(M, \partial M) \to (X, \partial X)$, with $c = h$ or s as usual, such that ∂M maps to ∂X by a diffeomorphism.

The basic aim of isovariant surgery theory is to provide a setting that is broad enough to include both the Browder-Quinn theory and the work of Browder-Petrie and Rothenberg-Sondow, but is also more or less computable, at least in some cases beyond those of [BP], [RSo], and [BQ].

We shall begin by relating [BQ] to [BP] and [RSo] in the case of homotopically linear semifree group actions on spheres. The tangential representation at a fixed point will be assumed to have the form $V \approx \mathbb{R}^k \oplus \alpha$, where the representation α has no trivial summands (hence G acts freely on the unit sphere $S(\alpha)$). Deviating slightly from the the notation of [Sc3], let $CS_k(G, \alpha)$ be the set of equivariantly oriented h-cobordism classes of homotopically linear semifree group actions as described above, where the tangent space at a fixed point is G-isomorphic to the representation V; as noted in [RSo] these sets have natural abelian group structures if the dimension of the fixed point set is at least 2. There are also canonical abelian group structures on the relative Browder-Quinn structure sets

$$\mathcal{S}_G^{h,BQ}(\,D(V), S(V)\,),$$

and there is a natural forgetful map $\mathcal{S}_G^{h,BQ}(\,D(V), S(V)\,) \to CS_k(G,\alpha)$, given by gluing a copy of $D(V)$ to the boundary, that is additive. As in Part I let $F_G(\alpha)$ be the space of equivariant self maps of the unit sphere $S(\alpha)$. The orthogonal centralizer of α is a compact subgroup of the topological monoid $F_G(\alpha)$ and will be denoted by $C_G(\alpha)$; it follows that the quotient space construction defines a principal bundle

$$C_G(\alpha) \subset F_G(\alpha) \to F_G(\alpha)/C_G(\alpha).$$

One can then define a *knot invariant* homomorphism

$$\omega : CS_k(G,\alpha) \to \pi_k(F_G(\alpha)/C_G(\alpha)\,)$$

as in [Sc3, top of p. 311] or [Sc4, Sec. 2].

Proposition 1.1. *In the notation of the preceding paragraph, there is a long exact sequence of the following form:*

$$\cdots \pi_{k+1}(F_G(\alpha)/C_G(\alpha)) \to \mathcal{S}_G^{h,BQ}(D(V), S(V)) \to CS_k(G,\alpha) \to \pi_k(F_G(\alpha)/C_G(\alpha))$$

In other words, the forgetful map from the Browder-Quinn groups to the $CS_k(G,\alpha)$ groups is the "homotopy fiber of the knot invariant."■

The next step in the comparison is to note that each element of $CS_k(G,\alpha)$ is canonically isovariantly homotopy equivalent to $S(V \oplus \mathbb{R}) \approx D_+(V) \cup_\partial D_-(V)$. This suggests that the groups $CS(G,V)$ should be viewed as structure sets for relative G-isovariant homotopy structures on $(D(V), D(V))$. In fact, it is possible to describe structure set theories for arbitrary smooth semifree G-manifolds in the spirit of [BP] and [RSo]. As before there are two versions $\mathbf{I}\mathcal{S}_G^s$ and $\mathbf{I}\mathcal{S}_G^h$ for isovariant simple homotopy and ordinary isovariant homotopy equivalences respectively. There are also relative versions of these structure sets for isovariant homotopy structures that are diffeomorphisms on the boundary.

The exact sequence of [Sc3, (1.1), p. 311] plays an important role in many studies of the groups $CS_k(G,\alpha)$, and therefore one would like to have analogs of this for the structure sets $\mathbf{I}\mathcal{S}_G^c(M)$. In order to do this it is necessary to generalize the knot invariant, and this in turn requires a suitable analog of the homotopy group $\pi_k(F_G(\alpha)/C_G(\alpha)\,)$. The approach below is an adaptation of ideas from [Sc9, Secs. 2–3].

Definition. Let X be a G-space, let $A \subset X$ be G-invariant, and let ξ be a G-vector bundle over X. A *G-isovariant fiber homotopy linearization* of ξ is a pair (ξ, h) consisting of a G-vector bundle $\omega \downarrow X$ and a G-isovariant fiber homotopy equivalence $h : S(\omega) \to S(\xi)$ that is an orthogonal isomorphism over A. The set $F/O_{G,iso}(\xi \text{ rel } A)$ is the set of Such objects modulo the equivalence relation generated by fiber preserving orthogonal vector bundle isomorphisms $S(\omega') \to S(\omega)$ and isovariant homotopies $H_t : S(\omega) \to S(\xi)$ that are orthogonal over A.

Note. If G acts semifreely on M with $X = M^G$ and $\xi = \alpha_M$, then G acts freely on $S(\xi)$ and the isovariance condition merely requires that G act freely on $S(\omega)$.

If $f : M \to N$ is an isovariant homotopy equivalence of semifree smooth G-manifolds, then one can define a generalized knot invariant of f in $F/O_{G,iso}(\alpha_N)$ as follows: By the results of Section I.3 we can deform f isovariantly so that f maps $S(\alpha_M)$ to $S(\alpha_N)$, and the construction yields a unique isovariant homotopy class of maps $S(\alpha_M) \to S(\alpha_N)$; this map can be further deformed, again uniquely up to isovariant homotopy, to a map f' such that the following diagram commutes:

$$\begin{array}{ccc} S(\alpha_M) & \xrightarrow{f'} & S(\alpha_N) \\ \downarrow & & \downarrow \\ M^G & \xrightarrow{f^G} & N^G \end{array}$$

Since f^G is a homotopy equivalence there is a unique G-vector bundle β (up to isomorphism) such that $\alpha_M \cong \{f^G\}^*\beta$, and it follows that f' factors through an isovariant fiber homotopy equivalence $\eta : S(\beta) \to S(\alpha_M)$; by construction the class of (β, η) in the set $F/O_{G,iso}(\alpha_N)$ is well defined, and this is the generalized knot invariant of f.

