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PART III

FINITE GROUP ACTIONS ON HOMOLOGY 3-SPHERES

Background references

In contrast to Parts I and II, the emphasis in this final part will be on specific geometric problems involving transformation groups on 3-manifolds. The following references contain most of the background material we shall need:

(1) Article by M. Davis and J. Morgan in Bass–Morgan (eds.), *The Smith Conjecture* (:= [DaMo]).

(2) Article by Edmonds in Contemp. Math. Vol. 36 (1985) (:= [Ed]).

(3) Article by Raymond in Transactions Amer. Math. Soc. Vol. 131 (1968) (:= [Ra]).

(4) Notes from lectures of Thurston on the geometry and topology of 3-manifolds (:= [Th1]; the main points are summarized in [Th2]).

During the past fifteen years the concepts of *orbifold* and *orbifold fundamental group* have become fairly standard in 3-dimensional topology. Since both arose from considerations involving transformation groups, we shall use these concepts as needed. The basic definitions and examples can be found in Thurston's notes [Th1, Ch. 13, especially p. 13.5] or the first few sections of [DaMo].

1. A survey of known results

For several decades topologists have known that dimensions 3 and 4 form a transitional range from the geometric rigidity of line and surface topology to the freedom of movement one has in dimensions ≥ 5. References on this topic are numerous and include an old article of Siebenmann [Si1], work of A. Casson and C. Gordon [CG], and books of M. Freedman and F. Quinn [FQ], S. Donaldson and P. Kronheimer [DK], and S. Akbulut and J. McCarthy [AM]. Our interest in this article lies with transitional properties of group actions on spheres and manifolds closely resembling spheres (*e.g.*, manifolds homeomorphic but not necessarily diffeomorphic to the standard sphere). As indicated by the title above, we shall concentrate on the 3-dimensional case; some references for the 4-dimensional case include [Ed] for work done through the early nineteen eighties, [BKS] and [DM] for results about the fixed point sets of smooth actions on S^4, [KwS2–4] for results on topological actions on S^4 with isolated singular points, and [KwS1] and [KwS5] for results concerning topological circle actions on S^4 and other 4-manifolds.

In 1- and 2-dimensional topology all group actions on manifolds are equivalent to smooth actions that preserve nontrivial geometric structures (*cf.*, [Ed, p. 341]). For example, all compact Lie group actions on S^1 and S^2 are equivalent to orthogonal actions. On the other hand, it is well known that a smooth action of a finite cyclic group on S^4 need not be orthogonal because the fixed point set can be a nontrivially knotted 2-sphere (see Giffen [Gi] and Gordon [Go1]). The known results for dimension 3 lie somewhere between these two adjacent cases.

Fact 1.1. *There is a continuum of pairwise inequivalent topological $\mathbb{Z}_k$ actions on S^3 for every prime k.*

As noted in [Ed], this is due to Bing [Bi] and Alford [Al].■

Fact 1.2. *All smooth actions of compact Lie groups on S^3 with nonempty fixed point sets are orthogonal. Modulo some possibly exceptional cases, all actions of compact Lie groups on S^3 with positive-dimensional singular sets are orthogonal.*

If $G = S^1$ this result is contained in [Ra], and for larger Lie groups the result is a straightforward exercise. When G is finite cyclic and the fixed point set is a circle this was conjectured by P. A. Smith and solved in the late nineteen seventies by the combined efforts of several mathematicians (see the book containing [DaMo]). Results for other finite groups appear in several different places, including papers of M. Davis and J. Morgan [DaMo], M. Feighn [Fn], and S. Kwasik and the author [KwS6]. About ten years ago W. Thurston announced that the second assertion in 1.2 holds without exception [Th3]; although workers in the area have few doubts about the correctness of this statement, it is not clear when a complete written proof will be available.□

Fact 1.3. *There are many smooth group actions on integral homology* 3*-spheres that are analogous to important examples of smooth actions on higher-dimensional spheres.*

