Isovariant Homotopy Theory, the Gap Hypothesis, and Rational Classification Problems

Two basic types of problems in geometric topology:

(1) Existence of objects with certain properties.

(2) Classification of objects similar to standard examples up to isomorphism.

For (topological or smooth) manifolds, surgery theory is a collection of tools for analyzing such problems.

Want to modify and extend things to manifolds with group actions (all smooth here). This ultimately requires a well-developed understanding of G-equivariant algebraic topology when G is a compact Lie group.

SIMPLIFYING ASSUMPTION. Default hypothesis that actions be semifree (the group acts freely off the fixed point set). Note that if p is prime, then all actions of $\mathbb{Z}/p$ must be semifree.—Can handle more complicated orbit structure using the ideas in the special case plus inductive machinery.

FOCUS ON CLASSIFICATION. In geometric topology, one productive topic is the classification of (topological or smooth) manifolds with a given homotopy type. One can ask a similar question for manifolds with smooth actions of finite, or more generally compact Lie, groups. As the talk's title should suggest, this is the focus here.

Two approaches which developed fairly early:

(A) Look at a refinement of equivariance called **isovariance**. An equivariant map f satisfies $f(g \cdot x) = g \cdot f(x)$, so the isotropy subgroups satisfy $G_x \subset G_{f(x)}$ for all x. Isovariant means that equality always holds.

(B) The standard **Gap Hypothesis**: For semifree actions this reduces to $\dim M \geq 2 \cdot \dim M^G + 2$.

Note that the fixed set M^G is a union of pairwise disjoint submanifolds for smooth actions; the dimensions of the pieces may vary, and $\dim M^G$ is the highest of these numbers.

Question. How are these related?

Need tools to work with isovariant homotopy theory as "a homotopy theory." By Dula-RS, for smooth actions this is equivalent to a category of diagrams in equivariant homotopy theory. [Figure 1] We then have the following evidence of some relationship:

Theorem. (S. Straus, W. Browder) *Aside from some low-dimensional issues, an equivariant homotopy equivalence of smooth semifree G-manifolds is equivariantly homotopic to an isovariant homotopy equivalence* PROVIDED *the (standard) Gap Hypothesis holds. A similar result holds for equivariant homotopies of such maps under a slightly stronger hypothesis.* [Figure 2]

What happens if the Gap Hypothesis does not hold?

Some differences almost certain. However, if $\Delta = \dim M - 2 \cdot \dim M^G$ is not too negative, things sometimes do not behave too badly. (A. Bak - M. Morimoto in odd dims, K. H. Dovermann - RS in even dimensions). But if the difference gets a little larger, then things really start to go wrong.

> (M. Rothenberg - S. Weinberger) Breakdown of a fundamental surgery result (an equivariant $\pi - \pi$ Theorem).
>
> (RS) Examples of equivariantly homotopy equivalent $\mathbb{Z}/2$-manifolds which are not isovariantly homotopy equivalent.

A recent result of S. Safii (Ph. D. Thesis, UC Riverside, 2015) shows that some things go wrong right away:

Theorem. *Let $\Delta \leq 1$. Then there is a smooth simply connected G-manifold M with $\Delta(M) = \Delta$, an orientation preserving (semifree) smooth G-action on M, and a G-equivariant homotopy equivalence $f : N \to M$ which is not equivariantly homotopic to an isovariant homotopy equivalence.*

One consequence: The conclusion of the (general) equivariant $\pi - \pi$ Theorem is false for orientation preserving actions whenever the Gap Hypothesis fails. — If it were true for, say, $\Delta = 1$, then one could improve the range for which the result of Straus and Browder is valid.

One conclusion might be to look more closely at the classification of smooth G-manifolds with a given **isovariant** homotopy type if we want to go beyond the Gap Hypothesis range.