A similar construction is valid for relative homotopy structures on a compact smooth semifree G-manifold with boundary, and in this case the knot invariant lies in the relative set $F/O_{G,iso}(\alpha_N \text{ rel } \partial N)$. The following result is a natural extension of Proposition 1.1 to arbitrary smooth semifree G-manifolds:

Theorem 1.2. *If X is a closed smooth semifree G-manifold such that each component of X^G is at least 5-dimensional, then there is an exact sequence of structure sets*

$$\cdots F/O_{G,iso}(\alpha_X \times I \ \mathit{rel}\ X \times \{0,1\}) \to \mathcal{S}^{c,BQ}(X) \to \mathbf{I}\mathcal{S}^c_G(X) \to F/O_{G,iso}(\alpha_X)$$

that extends infinitely to the left. All objects to the left of $\mathcal{S}^{c,BQ}(X)$ are groups, and all maps are compatible with group structures as in the Sullivan-Wall exact sequence.■

Reminder. In the Sullivan-Wall exact sequence the source and target for the surgery obstruction map $[X, F/O] \to L^c_n(\pi, w)$ are abelian groups but the map itself is not additive in general.

The preceding sequence relates $\mathcal{S}^{c,BQ}(X)$ to $\mathbf{I}\mathcal{S}^c_G(X)$. There is also an exact sequence for $\mathbf{I}\mathcal{S}^c_G(X)$ that generalizes the exact sequence for homotopy linear actions in [Sc3, (1.1), p. 311]. Before stating this result we need a notational convention:

> If Y is a compact bounded manifold then $\partial^* : \mathcal{S}^c(Y) \to \mathcal{S}^c(\partial Y)$ is given by restriction to the boundary.

Theorem 1.3. *Let X satisfy the conditions of Theorem 1.2. Then there is an exact sequence of sets*

$$\begin{tikzcd}
\cdots \mathcal{S}^{c,Diff}(X - Int(D(\alpha_X))/G, S(\alpha_X)/G) \arrow[r] & \mathbf{I}\mathcal{S}^{c}_{G}(X) \arrow[d] \\
& \mathcal{S}^{c,Diff}(X^G) \times F/O_{G,iso}(\alpha_X) \arrow[d] \\
& \mathcal{S}^{c,Diff}(S(\alpha_X)/G)/Image\ \partial^*
\end{tikzcd}$$

that extends infinitely to the left. All objects to the left of the raised dots are groups and the corresponding maps are compatible with group structures as in Theorem 1.2.■

As usual, there is a variant of this exact sequence for relative structure sets.

Extensions to more general actions

Theorems 1.2 and 1.3 provide a means for analyzing isovariant structure sets in terms of ordinary structure sets and equivariant/isovariant homotopy theory, provided the group action is semifree. Each of these extends to actions with more complicated orbit structure. In particular, a generalization along the lines of 1.2 was considered in earlier work by the author [Sc13]; the necessary modifications include

(i) an extension of $F/O_{G,iso}(-)$ from vector bundles to the vector bundle systems (known as Π-*bundles* in the papers of Dovermann-Petrie-Rothenberg [DP1–2, DR]) over $\mathrm{Sing}(X)$,

(ii) the definition of a generalized knot invariant for an isovariant homotopy equivalence, taking its value in the set described above.

With this machinery in place, it is a formal exercise to prove that the forgetful map $\mathcal{S}^{BQ,c}(X) \to \mathbf{I}\mathcal{S}^{c}_{G}(X)$ is essentially the homotopy fiber of the knot invariant constructed by (ii).

Theorem 1.3 is essentially a means for analyzing isovariant homotopy structures on X by splitting them into two pieces; namely, pieces over a tubular neighborhood of X^G and pieces over the free G-manifold $X \overset{*}{-} X^G$. There are several ways of extending this to more general actions; we shall only discuss two extreme cases here. The first approach is to split an arbitrary smooth G-manifold into a smooth equivariant regular neighborhood $\mathcal{R}_X$ of the singular set $\mathrm{Sing}(X)$ and the free G-manifold $X \overset{*}{-} \mathrm{Sing}(X)$. This approach was discussed in [Sc6]; we shall not attempt to provide a precise description because it requires a notion of isovariant structure set for the singular set $\mathrm{Sing}(X)$, which is generally not a smooth G-manifold (with a possibly noneffective group action). To describe a complementary approach, we shall assume for the sake of simplicity that all isotropy subgroups are normal (*e.g.*, this happens if G is abelian). Suppose that H is a maximal isotropy subgroup, and let α_H be the equivariant normal bundle of X^H in X. Then one has the following analog of Theorem 1.3:

Theorem 1.4. *Let X satisfy the conditions of the preceding paragraph. Then there is an exact sequence of sets*

$$\begin{array}{ccc}
\cdots \mathbf{I}\mathcal{S}^c_G(X - Int(D(\alpha_X)), S(\alpha_X)) & \longrightarrow & \mathbf{I}\mathcal{S}^c_G(X) \\
 & & \downarrow \\
 & & \mathbf{I}\mathcal{S}^c_{G/H}(X^H) \times F/O_{G,iso}(\alpha_X) \\
 & & \downarrow \\
 & & \mathbf{I}\mathcal{S}^c_G(S(\alpha_X))/Image\ \partial^*
\end{array}$$

that extends infinitely to the left. All objects to the left of the raised dots are groups and the corresponding maps are compatible with group structures as in Theorems $1.2 - 1.3$.■

This result provides a means for analyzing isovariant structure sets inductively with respect to the number of orbit types, for the two structure sets in the sequence aside from $\mathbf{I}\mathcal{S}^{BQ,c}_G(X)$ have fewer orbit types than the original action, and the same is true for the data in $F/O_{G,iso}(-)$. In subsequent work we shall study special cases of this sequence in connection with questions from Section 3 below.

