Smooth Atlases and Global Smoothness

This note provides a more detailed account of some about smooth structures and related topics. Several of these are stated as exercises in Conlon. Although the proofs only involve fairly elementary considerations, they are sometimes relatively long and it might not be immediately apparent how to set up the arguments.

Unless stated otherwise, "smooth" means C^∞ in this note.

Given a topological n-manifold M, an *atlas* (more recisely, a *topological atlas*) for M is a collection $\mathcal{A}$ of ordered pairs (U_α, h_α) such that each U_α is homeomorphic to an open subset in $\mathbf{R}^n$, each h_α is a homeomorphism from U_α to an open subset of M, and the sets $h_\alpha(U_\alpha)$ form an open covering of M. The pairs (U_α, h_α) are called the *charts* or the atlas.

By construction every topological manifold has a topological atlas, and there is an obvious maximal topological atlas: Simply take all (U, h) so that U is open in $\mathbf{R}^n$ and h is a homeomorphism from U onto an open subset of M. Clearly this is also the **only** maximal topological atlas.

The related concept of *smooth atlas* is fundamental to this course. A topological atlas $\mathcal{A}$ is said to be *smooth* if for all pairs (U_α, h_α) and (U_β, h_β) in $\mathcal{A}$ the transition homeomorphisms

$$\psi_{\beta\alpha} : h_\alpha^{-1}\left(h_\beta(U_\beta)\right) \to h_\beta^{-1}\left(h_\alpha(U_\alpha)\right)$$

defined by $\psi_{\beta\alpha}(x) = h_\beta^{-1}(h_\alpha(x))$ are smooth diffeomorphisms.

A smooth atlas turns out to be the additional structure one needs to talk about smooth mappings on manifolds. However, if we simply say that a smooth structure is given by a topological manifold M and a smooth atlas for M, we will end up with a lot of redundancy that is at best clumsy and at worst confusing. Perhaps the simplest examples involve open subsets in Euclidean spaces. If U is open in $\mathbf{R}^n$ then the simplest example of a smooth atlas is just $(U, 1_U)$. However, if U_α is an arbitrary open covering of U and $i_\alpha : U_\alpha \to U$ is the inclusion map, then each family (U_α, i_α) is also a smooth atlas. Our definition of smooth structure should be formulated so that an open subset in $\mathbf{R}^n$ has a unique associated smooth structure rather than a multitude of smooth structures given by all possible open coverings as well as even larger atlases.

The same can be said for the sphere S^n. One example of a smooth atlas for S^n is given by the stereographic projection charts in Example 1.2.3 of Conlon (see p. 3). Another was given in the lectures, with charts of the form

$$h_{i\pm} : \{|x| < 1\} \to S^n$$

such that $h_{i\pm}$ sends the first $(i-1)$ coordinates to themselves, shifts the remaining coordinates to the $(i+1)$ through $(n+1)$ coordinates on $S^n \subset \mathbf{R}^{n+1}$ and inserts $\pm\sqrt{1-|x|^2}$ in the i-th coordinate.

In analogy with the topological case, one would like to have a **universal** smooth atlas associated to a smooth structure. This is given by the following result:

THEOREM. *If $\mathcal{A}$ is a smooth atlas for the n-manifold M, then there is a unique* MAXIMAL *smooth atlas $\mathcal{A}'$ containing $\mathcal{A}$. A chart (V, k) belongs to $\mathcal{A}'$ if and only if for each chart (U_α) in $\mathcal{A}$ the associated transition maps from $h_\alpha^{-1}(k(V)))$ to $k^{-1}(h_\alpha(U_\alpha))$ and vice versa are diffeomorphisms.*

Proof. By construction the set $\mathcal{A}'$ contains $\mathcal{A}$. There are three things to prove:

(1) If $\mathcal{A}'$ is defined as in the statement of the theorem, then it is a smooth atlas.
(2) $\mathcal{A}'$ is a maximal smooth atlas.
(3) $\mathcal{A}'$ is the only maximal smooth atlas containing $\mathcal{A}$.

Verification of (1). We need to show that if (V,k) and (W,ℓ) are charts in $\mathcal{A}'$ then the transition map "$\ell^{-1}\circ k$" from $k^{-1}(\ell(W))$ to $\ell^{-1}(k(V))$ is smooth. Let x be an arbitrary point in the first subset.