A similar—and related—phenomenon occurs in the theory of isolated singularities of complex hypersurfaces (*cf.* Mumford [Mum]): Given a complex polynomial $f(\mathbf{z})$ in $n+1$ variables such that the origin is an isolated singularity, let Σ_f be the intersection of the zero set $\{\mathbf{z} \in \mathbb{C}^{n+1} | f(\mathbf{z}) = 0\}$ with a sphere of sufficiently small radius; it follows that Σ_f is a closed smooth $(2n-1)$-manifold. Mumford's result deals with the case $n = 2$ and states that a 3-dimensional manifold of the form Σ_f is diffeomorphic to S^3 if it is simply connected; on the other hand, there are many examples where Σ_f is a nonsimply connected integral homology sphere. The analogs of the latter in higher dimensions are homeomorphic but not necessarily diffeomorphic to the standard $(2n-1)$-sphere (see Milnor's book [Mln3] for further information on this topic).

Examples for Fact 1.3. (1) Perhaps the most basic of these are the *pseudofree* smooth circle actions on Seifert homology 3-spheres. These actions are free on the complement of a finite set of pairwise disjoint circles, and the isotropy subgroups for points on these circles are pairwise relatively prime integers $d_i > 1$. The family of such actions includes all fixed point free orthogonal circle actions on S^3; in fact, the orthogonal actions are precisely those for which there are at most two exceptional circles, and in all remaining cases the underlying homology spheres are not simply connected. In contrast to the orthogonal case, the number of exceptional orbits for an arbitrary action on a Seifert homology 3-sphere can be an arbitrary positive integer. There are several excellent descriptions of these manifolds in the literature, including Raymond's article [Ra], the book by P. Orlik [Or], and lecture notes by M. Jankins and W. Neumann [JN]. In higher dimensions one also has orthogonal pseudofree circle actions on S^{2n-1}: If $(d_1, \cdots, d_n)$ is a sequence of pairwise relatively prime positive integers, then the unit sphere in $\mathbb{C}^n$ for the linear action

$$t \cdot (z_1, \cdots, z_n) := (t^{d_1} z_1, \cdots, t^{d_n} z_n)$$

is free on the complement of m circles, where m is the number of integers d_i that are greater than one. Of course, it follows that the number of exceptional circles for an orthogonal pseudofree circle action on S^{2n-1} is at most n, and it is natural to ask if it is possible to construct pseudofree smooth circle actions on S^{2n-1} with larger numbers of exceptional circles for larger values of n. In [MnY1–2] D. Montgomery and C.-T. Yang succeeded in producing such examples when $n = 3$; a survey of this and analogous work in dimensions $(2n + 1) \geq 9$ appears in [DPS, Sec. 2] (also see [Pet1]). To complete the picture, we note that the case $n = 2$ (*i.e.*, the 5-dimensional case) reflects the basic difficulties of 4-dimensional topology. One can use M. Freedman's work on 4-dimensional topological surgery [FQ] to construct topological pseudofree circle actions on S^5 that are locally linear, and applications of gauge theory to smooth pseudofree circle actions on S^5 have also been considered (*cf.* [FS]; see also *Math. Reviews* **88e:**57032).

(2) Let p be an odd prime, and let D_{2p} be the dihedral group of order $2p$. If we are given a fixed point free, linear action of D_{2p} on $S(V) \approx S^{2n-1}$. then the fixed point sets of the order two subgroups are all $(n-1)$-spheres, and if H and K are two distinct subgroups of order two then $S(V)^H$ and $S(V)^K$ have linking number ± 1. If we are given an arbitrary fixed point free D_{2p} action on $M \approx S^{2n-1}$ such that the fixed point set of every (equivalently, of some) order two subgroup is an $(n-1)$-sphere, then Smith theory implies that the linking numbers of M^H and M^K are congruent to ± 1 mod p; results of J. Davis and T. tom Dieck [DtD, tD3] show that one can realize exotic linking numbers by smooth D_{2p} actions on the $(2n-1)$-sphere if $n \geq 3$. In contrast, special cases of Fact 1.2 imply that no such examples can exist if $n = 2$ and $p > 5$ (*e.g.*, see [DaMo]). Despite this, C. Livingston has shown that fixed point free, smooth D_{2p}-actions with exotic linking numbers exist on integral homology 3-spheres [Liv].