2. Isovariance and the Gap Hypothesis

During the nineteen seventies and early eighties, Petrie and several other topologists (beginning with S. Straus [Str]) found many striking applications of surgery to smooth G-manifolds satisfying the following basic condition:

Gap Hypothesis. A smooth G–manifold M is said to satisfy the (standard version of the) *Gap Hypothesis* if for each pair of isotropy subgroups $H \supsetneqq K$ and each pair of components $B \subset M^H$, $C \subset M^K$ such that $B \subsetneqq C$ we have

$$(\ddagger) \qquad \dim B < \tfrac{1}{2}(\dim C).$$

This is basically a general position condition. Its usefulness arises because surgery theory involves the existence of smoothly embedded spheres whose dimensions are no more than half the dimensions of the ambient manifolds. If the Gap Hypothesis holds, then one can choose the appropriate embedded spheres in each fixed set component $C \subset M^K$ to miss all the components $D \subset M^H$ such that $D \subsetneqq C$. This means that all the constructions involving embedded spheres can be done equivariantly on the set $M^K \cap M_{(K)}$, which has only one isotropy type (namely, K). In effect, this reduces an equivariant surgery problem to a sequence of nonequivariant problems over the orbit spaces $M_{(K)}/G$. A similar reduction arises in the Browder-Quinn theory even if the Gap Hypothesis does not hold; this follows directly from the isovariance and transverse linearity conditions of [BQ].

Most of Petrie's work dealt with the existence of smooth G-actions on disks and spheres with properties quite unlike those of orthogonal actions (compare [Pet1–2] and [DPS]). In a somewhat different direction, K. H. Dovermann and M. Rothenberg modified Petrie's approach to construct classification theories for G-manifolds in a given equivariant homotopy type *provided the Gap Hypothesis holds* (see [DR] and [LüMa]).

One of the central problems in equivariant surgery is to understand the role of the Gap Hypothesis more clearly (*cf.* [Sc12, Sec. 4]), and thus it is natural to seek relationships between the isovariant homotopy structure theory of Section 1, which does not require the Gap Hypothesis, and equivariant surgery theories that somehow rely on the Gap Hypothesis as in [DR] or [LüMa] (related examples are also discussed in [DoS2, Sec. II.3]). As noted in [Daw], the isovariant structure sets of Section 1 lie somewhere between such equivariant surgery theories and the Browder-Quinn theories. The following result of S. Straus [Str] and W. Browder [Brw4] establishes a stronger and more precise relationship; in particular, the theories of [DR] and [LüMa] are equivalent to the theories of Section 1 when the Gap Hypothesis holds.

Theorem 2.1. *Let $f : M \to N$ be an equivariant homotopy equivalence of closed smooth G-manifolds that satisfy the Gap Hypothesis. Then f is equivariantly homotopic to an isovariant homotopy equivalence. Furthermore, if $M \times [0,1]$ satisfies the Gap Hypothesis then this isovariant homotopy equivalence is unique up to isovariant homotopy.*

This result and the machinery of Sections I.4 and II.1 suggest a two step approach to analyzing smooth G-manifolds within a given equivariant homotopy type if the Gap Hypothesis does not necessarily hold; namely, the first step is to study the obstructions to isovariance for an equivariant homotopy equivalence and the second step is to study the isovariant structure sets of the preceding section.

Sketch of the proof of Theorem 2.1. We shall only deal with semifree G-manifolds in order to illustrate the ideas without addressing the bookkeeping problems that arise for more general actions; furthermore, for the sake of simplicity we shall use a slightly stronger version of the Gap Hypothesis with $\dim B + \varepsilon < \frac{1}{2}(\dim C)$ for some small positive integer ε. Finally, we shall only consider the existence question; the uniqueness result follows by applying similar methods to $M \times [0,1]$.

The original proofs of Straus and Browder rely heavily on methods and results from nonsimply connected surgery. The argument presented here does not completely eliminate geometric topology, but it only requires simple considerations involving embeddings in the general position range and transversality. Of course it would be interesting to know if the proof can be done entirely with homotopy theoretic machinery.

The first step in the proof is to deform f equivariantly so that it maps $D(\alpha_M)$ isovariantly to $D(\alpha_N)$ such that $S(\alpha_M)$ is sent to $S(\alpha_N)$. This will follow quickly if $S(\alpha_M)$ and $S(\{f^G\}^*\alpha_N)$ are equivariantly fiber homotopy equivalent. To prove the latter, one first uses a result of K. Kawakubo [Ka] to show that the equivariant tangent bundle τ_M is stably equivariantly fiber homotopy equivalent to $f^*\tau_N$. Restricting to fixed point sets, we conclude next that the restrictions of these bundles to N^G are also equivariantly stably fiber homotopy equivalent; in other words, $\{f^G\}^*\tau_{N^G} \oplus \{f^G\}^*\alpha_N$

is equivariantly stably fiber homotopy equivalent to $\tau_{M^G} \oplus \alpha_M$. The classifying space versions of the standard splittings for equivariant stable homotopy theory (*e.g.*, the discussion at the end of [Se]) then imply that $\{f^G\}^*\alpha_N$ is equivariantly stably fiber homotopy equivalent to α_M. Since the dimensions of the latter bundles are at least somewhat larger than the dimensions of M^G and N^G, the stable range theorems of [Sc1] and [Sc5] imply that the unit sphere bundles of $\{f^G\}^*\alpha_N$ and α_M are already equivariantly fiber homotopy equivalent before stabilization.