Decent classification schemes exist, but they immediately run into formidable homotopy problems like understanding the higher homotopy groups of spheres. These can be handled in specific cases. To get a general picture one can follow D. Sullivan's approach and look for complete sets of **rational invariants** which classify objects up to finite ambiguity.

For example, higher dimensional simply connected closed manifolds are given up to finite ambiguity by

(a) a grade commutative differential graded algebra which describes the rational homotopy type,

(b) a bound on the torsion in the integral (co)homology,

(c) an integral lattice in the first item which gives integral information on the homotopy type,

(d) candidates for the rational Pontryagin classes.

Each of the last three conditions arises naturally. The rational homotopy type does not catch torsion in the cohomology, and one can have torsion of all orders. Likewise, if two cohomology classes have a nonzero cup product of infinite order, then the cup product in integral cohomology may be any nonzero multiple of an additive generator in cohomology. The first three specify the actual homotopy type up to finite ambiguity. Since homotopy equivalent manifolds may have different rational Pontryagin classes, the fourth piece of data is also needed. Simply connected surgery then shows that the four conditions suffice to describe the homeomorphism or diffeomorphism type up to finite ambiguity in higher dimensions.

M. Rothenberg and G. Triantafillou established an equivariant analog of this result in the 1990s.

Theorem. *Let G be a finite group, and consider closed smooth G-manifolds such that every fixed point set of every subgroup is simply connected and higher dimensional, and assume further that the Gap Hypothesis holds. Then there is a notion of equivariant simple homotopy type which refines the model in* (a) *such that analogs of the data yield a finite ambiguity classification for G-manifolds up to equivariant* almost diffeormorphism. [Figure 3]

An **almost diffeomorphism** means a homeomorphism which is a diffeomorphism except at a finite set of fixed points. — The "almost" can be removed for orientation preserving $\mathbb{Z}/p$ actions on even dimensional manifolds (RT), and in the odd dimensional case one can sometimes do this by introducing an invariant called a **G-signature defect** (M. Masuda-RS).

Two issues arise. (1) The need to assume the Gap Hypothesis. (2) The need to assume that every fixed point set of every subgroup is connected. — Even for standard examples, the fixed point sets might have many components. One simple class of examples arises by starting with finite linear actions of a group G on $\mathbb{C}^{n+1}$ and considering the induced quotient manifold actions on $\mathbb{CP}^n$.

"Work in progress" to suggest the usefulness of isovariant homotopy theory in dealing with both issues:

Start with some restrictions like semifree, sufficiently large dimensions, all components of fixed point sets are simply connected (true for previously described actions on $\mathbb{CP}^n$), codimension ≥ 3 Gap Hypothesis ($\dim M \geq \dim M^G - 3$), and (**technically useful but conceptually undesirable**) each map $x \to g \cdot x$ is homotopic to the identity (but the homotopy need not be equivariant).

Theorem. *Under these conditions, the following data yield a finite ambiguity classification for up to equivariant almost diffeormorphism:*

(*a*) *A diagram of grade commutative differential graded algebras resembling the Dula-Schultz diagrams.*

(*b*) *Reidemeister torsion invariants (which yield an equivariant simple homotopy type up to finite ambiguity).*

(*c*) *Torsion bounds, integral lattices, candidates for rational characteristic classes.*

(*d*) *For each component* M_α *of* M^G*, a twisting invariant* $\Phi_\alpha \in H^{q(\alpha)-1}(M_\alpha;\mathbb{Q})$*, where* $q(a) = \dim M - \dim M_\alpha$.

The last invariant is elusive, and it represents the obstruction to deforming an isovariant homotopy equivalence (isovariantly) to a transverse linear isovariant map in the sense of W. Browder and F. Quinn. There are many conditions under which it is automatically zero. For example, the invariant lies in a zero group if the Gap Hypothesis holds, and it automatically vanishes when $G = \mathbb{Z}/2$ or $\mathbb{Z}/3$. However, if M is a sphere with a linear G-action, this invariant can be nonzero.