(3) If A_5 denotes the alternating group on 5 letters, then A_5 can be viewed as the group of isometries of a regular dodecahedron or icosahedron, and consequently there are natural realizations of A_5 as a subgroup of SO_3. The homogeneous space SO_3/A_5 is the well-known *Poincaré homology 3-sphere* that we shall denote by $\Sigma(2, 3, 5)$; summaries of the properties of this manifold appear in [Bre3, Sec. I.8] and [KiSc]. As noted in [Bre3, pp. 55–56], the action of A_5 on $\Sigma(2, 3, 5)$ obtained by restricting the transitive action of SO_3 has exactly one fixed point. Although smooth actions with one fixed point cannot exist on a smooth homotopy 3-sphere [BKS], the methods of geometric topology

have produced numerous examples of smooth actions on higher dimensional spheres with one fixed point during the past two decades; the first examples were due to E. V. Stein [Stn], with additional families of examples due to T. Petrie [Pet1–2] appearing shortly afterwards. We shall not attempt to summarize subsequent work here, but many further references appear in [BKS], applications to algebraic group actions are discussed in [DMP], and an article by E. Laitinen, M. Morimoto, and K. Pawałowski [LMP] provides the most recent information available at this time.

In the remaining sections of Part III we shall concentrate on the following question:

(‡) *What is the role of the standard one fixed point action on* $\Sigma(2,3,5)$ *in the family of all smooth one fixed point actions on homology* 3*-spheres?*

Standard considerations involving P. A. Smith cohomological fixed point theory, the local linearity of smooth actions near fixed points, and the subgroups of SO_3 show that A_5 is the only group that can act on a closed integral homology 3-sphere with exactly one fixed point.

One motivation for studying such actions is that symmetry considerations have led to interesting classes of 3-manifolds such as Seifert manifolds. On the other hand, smooth group actions with one fixed point have been studied extensively over the past two decades, both as test cases for the sorts of exotic smooth group actions that can exist on spheres and in connection with questions from algebraic transformation groups. In particular, recent work has shown that smooth one fixed point actions on nonsimply connected homology 3-spheres are rather exceptional low-dimensional examples. Standard geometrization results (*cf.* [Ed]) imply that finite group actions with a single fixed point do not exist on (homology) spheres of dimension 1 or 2, and more recent results of [BKS], [DM], and [Mto2] show the nonexistence of smooth actions with exactly one fixed point on homology 4- and 5-spheres. In contrast, such smooth actions exist on genuine spheres in all dimensions ≥ 6 (*e.g.*, see [Stn], [Pet2], [Mto2], [BaMo], and related examples of [BKS]).

A result of G. Bredon [Bre1] states that $\Sigma(2,3,5)$ is the only integral homology sphere that admits a transitive action of a compact Lie group that is not equivalent to an orthogonal action on a standard sphere. Since the one fixed point action of A_5 on $\Sigma(2,3,5)$ is the restriction of this exceptional transitive action, the evidence in this and the preceding paragraph may suggest that all one fixed point actions on homology 3-spheres are closely related to the standard examples in some fashion (*e.g.*, perhaps there is an equivariant degree one map into $\Sigma(2,3,5)$). However, our main results show the existence of many smooth one fixed point actions on irreducible homology 3-spheres. Some of these actions are clearly related to the standard actions on $\Sigma(2,3,5)$, but others are quite different.

2. Equivariant surgery in three dimensions

Our objective is to construct exotic examples of smooth one fixed point actions on homology 3-spheres by means of equivariant surgery and other techniques. Of course, it is well known that many basic results of surgery theory fail in dimension three; the purpose of this section is to summarize some aspects of surgery theory that are both valid and useful for 3-manifolds.

For many years geometric topologists have known that surgery theory applies to dimension three if one is willing to settle for homology equivalences rather than genuine homotopy equivalences; informal discussions of this appear in the writeup by F. Quinn on page 225 of the book containing [Brw2] and also in [FQ, p. 200, lines –3 to –1]. Here is a more formal statement that "homology surgery works for 3-manifolds."