The second step is to analyze the set of points where the modified map fails to be isovariant. We can apply transversality on the complement of $D(\alpha_M)$, without changing the map on $S(\alpha_M)$, so that a further equivariant deformation of f is transverse to N^G on the complement of M^G. It follows immediately that the set Y of nonisovariant points is a smooth invariant submanifold such that $\dim Y = \dim M^G$. Using general position and the fact that f is an equivariant homotopy equivalence, one can then show that Y lies in some tubular neighborhood of the fixed point set (the inclusion of Y can be deformed into $D(\alpha)$ because f is an equivariant homotopy equivalence, and by general position one can modify this into an isotopy of Y into $D(\alpha)$).

The third step is to show that the map obtained in the previous step is equivariantly homotopic to an isovariant map if and only if the class of the nonisovariant set in an appropriate bordism theory vanishes. By the results of Section I.4, the obstruction to deforming the map f_2 obtained thus far is the obstruction to finding an equivariant lifting of $f_2|M \overset{*}{-} M^G$ from N to $N \overset{*}{-} N^G$. Because the Gap Hypothesis holds, one can use the Blakers-Massey Theorem to view this lifting obstruction as the obstruction to finding an equivariant nullhomotopy for the composite of $f_2|M \overset{*}{-} M^G$ with the collapse map $N \to N/N \overset{*}{-} N^G \approx D(\alpha_N)/S(\alpha_N)$.

The next to last step is to notice that the obstruction from the preceding step need not vanish, but it has a canonical indeterminacy given by the possible choices of the equivariant fiber homotopy equivalence from α_M to $\{f^G\}^*\alpha_N$. In fact, since we are in the stable range the homotopy classes of such equivalences are given by $[N^G, F_G]$. Finally, an analysis of the obstructions in the third step shows that one can kill the isovariance obstruction by choosing a (possibly) different equivariant fiber homotopy equivalence on the equivariant sphere bundles.$\blacksquare$

3. Homotopy linear actions on spheres

As indicated in Section 1, the original interest in classifying smooth manifolds in a given isovariant homotopy type involved certain smooth group actions on homotopy spheres. In this section we shall discuss some basic questions in this area that can be analyzed, at least to some extent, by the methods of the preceding sections.

From a purely formal viewpoint we are interested in smooth G-manifolds that are isovariantly homotopy equivalent to linear actions on spheres. However, for historical and practical reasons it is more useful to deal with actions satisfying apparently weaker assumptions and to prove that all such actions are isovariantly homotopy equivalent to the appropriate linear example.

The basic concepts and constructions for homotopy linear actions are summarized in [Sc10, Secs. 5–6]. We shall begin with a modified version of the definition in [Sc10, Sec. 5, p. 274].

Definition. Let φ_0 be a linear representation of G on $\mathbb{R}^n + 1$ that splits as $\varphi \oplus \mathbb{R}$ (with trivial action on the second summand). If H is a subgroup of G let $n(H) + 1$ denote the dimension of the real vector space φ_0^H (hence $n(H) \geq 0$). A smooth G-action γ on a smooth manifold Σ^n is said to be *strongly φ-homotopy linear* ($\equiv$ *strongly φ-homotopically linear* or *strongly φ-semilinear*) if the following hold:

(1) For each $H \subset G$ the fixed point set of H is homeomorphic (but not necessarily diffeomorphic) to $S^{n(H)}$.
(2) If $H \subset K \subset G$ and $n(H) - n(K) = 2$ then $\Sigma^H - \Sigma^K$ is homotopy equivalent to S^1.
(3) The induced G representations at the tangent spaces of points in Σ^G are all equivalent to φ.

It is fairly elementary to show that each such action is G-homotopy equivalent to the unit sphere $S(\varphi_0)$, or equivalently to the one point compactification of φ (*cf.* [Sc10, Prop. 5.1]); in fact, Σ^n is usually G-homeomorphic to this linear sphere (see the remarks on [Sc10,p. 274] following Proposition 5.1), and in the remaining cases the results of [DuS, Sec. 4] show that Σ is isovariantly homotopy equivalent to the linear action.

Connected sums. If $G = \{1\}$ then a strongly homotopy linear G-manifold is a manifold homeomorphic to S^n (*i.e.*, an *exotic sphere*) by the Generalized Poincaré Conjecture [Mln1, p. 109]; a diffeomorphism classification of such objects was developed in the previously mentioned work of Kervaire and Milnor during the late nineteen fifties and early nineteen sixties [KM]. An elementary but highly useful step in their program was the use of objects with orientations and the introduction of an abelian group structure on the oriented diffeomorphism classes of exotic spheres by means of connected sums (see [Sc10, p. 275] and the references cited there). One can proceed similarly with strongly homotopy linear φ-spheres: Given two such G-manifolds Σ_1 and Σ_2, let $D_i \subset \Sigma_i$ be G-diffeomorphic to the disk $D(\varphi)$ and glue $\Sigma_1 - \mathrm{Int}(D_1)$ to $\Sigma_2 - \mathrm{Int}(D_2)$ equivariantly along the common boundary; once again one needs a suitable concept of orientation to ensure this construction is well defined, and this can be done as in [Sc9, Sec. 1]. As noted in [Sc10, Prop. 5.2, p. 276], this yields a monoid structure on the set of all (suitably equivariantly oriented) diffeomorphism classes of strongly φ-semilinear spheres, the resulting monoid is abelian if the fixed point set dimension is at least 2, and if we factor out the submonoid of actions that bound equivariantly contractible G-manifolds, then the resulting quotient is a group (abelian if the fixed point sets are at least 2-dimensional). Following [Sc10] we shall denote this group by $\Theta^G(\varphi)$.

Digression—some motivation

Although one can certainly study the groups $\Theta^G(\varphi)$ for their own sake, these groups also arise naturally in connection with certain questions of independent interest. Before proceeding with further results on such actions we shall describe some of these contexts.