By the definition of a smooth atlas there is a smooth chart (U_α, h_α) in $\mathcal{A}$ such that $k(x) \in h_\alpha(U_\alpha)$; *i.e.*, $x \in k^{-1}\ell(W) \cap k^{-1}(h_\alpha(U_\alpha))$. Under the transition map from $k^{-1}(\ell(W))$ to $\ell^{-1}(k(V))$ the subset $k^{-1}(\ell(W)) \cap k^{-1}(h_\alpha(U_\alpha))$ is mapped to $\ell^{-1}(k(V)) \cap \ell^{-1}(h_\alpha(U_\alpha))$. On these subsets one can express "$\ell^{-1}\circ k$" as a composite ("$\ell^{-1}\circ h_\alpha$")$\circ$("$h_\alpha^{-1}\circ k$") and by the defining condition for $\mathcal{A}'$ it follows that both of these composites are diffeomorphisms. Therefore the transition map "$\ell^{-1}\circ k$" is smooth on the open set $k^{-1}(\ell(W)) \cap k^{-1}(h_\alpha(U_\alpha))$ and hence is smooth near x. Since x was arbitrary this implies that the transition map is smooth everywhere.

Verification of (2). If we try to add another chart (N,φ) to $\mathcal{A}'$ then the defining condition for the latter implies that at least one of the transition maps "$\varphi^{-1}\circ h_\alpha$" is not a diffeomorphism. But this means that $\mathcal{A}'$ with the extra chart does not form a smooth atlas.

Verification of (3). Suppose that $\mathcal{H}$ is an arbitrary maximal smooth atlas containing $\mathcal{A}$, and let (N,φ) be a chart in $\mathcal{H}$. Since the latter contains $\mathcal{A}$ it follows that all of the transition maps "$\varphi^{-1}\circ h_\alpha$" are diffeomorphisms. But this implies that (N,φ) is a chart in $\mathcal{A}'$, which in turn implies $\mathcal{H} \subset \mathcal{A}'$. Since both are maximal they must be equal.

We may now **DEFINE** a **smooth n-manifold** to be a pair $(M,\mathcal{A})$ where M is a topological n-manifold and $\mathcal{A}$ is a maximal smooth atlas on M. Frequently one says that $\mathcal{A}$ is a *smooth structure* on M. The charts in the maximal atlas are said to be *smooth charts* for the smooth manifold or smooth structure.

The following result is an elementary but very useful consequence of the definitions, and its proof is left as an exercise.

Lemma. *Let $\mathcal{A}$ be a maximal atlas for M, let (U,h) be a chart in $\mathcal{A}$, and let V be an open subset of U. Then $(V,h|V)$ is also a chart in $\mathcal{A}$.*

The next step is to define smooth maps of smooth manifolds. Suppose that $(M,\mathcal{A})$ and $(N,\mathcal{B})$ are smooth manifolds,let $\mathcal{B}_0$ be a smooth subatlas of $\mathcal{B}$, and let $f : M \to N$ be a continuous function. The preceding lemma and continuity imply the existence of a smooth atlas $\mathcal{A}_0 \subset \mathcal{A}$ such that for each smooth chart (U,h) in $\mathcal{A}_0$ there is a smooth chart $(V,k) \in \mathcal{B}_0$ such that $f(h(U)) \subset k(V)$ (explain why this is true!).

We now **DEFINE** a **smooth map of smooth manifolds** from $(M,\mathcal{A})$ to $(N,\mathcal{B})$ to be a continuous map $f : M \to N$ with the following additional property:

($\star$) Given smooth atlases $\mathcal{B}_0$ and $\mathcal{A}_0$ as above and charts (U,h) in $\mathcal{A}_0$ and $(V,k) \in \mathcal{B}_0$ such that $f(h(U)) \subset k(V)$, the associated map "$k^{-1}\circ f\circ h$": $U \to V$ is smooth.

This definition is very useful for proving that a continuous map is smooth because it allows one to choose the smooth atlases $\mathcal{B}_0$ and $\mathcal{A}_0$ in a convenient manner. However, this freedom of choice is also a potential shortcoming because one can ask what might happen if another pair of atlases is chosen. The next result implies that if ($\star$) holds for one choice of atlases it holds for all such choices.

Proposition. *Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manirolds, let $f : M \to N$ be continuous map, and assume that $(\star)$ holds for suitably chosen subatlases. Then $(\star)$ also holds for all smooth charts (U_α, h_α) in $\mathcal{A}$ and $(V_\beta, k_\beta) \in \mathcal{B}$ such that $f(h_\alpha(U)) \subset k_\beta(V)$.*

Proof. Take arbitrary charts (U_α, h_α) in $\mathcal{A}$ and $(V_\beta, k_\beta) \in \mathcal{B}$ such that $f(h_\alpha(U)) \subset k_\beta(V)$, and let $x \in U_\alpha$. Choose charts (N, p) and (Ω, q) in $\mathcal{A}_0$ and $\mathcal{B}_0$ respectively such that $x \in p(N)$, $f(x) \in q(\Omega)$ and $f(p(N)) \subset q(\Omega)$. It will suffice to show that the restriction of "$k_\beta^{-1} \circ f \circ h_\alpha$" to $h_\alpha^{-1}(p(N))$ is smooth (because x is an arbitrary point in U_α and the set in question is an open neighborhood of x).