Folk Theorem 2.1. *Suppose that $f : (M, \partial M) \to (X, \partial X)$ and appropriate bundle data determine a degree one normal map of compact 3-manifolds with boundary such that $\partial f : \partial M \to \partial X$ induces a $\mathbb{Z}[\pi_1(X)]$-homology equivalence (with possibly twisted coefficients). Then f is normally cobordant rel boundaries to a simple $\mathbb{Z}[\pi_1(X)]$-homology equivalence if the ordinary Wall surgery obstruction $\sigma(f) \in L_3^s(\mathbb{Z}[\pi_1(X)], w)$ is trivial.*

The proof of this follows directly from the methods of Wall's book for (2n+1)-manifolds with $n \geq 2$ subject to one complication: If one removes an embedded n-sphere from a $(2n+1)$-manifold, the fundamental group does not change if $n \geq 2$ but it usually changes drastically if $n = 1$. This means that one loses control of the fundamental group of the source manifold for the normal map, but the underlying homological arguments remain valid if we we work with twisted coefficients in the group ring $\mathbb{Z}[\pi_1(X)]$.■

In [BaMo] Bak and Morimoto formulate a version of this for equivariant surgery on 3-manifolds with orientation-preserving actions.

Theorem 2.2. *Let M and X be closed smooth G-manifolds with orientation-preserving actions of a finite group G, let $f : M \to X$ and suitable bundle data define a degree one equivariant surgery problem, where X is simply connected and f maps $\mathrm{Sing}(M)$ to $\mathrm{Sing}(X)$ by an equivariant homotopy equivalence. Furthermore, assume that the projective class group $\widetilde{K}_0(\mathbb{Z}[G])$ is zero. Then f is equivariantly normally bordant to a G-homotopy equivalence if an obstruction in the Bak group*

$$\mathbf{W}_3^h(\mathbb{Z}[G], \mathit{\Gamma} G(X); 1)$$

is trivial.■

As noted in Section 2.4, the Bak group $\mathbf{W}_3^h(\mathbb{Z}[G], \mathit{\Gamma} G(X); 1)$ is defined in [Ba] as a quotient of the Wall group $L_3^h(\mathbb{Z}[G], 1)$.

We shall need a slight extension of the preceding result:

Complement 2.3. *The preceding result remains valid if one merely assumes that X is obtained from a compact smooth 3-manifold by attaching G-free equivariant cells on the free part of X.*■

We shall also need some computational results for Bak groups and certain other algebraic K-theoretic groups. These are all established in [BaMo].

Theorem 2.4. *Let G be the alternating group A_5. Then the projective class group of $\mathbb{Z}[G]$, the Whitehead group of G, and the Bak group $\mathbf{W}^h_3(\mathbb{Z}[G], \mathit{\Gamma} G(X); 1)$ are all trivial.*■

In contrast to the preceding result, the Wall group $L^h_3(\mathbb{Z}[A_5], 1)$ is nontrivial, for $L^h_3(\mathbb{Z}[\mathbb{Z}_2], 1) \approx \mathbb{Z}_2$ implies that $L^h_3(\mathbb{Z}[\mathbb{Z}_2 \times \mathbb{Z}_2], 1) \supset \mathbb{Z}_2 \times \mathbb{Z}_2$, and by transfer considerations it follows that $L^h_3(\mathbb{Z}[A_5], 1)$ also contains a copy of $\mathbb{Z}_2 \times \mathbb{Z}_2$.■

3. Construction of exotic examples

Elementary considerations show that the one fixed point action on $\Sigma(2, 3, 5)$ is not the only smooth action on a homology 3-sphere with exactly one fixed point. For example, one can construct many examples by taking an equivariant connected sum of $60 = |A_5|$ copies of some homotopy sphere P^3 over the free part of the action on $\Sigma(2, 3, 5)$; this yields an infinite family of pairwise inequivalent one fixed point A_5-actions on homology 3-spheres. More generally, if H is an isotropy subgroup of the action on $\Sigma(2, 3, 5)$ and P^3 has a smooth action of H, then one can often form a *stratumwise equivariant connected sum* of $\Sigma(2, 3, 5)$ with $|G/H|$ copies of P along the fixed point sets of the conjugates of H; constructions of this type are used extensively by Meeks and Yau in [MeY1, Sec. 9]. In particular, one can take $H = A_4$ and obtain a one fixed point action on a connected sum of 6 copies of $\Sigma(2, 3, 5)$; we mention this because it appears to be the simplest example of an integral homology 3-sphere that supports a smooth one fixed point action and has a vanishing Rochlin invariant.