Examples. **1.** *Smooth actions of arbitrary p-groups on exotic spheres.* In fact, as noted in [Sc6] this was one of the original motivations for studying the classification of smooth G-manifolds in a given isovariant homotopy type. If one is given an action of a finite abelian p-group on an exotic sphere, then Smith theory shows that all the fixed point sets are mod p homology spheres. This implies that an arbitrary such action admits an isovariant map to a linear model with degree prime to p. If the dimension of the fixed point set of G is at least 2, then one can use these maps and the methods of Section 1 to define p-localized versions of the knot invariant with values in the abelian groups

$$F/O_{G,iso}(S(\varphi^H \oplus \mathbb{R}) \times [\varphi/\varphi^H] \text{ rel } \{\text{basept.}\})_{(p)}$$

where H is an arbitrary isotropy subgroup of the action. For the special case of cyclic p-groups where H is the minimal nontrivial isotropy subgroup, this was done previously in [Sc4] and [Sc9, Sec. 4], where the invariants were used to obtain restrictions on the fixed point structure of smooth $\mathbb{Z}_{p^r}$-actions on exotic spheres. In subsequent work the more general invariants will be used to study actions of other abelian p-groups on exotic spheres in relatively low dimensions.

2. *Fixed point sets of differentiable actions on spheres.* Results of L. Jones [Jo], A. Assadi [As] and others show that certain variants of the groups $\Theta^G(\varphi)$ carry the obstructions to realizing a mod p homology k-sphere as the fixed point set of a smooth semifree $\mathbb{Z}_{p^r}$-action on some homotopy $(k+2m)$-sphere, where $m \geq 2$ (*cf.* [As]). The basic idea is simple: If A is the homology sphere, let A_0 denote A with the interior of a closed disk removed. Then one can realize A_0 as the fixed point set of a smooth semifree $\mathbb{Z}_{p^r}$-action on D^{k+2m}. The induced action on the boundary then determines an element of the appropriate variant group $\mathrm{Var}\Theta^G(\varphi)$, and one can extend the action to an action on a homotopy sphere if and only if this element is zero. Related ideas are used in [Sc14] to construct examples of smooth $\mathbb{Z}_{pq}$-actions on spheres, where p and q are distinct odd primes, such that the Pontrjagin numbers of the fixed point set are nontrivial.

3. *Equivariant smoothings of topological G-manifolds.* Results of Lashof and Rothenberg [LaR] show that the smoothability of a G-manifolds and the classification of equivariant smoothings reduce to equivariant bundle-theoretic questions, at least if there are no 4-dimensional components in the fixed point sets of the isotropy subgroups. This is formally parallel to ordinary smoothing theory for topological manifolds (*cf.* [KiSb]). However, in ordinary smoothing theory the results of [KM] and [KiSb] translate the bundle-theoretic problems into well known questions of homotopy theory, but comparable insights into equivariant smoothing theory only exist in special cases. This is already evident in known results on the topological classification of linear representations (*e.g.*, see [CS1–3, CSSW, CSSWW]), which is the first step in analyzing the bundle-theoretic problems in [LaR]. Partial results on the higher order steps appear explicitly in [LaR] and [MR], and implicitly in [Sc7], [KL], and [KwS6]. Standard techniques of engulfing theory ([Hud, Ch. VII] or [RSa, Ch. 7]) imply that a strongly φ-homotopically linear G-manifold is equivariantly homeomorphic to $S(\varphi \oplus \mathbb{R})$ if the dimension of φ^G is sufficiently large [CMY, Il5, Ro, Sc7], and thus information about the groups $\Theta^G(\varphi)$ has immediate implications for equivariant smoothing theory (*cf.* the results on rational characteristic classes in [Sc7]). It is conceivable that information on the groups $\Theta^G(\varphi)$ can also shed light on equivariant smoothing theory for more general G-manifolds; in particular, the results of [LaR, pp. 215 and 264–265] suggest this.

4. *Rational invariants for classifying smooth G-manifolds up to finite ambiguity.* In a sequence of papers culminating with [RT], Rothenberg and Triantafillou described an equivariant analog of D. Sullivan's rational invariants for diffeomorphism classification of certain smooth simply connected manifolds up to finite ambiguity [Su]. However, their invariants only provide an equivariant *almost diffeomorphsim* classification up to finite ambiguity in many cases; in other words, to complete the picture one needs a smooth equivariant classification up to finite ambiguity for all G-manifolds that are equivariant connected sums $M_0\#\Sigma$, where M_0 is fixed and Σ is a homotopy linear G-sphere. Questions of this type have been studied by M. Masuda [Ms] and will be considered further in joint work of Masuda and the author [MSc].

Exact sequences

Of course, the usefulness of the groups $\Theta^G(\varphi)$ depends on the extent to which they can be computed. The original work of [KM] can be summarized in a long exact sequence

$$\cdots \to P_{n+1} \to \Theta_n \to \pi_n(F/O) \to \cdots$$

where the groups P_k are 0 if k is odd, infinite cyclic if k is divisible by 4, and cyclic of order two if $k \equiv 2 \mod 4$ (compare [Lev] or [Sc10, Thm. 6.1, p. 277]). One particular consequence of this sequence is the finiteness of the groups Θ_n if $n \geq 4$. The subsequent work of [BP] and [RSo] yielded somewhat different exact sequences for $\Theta^G(\varphi)$ when G acts semifreely on φ (see [Sc3, (1.1)]; also compare [Sc10, Thm. 6.3, p. 277]). As indicated in [Sc10, Sec. 6] one can use these exact sequences to obtain fairly complete information on the rationalized group $\Theta^G(\varphi)\otimes\mathbb{Q}$. In particular, the following conclusion is an elementary consequence of the exact sequences in [Sc10, Thm. 6.3]:

Proposition 3.1. *If G acts semifreely on φ such that $\dim\varphi \geq 5$ and $\dim\varphi - \dim\varphi^G \geq 3$, then the dimension of the rational vector space $\Theta^G(\varphi)\otimes\mathbb{Q}$ is at most $|G|+3$.* ■

This estimate is not really the best possible, but it shows that the ranks of the rationalized groups have uniform bounds depending only on the order of the group.