By hypothesis the map "$q^{-1} \circ f \circ p$" is smooth. Furthermore, as in the proof of the preceding theorem, on the open subset $h_\alpha^{-1}(p(N))$ the composite "$k_\beta^{-1} \circ f \circ h_\alpha$" may be written as a composite ("$k_\beta^{-1} \circ q$")$\circ$("$q^{-1} \circ f \circ p$")$\circ$("$p^{-1} \circ h_\alpha$"); the middle factor is smooth by hypothesis, and the first and last factors are smooth because they are transition maps in the maximal atlases. Therefore the factors of the threefold composite are all smooth maps, and hence the cmposite itself is smooth.

One can now define diffeomorphisms exactly as for open sets in Euclidean spaces: They are smooth maps with smooth inverses. It is also possible to formulate definitions of submersions and immersions that work for smooth manifolds, but these will be easier to state in terms of tangent spaces, which are constructed in Chapter 3 of Conlon.

The following result is left to the reader as an exercise:

THEOREM. *If $(M, \mathcal{A})$ is a smooth manifold, then the identity map on M is smooth. If $f : (M, \mathcal{A}) \to (N, \mathcal{B})$ and $g : (M, \mathcal{B}) \to (L, \mathcal{C})$ are smooth maps of smooth manirolds, then the composite $g \circ f$ is smooth.*

Constructions for smooth manifolds

In the theory of topological spaces it is often useful and important to construct new examples of topological spaces out of old ones. Not all of the constructions from point set topology have analogs for smooth manifolds, but there are several important cases where such analogs exist.

RESTRICTIONS TO OPEN SUBSETS. *Let $(M, \mathcal{A})$ be a smooth manifold, and let Ω be an open subset of M. Then there is a smooth atlas $\mathcal{A}|\Omega$ on Ω such that*

(i) the inclusion map $j : (\Omega, \mathcal{A}|\Omega) \to (M, \mathcal{A})$ is smooth,

(ii) if $(L, \mathcal{C})$ is a smooth manifold and $g : L \to \Omega$ is continuous, then g is smooth if and only if $j \circ g$ is smooth.

One simply takes $\mathcal{A}|\Omega$ to be the set of all charts (U, h) in $\mathcal{A}$ such that $h(U) \subset \Omega$.

PRODUCTS. *Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds. Then there is a smooth atlas $\mathcal{P}$ on $M \times N$ such that the coordinate projection maps $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are smooth, and more generally a continuous map $f : P \to M \times N$ is smooth if and only if the coordinate functions $\pi_M \circ f$ and $\pi_N \circ f$ are smooth.*

The idea of the construction is simple: A smooth atlas for the product $M \times N$ can be constructed using charts of the form $(U_\alpha \times V_\beta, h_\alpha \times k_\beta)$ where (U_α, h_α) ranges over all charts in the maximal atlas for M and (V_β, k_β) ranges over all charts in the maximal atlas for N.

Note. Product constructions are possible for smooth manifolds only if the number of factors is finite.

COVERING SPACES. *Let $p : E \to M$ be a covering space projection where M is a topological manifold and E is Hausdorff, and let $\mathcal{A}$ be a maximal smooth atlas for M. Then there is a smooth atlas $\mathcal{E}$ on E such that if (U, h) is a coordinate chart for M such that $h(U)$ is evenly covered, then the restriction of p to each sheet of $p^{-1}(h(U))$ is a diffeomorphism.*

Here again the basic idea is fairly straightforward. Consider the subatlas $\mathcal{A}_0$ of $\mathcal{A}$ consisting of all charts (U, h) so that $h(U)$ is evenly covered. For each sheet W_γ of $p^{-1}(h(U))$ let s_γ be an inverse to $p|W_\gamma$. Then $\mathcal{E}_0$ is defined to be the set of all pairs $(U, s_\gamma \circ h)$ where (U, h) is a chart in $\mathcal{A}_0$; $\mathcal{E}$ is then defined to be the unique maximal atlas containing $\mathcal{E}_0$.

The following result is left to the reader as an exercise:

Proposition. *In the setting above, suppose that $(N, \mathcal{B})$ is a smooth manifold and that $f : N \to E$ is continuous. Then f is smooth if and only if $p \circ f$ is smooth.*

In particular, if we have a smooth map into the base of a covering space projection and the map lifts continuously, then the lifting is always smooth.