The preceding examples are all obtainable from $\Sigma(2, 3, 5)$ by familiar sorts of constructions. In fact, there have been some informal regularity conjectures that all one smooth fixed point actions on homology 3-spheres are somehow modeled after $\Sigma(2, 3, 5)$. Perhaps the weakest of these conjectures is that the singular set of such an action is always equivalent to $\mathrm{Sing}\,(\,\Sigma(2, 3, 5)\,)$. The main result of this section provides a negative answer to this particular question and describes all possible fixed point sets.

Theorem 3.1. *There are exactly four equivariant homeomorphism classes of singular sets for smooth A_5-actions on $\mathbb{Z}$-homology 3-spheres with exactly one fixed point.*

The possibilities for the singular set may be described as follows: Suppose we are given a smooth action of A_5 on the homology 3-sphere Σ^3 with exactly one fixed point. For each subgroup C of order 2 the fixed set of C is the union of two semicircles with two endpoints in common; one of the endpoints is the fixed point of the A_5-action, and the other is fixed under the unique subgroup of A_5 that contains C and is isomorphic to the alternating group on four letters. Each semicircle also contains a point that is fixed under a subgroup of order 6 containing C and a point that is fixed under a subgroup of order 10 containing C. The union of all fixed sets of order 2 subgroups consists of 30 semicircles, and A_5 acts transitively on this set of semicircles. On the other hand, a direct analysis shows that there are exactly four ways of constructing an A_5-orbit of data $(\Gamma_C; x_6, H_6, x_{10}, H_{10})$, where C is a subgroup of order 2, Γ_C is homeomorphic to [0,1], the points x_6 and x_{10} belong to $\{\frac{1}{3}, \frac{2}{3}\}$, and H_6 and H_{10} are subgroups of order 6 and 10 respectively containing C. Furthermore, Smith theory implies that for each class of semicircles there is a unique 1-complex with cell-preserving A_5-action that is a potential singular set for a smooth action on a homology 3-sphere with one fixed point.

Sketch of proof of Theorem 3.1. Each of the 1-complexes K in the preceding paragraph can be realized as the singular set of a smooth action on *some* closed oriented 3-manifold by an elementary "rewiring" construction on $\Sigma(2,3,5)$ with its standard one fixed point action. Let A_K be the manifold so obtained; it follows from the construction that $H_1(A_K;\mathbb{Q}) \neq 0$. One can then use machinery developed by R. Oliver [OL] to add equivariant cells along the free part of A_K and obtain an equivariant CW complex B_K with the same structure on and near the singular set and such that B_K is homotopy equivalent to S^3; if we split A_K equivariantly as $D \cup_\partial E$ where D is a linear disk about a fixed point, the equivariant cells can all be added over E and one obtains a corresponding splitting $D \cup_\partial E'$ where E' is contractible. Therefore the inclusion of A_K in B_K can be viewed as a map of triads, and from this it follows that the inclusion is an isomorphism on $H_3(-;\mathbb{Z})$ and can be viewed as a map of degree one. In order to make this into an equivariant surgery problem, it is necessary to introduce some equivariant bundle data; the details of the rewiring construction imply that the equivariant tangent bundle is stably isomorphic to a product bundle on the complement of a finite invariant subset F, and it follows that equivariant bundle data can be given by crossing the map $A_K - F \to B_K$ with the identity on Ω for a suitable A_5-representation Ω. This suffices for surgery-theoretic purposes because the latter involve maps from positive-codimensional manifolds into A_K and such maps can always be deformed to avoid a finite subset (similar considerations arise in [DR], where bundle data with deficiencies are discussed in greater detail). The results of Bak and Morimoto [BaMo] (cited in Section 2) now show that

- (i) one can do equivariant surgery away from the singular set of A_K to convert the map $A_K \to B_K$ into a $\mathbb{Z}$-homology equivalence if an obstruction in some quotient group of the Wall group $L_3^h(A_5;1)$ – specifically, the associated Bak group $W_3^h(A_5, \varGamma G(X);1)$ – is zero,
- (ii) the Bak group in (i) is equal to zero.