In [Sc9] the approach for semifree actions is extended to a more general class of actions that are called *ultrasemifree*; precise descriptions of the basic exact sequences appear in [Sc9, (6.2), p. 275], and rational computations with these exact sequences are discussed in [Sc9, Sec. 7]. For these cases one again obtains bounds for the ranks of the groups $\Theta^G(\varphi)$ that only depend upon the order of G.

The machinery of Section 1 allows one to extend everything to more general actions in a straightforward manner:

Theorem 3.2. *Let φ be a G-representation such that all isotropy subgroups are normal (e.g., suppose G is abelian) and $\dim\varphi^G \geq 2$. Let H be an isotropy subgroup for the* *action on* φ*, and let* $\varphi_H := \varphi/\varphi^H$*. Then there is the following long exact sequence of abelian groups:*

$$
\begin{array}{c}
\mathbf{I}\mathcal{S}^h_G(D(\varphi^H \oplus \mathbb{R}) \times S(\varphi_H), \partial(-)) \\
\big\downarrow \\
\Theta^G(\varphi) \\
\big\downarrow \\
\Theta^{G/H}(\varphi^H) \oplus F/O_{G,iso}(S(\varphi^H \oplus \mathbb{R}) \times \varphi_H, \text{basept.}) \\
\big\downarrow \\
\mathbf{I}\mathcal{S}^h_G(D(\varphi^H) \times S(\varphi_H), \partial(-)) \\
\big\downarrow
\end{array}
$$

In particular, it seems likely that the preceding exact sequence should imply that the abelian groups $\Theta^G(\varphi)$ are finitely generated, but we have not verified this.

Although the exact sequence in Theorem 3.2 has not yet been used to do many computations beyond those for semifree and ultrasemifree actions, the following result from [Sc16] indicates the potential usefulness of Theorem 3.2.

Proposition 3.3. *Let* p *and* q *be distinct odd primes, and let* ω *be an orthogonal representation of* $\mathbb{Z}_{pq}$ *such that the following hold:*
(i) *If* ω_p *and* ω_q *are the fixed sets of* $\mathbb{Z}_p$ *and* $\mathbb{Z}_q$ *respectively, then each has dimension at least* 4 *and their intersection is the zero subspace.*
(ii) *The dimension of* $\omega_1 := \omega/(\omega_p + \omega_q)$ *is at least 4.*
Then there are finitely generated subgroups $V_k \subset \Theta^G(\mathbb{R}^k + \omega)$ *such that if* $v_k = \dim V_k \otimes \mathbb{Q}$ *then for all positive integers* n *the sequence* $\{v_k/k^n\}$ *is unbounded.*■

This contrasts sharply with the results on $\dim \Theta^G(\varphi)$ in the semifree and ultrasemifree cases (refer back to Proposition 3.1).

The groups $\Theta^G(\varphi)$ *and the Gap Hypothesis*

If the G-manifold φ satisfies the Gap Hypothesis, then the "rather long" equivariant surgery sequence of Dovermann and Rothenberg [DR] provides another means for computing $\Theta^G(\varphi)$. In particular, the methods and results of [DR] yield a canonical bound on the dimensions of the rational vector spaces $\Theta^G(\varphi) \otimes \mathbb{Q}$ in terms of $|G|$; consequently, Proposition 3.3 shows the existence of many new rational classification invariants for G-homotopically equivalent smooth G-manifolds beyond the usual invariants that appear when the standard Gap Hypothesis holds.

4. Borderline cases of the Gap Hypothesis

To simplify the discussion, in this section we shall only consider degree one equivariant normal maps ($f : M \to X,$ bundle data) such that f maps the singular set of M to the singular set of X by an equivariant homotopy equivalence. Since the key inductive step in equivariant surgery involves situations of this type, our hypothesis is basically a way of concentrating on a single inductive step. In any case, for such maps the appropriate Gap Hypothesis assumption is that

$$\dim X \geq 2 \cdot \dim(\mathrm{Sing}(X)) + 2.$$

In this section we are interested in examples that lie just outside this Gap Hypothesis range but have been studied effectively by the standard techniques of equivariant surgery. There are two reasons for our interest in such cases. First of all, some are needed in Part III. Second, special cases of these results have implications for equivariant and isovariant homotopy theory that are not presently obtainable by purely homotopy-theoretic methods; needless to say, it would be enlightening to have intrinsically homotopy theoretic proofs for such results.

Following [DoS2, Section III.2] we define the *Gap Hypothesis balance* to be $\Delta(X) := \dim X - 2 \cdot \dim(\mathrm{Sing}(X))$; with this terminology the appropriate version of the Gap Hypothesis is $\Delta(X) \geq 2$. The cases of interest here are $\Delta(X) = 0$ or 1; as one might expect, the similarities with the Gap Hypothesis range decrease as $\Delta(X)$ gets smaller. In particular, examples of M. Rothenberg and S. Weinberger (described in [DoS2, Sec. I.6]) indicate that the situation becomes even more complicated if $\Delta(X)$ is negative. We begin by discussing the situation when the Gap Hypothesis holds.