FREE QUOTIENTS. *Let $(M, \mathcal{A})$ be a smooth manifold, let G be a finite group, and let G act freely on G by diffeomorphisms: More precisely there is a family of diffeomorphisms Φ_g indexed by G so that $\Phi_{gh} = \Phi_g \circ \Phi_h$, Φ_1 is the identity, and if $g \neq 1$ then $\Phi_g(x) \neq x$ for all $x \in M$. Let M/G be the quotient space of M by this action; then topological considerations imply that the quotient projection map $p : M \to M/G$ is a covering space projection and M/G is a topological manifold of the same dimension as M. Then there is a smooth atlas $\mathcal{A}^*$ on M/G such that*

(i) $p : (M, \mathcal{A}) \to (M/G, \mathcal{A}^*)$ is smooth,

(ii) if $(N, \mathcal{B})$ is a smooth manifold and $f : M/G \to N$ is continuous, then f is smooth if and only if $f \circ p$ is smooth.

In this case one chooses the atlas $\mathcal{A}^*$ to consist of all charts (U, k) such that $k(U)$ is evenly covered and there is a smooth chart (U, h) in $\mathcal{A}$ such that $k = p \circ h$.

The most basic example of this situation is given by the *real projective plane* $\mathbf{RP^2}$, which can be viewed as the quotient of S^n by the action of the group $G = \{\pm 1\}$ by scalar multiplication.

Generalization of the preceding concept. Suppose we have a *regular* covering space projection $p : E \to B$ where both spaces are topological n-manifolds; the regularity condition implies that there is a group of covering transformations Γ acting on E such that M is homeomorphic to the qultient space E/Γ. Assume also that we have a smooth structure $\mathcal{B}$ on E such that the covering transformations are all *diffeomorphisms* from E to itself. If we let $\mathcal{A}$ be the set of all charts (U, k) on M such that $k(U)$ is evenly covered and there is a smooth chart (U, h) in $\mathcal{B}$ such that $k = p \circ h$, then $\mathcal{A}$ is a smooth atlas for M. Properties (i) and (ii) above remain true in this setting. Furthermore, if we construct the maximal smooth atlas $\mathcal{E}$ on E asociated to $\mathcal{A}$ as above, then $\mathcal{E}$ is equal to $\mathcal{B}$.

Important special case. Let $f : M \to M$ be a diffeomorphism, and consider the regular quotient space

$$T_f = M \times \mathbf{R}/\mu_f$$

where μ_f is the equivalence relation $(x, s) \sim (y, t)$ if and only if there is an integer n such that $s = t + n$ and $y = f^n(x)$. The quotient space is called the *mapping torus* of f. Perhaps the simplest nontrivial example of this is the Klein bottle, for which $M = S^1$ and f is complex conjugation.

Exercise. Prove that T_f is Hausdorff. [*Hint:* First show that there is a well-defined continuous map q from T_f to S^1 taking the equivalence class of (x, t) to $\exp(2\pi i t)$. Suppose $u \neq v$ in T_f. If $q(u) \neq q(v)$ then there are disjoint open neighborhoods U and V of these points in S^1, and their inverse images $q^{-1}(U)$ and $q^{-1})(V)$ are disjoint open neighborhoods of u and v. On the other hand, if $q(u) = q(v) = z$ and W is the open semicirular arc centered at z, then $q^{-1}(W)$ is homeomorphic to $(-1, 1) \times M$, which is Hausdorff.] Note that this proof only requires f to be a homeomorphism.

One can check directly that the quotient space projection is a regular covering space projection for which the group of covering transformations is the infinite cyclic group generated by $\varphi(x, t) = (f(x), t+1)$. Therefore the preceding discussion yields a smooth structure on the mapping torus of a smooth self-diffeomorphism.

Exercise. Suppose that M is connected with fundamental group G, and suppose that f is basepoint preserving. Show that T_f has a fundamental group Γ such that G is a normal subgroup, Γ/G is infinite cyclic, and there is a generator γ for the quotient group such that $\gamma g \gamma^{-1} = f_*(g)$, where f_* is the automorphism of the fundamental group defined by the homeomorphism f.

Note. Quotient constructions are possible only in a limited number of situations for smooth manifolds.

Final example

Although the disjoint union construction is usually not included in point set topology courses or texts, it is important for the theory of manifolds. The notes, *Constructing Topological Spaces from Pieces*, describe this construction and some of its elementary properties explicitly.

DISJOINT UNIONS. *Let $(M_j, \mathcal{A}_j)$ be a family of smooth manifolds, and let $\coprod M_j$ be their disjoint union. Then $\coprod_j \mathcal{A}_j$ defines a smooth atlas for $S = \coprod_j M_j$ such that*

(i) the injections $i_j : M_j \to S$ are smooth mappings,

(ii) if $(P, \mathcal{C}$ is a smooth manifold and $h : S \to P$ is continuous, then h is smooth if and only if each composite $h \circ i_j$ is smooth.