Therefore one can modify A_K by equivariant surgery away from the singular set to obtain a homology sphere with a smooth one fixed point action.■

Fixed point free actions on homology 3-spheres

The methods of this section also yield results about fixed point free actions of finite groups on homology 3-spheres. Some of this is work in progress, but the results for

$G = A_5$ can be stated fairly simply. There is a unique linear action of this type; namely, let V be the orthogonal complement of the diagonal in $\mathbb{R}^5$ where A_5 acts on the latter by permuting the coordinates, and take the induced action on the unit sphere $S(V)$. It is immediately clear that one can find examples with exotic orbit structures; in particular, this can be done by taking an equivariant connected sum of two one fixed point actions on homology 3-spheres at the fixed points. The methods of this section show that *the singular set of a fixed point free A_5-action on an integral homology 3-sphere can be an arbitrary 1-dimensional A_5-complex that satisfies the necessary conditions imposed by Smith theory*; the list of all such possibilities is fairly short, but it does contain more than the singular set of the linear action and the singular sets obtained from the equivariant connected sums described above. One reason for interest in such actions involves the linearity question for smooth actions of finite groups on S^3 (*cf.* [DaMo], [Fn], [KwS6]); fixed point free A_5-actions represent one basic type that is included in Thurston's announcement [Th3] but has not yet been verified elsewhere.

4. Actions on hyperbolic homology spheres

One obvious drawback of Theorem 3.1 is that the argument does not yield explicit examples of actions with exotic singular sets. In particular, it is natural to ask if such examples can be found on homology 3-spheres that are *irreducible* and geometric in the sense of Thurston [Th2]. More specifically, Thurston's hyperbolization theorem [Th4] implies that one can often find hyperbolic 3-manifolds with certain topological or geometric properties by applying suitable conversion procedures to general 3-manifolds with such properties, and in this connection one would like to know if smooth one fixed point actions can be found on hyperbolic homology 3-spheres. According to the main result of this section (Theorem 4.3), such examples can be found. The results of this section generate a variety of questions; some examples are presented after the proof of Theorem 4.3.

One can interpret Theorem 3.1 as a negative answer to questions about the existence of a single basic model for one fixed point actions on homology 3-spheres. However, the following result shows that the one fixed point actions on irreducible homology 3-spheres form a family of models for all such actions.

Theorem 4.1. *If Σ^3 is a closed integral homology 3-sphere with a smooth one fixed point action of A_5, then Σ^3 is equivariantly diffeomorphic to an iterated equivariant connected sum along strata of the form $\Sigma_0 \# |G/H_1|\Sigma_1 \cdots \# |G/H_r|\Sigma_r$, where Σ_i is an irreducible homology 3-sphere with a smooth action of H_i if $i > 1$ and Σ_0 is an irreducible homology 3-sphere with a smooth one fixed point action of A_5.*

In particular, it follows that every one fixed point action on a homology 3-sphere has an *irreducible nucleus* given by a smooth one fixed point action on some connected summand of Σ.

The proof of Theorem 4.11 follows from the standard equivariant analogs of the Papakyriakopoulos Sphere Theorem [MeY2, JR] and an analysis of the ways in which the invariant separating spheres in Σ can meet the singular set of the action; this set turns out to be a 1-dimensional finite cell complex with a cell-preserving group action (that can be equivariantly subdivided to yield an equivariant regular simplicial action in the sense of, say, [Bre3, Ch. III]).■

In view of Theorem 4.1 and Thurston's Geometrization Conjecture [Th1–2], it is natural to ask next about one fixed point actions on irreducible geometric homology spheres. These manifolds fall into three distinct classes; namely, *Seifert fibered*, *non-simple Haken* (in other words, containing an incompressible torus), and *hyperbolic*. Since the Poincaré homology 3-sphere $\Sigma(2,3,5)$ is an example of a Seifert fibered 3-manifold (in fact, it is a Brieskorn variety), the most immediate question is whether other Seifert fibered homology 3-spheres support smooth one fixed point actions. This question has a simple negative answer:

Proposition 4.2. *Let Σ^3 be a Seifert fibered homology 3-sphere. Then Σ^3 admits a smooth action of a finite group with one fixed point if and only if Σ^3 is diffeomorphic to the Poincaré homology 3-sphere $\Sigma(2,3,5)$.*

This follows from considerations involving the fundamental group of Σ and the orbifold fundamental group associated to the group action.■

In contrast to the preceding result, situation is completely different for hyperbolic and Haken homology 3-spheres:

Theorem 4.3. (i) *There exist infinitely many irreducible non-simple Haken homology 3-spheres that support smooth actions of A_5 with exactly one fixed point.*

(ii) *There exist infinitely many irreducible hyperbolic homology 3-spheres that support smooth actions of A_5 with exactly one fixed point.*

Sketch of proof of Theorem 4.3. The constructions of examples depend heavily on Theorem 3.1, Theorem 4.1, and the nonexistence of smooth one fixed point actions on homotopy 3-spheres [BKS]. When combined, these imply that *every possible singular set is realized by a smooth one fixed point A_5-action on a nonsimply connected, irreducible homology 3-sphere.*

To prove statement (i), one first takes a simple closed curve in the free part of $\Sigma(2,3,5)$ that represents a nonzero element of $\pi_1(\Sigma(2,3,5))$, then forms a connected sum with a knot in some small coordinate 3-disk, and afterwards deforms it to be disjoint from all its translates under the action of A_5. Next, one takes a closed invariant tubular neighborhood U of these $|A_5|$ pairwise disjoint curves and replaces the interior of U with the $|A_5|$ copies of the interior of some nontrivial knot complement. One can do this such that the manifold in question becomes an irreducible homology sphere and the components of the boundary of U become incompressible tori.

The proof of statement (ii) is somewhat more delicate. As in the preceding discussion, one removes a suitably chosen union $\operatorname{Int} U$ of invariant open solid tori from the free part of the action to obtain a bounded Haken manifold V with a smooth one fixed point action and $\partial V = A_5 \times T^2$. Next one uses the splitting theorems of W. Jaco and J. H. Rubinstein [JR] to construct an equivariant splitting of V along incompressible tori into Seifert fibered and hyperbolic pieces; the hyperbolicity assertion uses Thurston's recognition principle for hyperbolic 3-manifolds ([Th4]; also see C. McMullen's article [McM] and the references cited there). The equivariant geometrization results of W. Meeks and G. P. Scott [MS] then imply that the fixed point of the induced action on V lies in a hyperbolic piece, say V_0, and an argument involving the dual graph of the splitting implies that one can attach solid tori to the boundary components of V_0 (nonequivariantly) to obtain a homology 3-sphere. Thurston's results on Dehn fillings [Th1–2] then imply that infinitely many such attachments yield a hyperbolic manifold; furthermore, elementary considerations imply that these Dehn fillings will yield integral homology spheres, and a more detailed analysis also shows that infinitely many of these constructions can be done equivariantly.■

The preceding results generate a variety of questions. Here are two examples:

(1) If Σ is an irreducible homology 3-sphere with a smooth A_5-action with one fixed point, is the Rochlin invariant always equal to 1? This is true for all examples checked thus far. Similarly, one can ask about the Casson invariant [AM] or other invariants from topological quantum field theory.

(2) In [Th3] Thurston announced results implying that the one fixed point actions on the hyperbolic homology 3-spheres of Theorem 4.3 are hyperbolic structure preserving. There are many examples of one fixed point actions on spheres in dimensions ≥ 6. Can one use equivariant surgery to convert such actions into one fixed point actions on hyperbolic homology spheres that preserve a hyperbolic structure? Results of M. Davis and T. Januszkiewicz [DJ] provide a means for converting manifolds and orbifolds to objects that are hyperbolic in the sense of M. Gromov [Gr] (also see [Bow], [GH]), and these suggest that one can find at least some hyperbolic one fixed point actions on higher dimensional integral homology spheres. More generally, it would be interesting to know which of the many exotic smooth finite group actions on spheres are simply connected analogs of hyperbolic actions on hyperbolic homology spheres.