The cases $\Delta(X) \geq 2$

In these cases the methods of equivariant surgery yield the following conclusion (compare [DR] or [DoS2, Sec. I.5]):

Theorem 4.1. *Suppose that $f : M \to X$ and appropriate bundle data determine an equivariant degree one normal map of closed smooth 1-connected n-manifolds ($n \geq 5$) such that the associated map of singular sets is an equivariant homotopy equivalence and $\Delta(X) \geq 2$. Then f is normally cobordant to an equivariant simple homotopy equivalence if an ordinary Wall surgery obstruction $\sigma(f) \in L_n^s(\mathbb{Z}[G], w)$ is trivial.*

As indicated in [DoS2, Sec. I.4], the crucial idea in the proof is that the pair ($M, M - \mathrm{Sing}(M)$) is highly connected, and this allows one to deform all surgical constructions into the complement of the singular set. It follows that the equivariant surgery problem essentially reduces to an ordinary surgery problem on the orbit space of the free part of the action.■

The case $\Delta(X) = 1$

The crucial new insights in this case are due to M. Morimoto. In [Ba] A. Bak defines a quotient of the Wall group $L_{2k+1}^c(\mathbb{Z}[G], w)$ that is denoted by $\mathbf{W}_{2k+1}^c(\mathbb{Z}[G], \mathit{\Gamma} G(X); w)$;

the set $\Gamma G(X)$ is a set of order two elements $g \in G$ such that $\dim X^g = k$ (*cf.* [Mto1, p. 467]) and w denotes the first Stiefel-Whitney class. Following Morimoto, we shall call the group $\mathbf{W}^c_*(-)$ the *Bak group* associated to the given data. Frequently the Bak group is isomorphic to the corresponding Wall group. In particular, this is true if G has odd order or k is even and the orientation homomorphism is trivial. However, the example in [Mto1, Corollary C, page 468] shows that the projection from the Wall group to the Bak group has a nontrivial kernel in some cases, and the results of [BaMo] yield additional examples.

Theorem 4.2. *Suppose that $f : M \to X$ and appropriate bundle data determine an equivariant degree one normal map of closed smooth 1-connected $(2k+1)$-manifolds ($k \geq 2$) such that the associated map of singular sets is an equivariant homotopy equivalence and $\Delta(X) = 1$ (hence the singular set is k-dimensional). Then f is normally cobordant to an equivariant simple homotopy equivalence if the image of an ordinary Wall surgery obstruction $\sigma(f) \in L^s_n(\mathbb{Z}[G], w)$ in the Bak group $\mathbf{W}^s_{2k+1}(\mathbb{Z}[G], \Gamma G(X); w)$ is trivial.*

An analogous result holds for equivariant homotopy equivalences if one replaces L^s by L^h and $\mathbf{W}^s$ by $\mathbf{W}^h$.

Under the hypotheses of Theorem 4.2 the pair $(M, M - \mathrm{Sing}(M))$ is not quite so highly connected, and one can deform some but not all surgical constructions back to the free part of the action on M. The methods employed in the proof of Theorem 4.1 still yield a surgery obstruction in the appropriate Wall group, but this obstruction is not necessarily well-defined. However, if one passes to the Bak group, then one does obtain a well-defined obstruction.■

Results of Dovermann [Do2] relate the preceding to questions involving isovariance. Namely, if f is isovariant on the singular set then the given conditions allow one to surger f into an isovariant map that is an equivariant homotopy equivalence on the singular set. In this setting the Wall group element represents the obstruction to surgering f into an isovariant homotopy equivalence. Thus the kernels of the maps from Wall groups to Bak groups carry obstructions for transforming certain equivariant equivalences into isovariant equivalences.

The case $\Delta(X) = 0$

In this case results are only known for the case $G \approx \mathbb{Z}_2$, and the main results are due to Dovermann [Do1] (see also [DoS1] and [Sc15, Thm. 2.5]).

Theorem 4.3. *Let $f : X \to Y$ come from a suitably defined degree one $\mathbb{Z}_2$-normal map of smooth 1-connected $2k$-dimensional $\mathbb{Z}_2$-manifolds, where f induces a homotopy equivalence of fixed point sets and the latter are k-dimensional. Then f is normally cobordant to a $\mathbb{Z}_2$-homotopy equivalence, relative to the fixed point sets, if and only if the following hold:*

(*i*) *If k is even, the $\mathbb{Z}_2$-signatures of X and Y are equal.*

(*ii*) *If k is odd, the ordinary Kervaire invariant of f is trivial and a mod 2 rank invariant of the surgery kernel of f is also trivial.*

Unlike the preceding cases, one must now consider homotopy classes in $\pi_q(M)$ that cannot be deformed into the complement of singular set; the obstruction to doing this is measured by a homological intersection number. One needs a modified concept of Hermitian form (called *quasi-quadratic* in [Do1]); the invariants of such forms turn out to be the algebraic invariants described in the statement of the theorem.■

Theorem 4.3 has some curious homotopy theoretic implications that are not yet derivable from other techniques. As indicated earlier, it would be enlightening to have intrinsically homotopy theoretic arguments, both for the sake of making everything more self contained and also in the interests of proving further results along the same lines.

The first implication involves equivariant function spaces. Following [BeS], if the finite group G acts freely and orthogonally on the unit sphere $S(V)$ in the Euclidean space V, let $F_G(V)$ be the space of G-equivariant self maps of $S(V)$. Of course, if H is a subgroup of G there is a natural forgetful map ρ from $F_G(V)$ to $F_H(V)$. Also, there is a stabilization map from $F_G(V)$ to a space $F_G := \lim F_G(V \oplus W)$ where W runs through all isomorphism classes of free G-representations. The main results of [BeS] state that F_G is homotopy equivalent to the free infinite loop space $\Omega^\infty S^\infty(K(G,1)_+)$ and the forgetful map from F_G to F is induced by the transfer map in stable homotopy (*cf.* [BG]). In particular, by the Kahn-Priddy Theorem [KP] the map $F_G \to F$ induces a split surjection in positive dimensional homotopy groups if $G \approx \mathbb{Z}_2$. Dovermann's results yield the following unstable analog of the Kahn-Priddy theorem.

Proposition 4.4. *In the terminology above, suppose that $G \approx \mathbb{Z}_2$ and $V = \mathbb{R}^n$ with the antipodal involution. If $j : F_G(V) \to F_G \to F$ is the composite of stabilization with the forgetful map, then the image of $j_* : \pi_n(F_G(V)) \to \pi_n(F) \cong \pi_n$ contains all positive-dimensional elements whose Hopf invariants are even and whose Kervaire invariants are zero.*

In particular, it follows that the image of j_* has index at most two in every dimension and the index is one except for a very sparse set of dimensions (recall that an element can have an odd Hopf invariant only in dimensions 1, 3, and 7, and an element can have a nonzero Kervaire invariant only in dimensions of the form $2^m - 2$).

Proposition 4.4 follows by combining the results of [Sc4] on knot invariants with the results of [Sc8] on realizing exotic spheres as fixed point sets of involutions on homotopy spheres (in this connection also see [Lö]); the role of Dovermann's work is that the crucial results from [Sc8] and [Lö] depend upon [Do1].

Questions: Can one eliminate the condition on Kervaire invariants in Proposition 4.4? Basic results in homotopy theory show that the condition on Hopf invariants CANNOT be eliminated. Also, can one prove a result similar to 4.4 for the image of $\pi_n(F_G(V)) \to \pi_n(F)$ when $V = \mathbb{R}^{n-1}$? An answer to this question would have implications for realizing exotic n-spheres as fixed point sets of smooth involutions on homotopy $(2n-1)$-spheres.

Here is another consequence of Theorem 4.3 to a question of interest in nonequivariant homotopy theory (compare [Str], [Sc15]):

Proposition 4.5. *Let M and N be closed, homotopically equivalent 1-connected n-manifolds where $n \geq 3$. Then the deleted symmetric squares $\{M \times M - \Delta_M\}/\mathbb{Z}_2$ and $\{N \times N - \Delta_N\}/\mathbb{Z}_2$ are homotopically equivalent.*■

Question: Can one eliminate the simple connectivity hypothesis in this result? Straus proves an analogous result for deleted reduced cyclic p-th powers where p is an odd prime, and the latter result has no hypothesis on the fundamental group [Str]. Results of P. Löffler and R. J. Milgram [LöMi] suggest that the answer to the question is yes. An obvious suggestion for approaching this is to prove a version of Dovermann's result for involutions on nonsimply connected manifolds with $\Delta(X) = 0$.

5. Isovariance and nonsmoothable group actions

Many questions arising in Parts I and II are also meaningful and interesting for group actions that are not smooth. In this section we shall describe some results along these lines.

Beginning in the mid nineteen eighties Cappell and Weinberger developed a variety of surgery-theoretic techniques for constructing exotic topological or PL group actions with a given isovariant homotopy type. Some of their results for circle actions are summarized in [CW1]. The following *replacement theorem* is a simple but basic example of their results for finite group actions:

Theorem 5.1. *Let $G = \mathbb{Z}_p$ where p is an odd prime, and let W^n be a closed simply connected manifold with a locally linear topological or PL G-action Φ such that $M + W^G$ is also simply connected and* $\dim M \geq 5$. *If $h : M' \to M$ is a homotopy equivalence, then there is another locally linear topological or PL (resp.) G-action Φ' on W such that*

(i) the two actions are equivalent on the complements of their fixed point sets,

(ii) the fixed point set of Φ' is M',

(iii) the G-manifolds (W, Φ) and (W', Φ') are G-isovariantly homotopy equivalent (but the equivalence is not necessarily isovariantly homotopic to a transverse linear map in the sense of Browder and Quinn).■

Cappell and Weinberger also obtain several extensions and refinements of Theorem 5.1; details of this work appear in [CW2], and further results on obstructions to replacement appear in [DW]. In [Daw] Dawson uses his version of the results in Section 1 to study similar replacement questions for smooth actions when the codimension of the fixed point set is small. Dawson has also obtained results on *replacement of tangential representations* up to homotopy, where one replaces the G-representation Ω at the tangent space of a fixed point by an isovariantly homotopy equivalent representation Ω' and attempts to find an action G-homotopy equivalent to the orignal one with the same fixed point set and the modified tangential data.

In a forthcoming book [Wb1] Weinberger develops powerful and fairly general machinery for classifying certain topological actions up to isovariant homotopy equivalence. Specifically, his results apply to group actions for which the fixed point sets of subgroups define a weak analog of a Thom-Mather stratification (*e.g.*, a CS stratification in the sense of Siebenmann [Si2] or a homotopy stratification in the sense of Quinn [Q2]). Since this work involves several deep concepts that are not needed in the smooth category (for example, results on ends of maps [Q1]), we shall not attempt to explain the main ideas here. This work has already produced some further developments and applications due to Weinberger and M. Yan [Wb2, WY, Y], including counterexamples to equivariant analogs of the Borel rigidity conjecture for aspherical manifolds [Wb2].

In view of [Wb1] it would be useful to have an extension of the results of Part I to isovariant maps of G-manifolds with weak stratifications of an appropriate type. Perhaps the most obvious complication is that fixed point sets need not have closed tubular neighborhoods (*e.g.*, see the examples near the beginning of [Q1]). An extension of the results in [DuS] to nonsmoothable actions will probably require diagrams involving Quinn's notion of *homotopy collar* [FQ, pp. 214–215] for the sets $M_{(H)}$ associated to a G-manifold M (homotopy collars are called *homotopy completions* in [Q1, Sec. 7.8]) and *homotopy links of* ∞ *in the one point compactification* in [Q2]).

Finally, we note that the Browder-Straus proof of Theorem 2.1 goes through for certain classes of nonsmoothable actions; for example, the proof applies to semifree PL actions on manifolds such that the fixed point sets are also manifolds. Of course, it would be enlightening to have an alternate proof as in the smooth category, with little or no input from surgery theory.