Selected answers for

Mathematics 205C, Spring 2003, Examination 1

Each problem is worth 25 points.

1. (i) If M is a topological n-manifold and $x \in M$, then for each open neighborhood U of x there is an open subneighborhood $V \subset U$ such that V is connected and $V - \{x\}$ has C_n connected components, where C_n is a positive integer that depends only on n. Give the exact values for the constants C_n.

(ii) State the definitions of a smooth atlas for a topological manifold M and of a smooth manifold.

SOLUTION TO PART (i)**:**

We have $C_1 = 2$ and $C_n = 1$ for $n \geq 2$. The correct value for C_0 is 0 because a 0-manifold is a discrete set (and therefore we can take $V = \{x\}$).

SOLUTION TO PART (ii)**:**

A smooth atlas for a topological n-manifold is a collection $\mathcal{A}$ of objects (U_α, h_α) such that each U_α is open in $\mathbf{R}^n$, each h_α is a map from the corresponding U_α to $\mathbf{R}^n$ that is a homeomorphism onto an open subset, the image sets $h_\alpha(U_\alpha)$ form an open covering of M, and the transition maps "$h_\beta^{-1} \circ h_\alpha$" from $h_\alpha^{-1}\left(h_\beta(U_\beta)\right)$ to $h_\beta^{-1}\left(h_\alpha(U_\alpha)\right)$ are smooth (C^∞) diffeomorphisms.

A smooth manifold is a pair $(M, \mathcal{A})$ consisting of a topological n-manifold M and a *maximal* smooth atlas $\mathcal{A}$ for M.

2. Let U and V be open subsets of Euclidean spaces and let $q : U \to V$ be a smooth map. A smooth map $s : V \to U$ is said to be a *smooth cross section* of q if the composite $q \circ s$ is the identity. Prove that a smooth cross section is always a 1–1 immersion.

SOLUTION:

First of all, s is $1-1$ because $s(y) = s(z)$ implies $q(s(y)) = q(s(z))$ and since $q \circ s$ is the identity the second equation reduces to $y = z$.

To see that s is an immersion, note that $qs = 1_V$ implies that

$$I = D(1_V) = D(q \circ s)$$

where D denotes the derivative, so that the Chain Rule implies

$$I = Dq\,(s(y)) \circ Ds(y)$$

for all $y \in V$. By elementary linear algebra, this implies that the kernel $Ds(y)$ is always the zero subspace and that $Ds(y)$ is $1-1$.

Note. The map q is a submersion on a neighborhood of $s(V)$ because q has maximum rank on the latter the set of all points $u \in U$ for which $Dq(u)$ has maximum rank is always open. However, it need not be a submersion everywhere. Take $U = \mathbf{R}^2$, $V = \mathbf{R}$, $s(v) = (v, 0)$. Let h be a smooth real valued funtion on the real line so that $h(t) = 1$ for $|t| \leq 1$ and $h(t) = 0$ for $|t| \geq 2$. Then the map $q(x, y) = (xh(y), yh(y))$ is not a submersion (it is zero for $|y| \geq 2$) but qs is the identity.

3. Outline the construction of a smooth function $\varphi : \mathbf{R} \to \mathbf{R}$ such that $\varphi(x) = 0$ for $x \leq 0$, φ is increasing for $x \in [0, 1]$, and $\varphi(x) = 1$ for $x \geq 1$.

SOLUTION:

The function $f(x) = \exp(-\frac{1}{x^2})$, which is defined for all real $x \neq 0$, has the property that

$$\lim_{x \to 0} f^{(n)}(x) = 0$$

for all nonnegative integers n, where as usual $f^{(n)}$ denotes the n-th derivative of f.

If we define g by $g(x) = f(x)$ if $x > 0$ and $g(x) = 0$ if $x \leq 0$ then g will be a C^∞ function.

Define $h(x) = g(x) \cdot g(1 - x)$ to obtain a C^∞ function that is positive if $x \in (0, 1)$ and zero otherwise.

Take $k(x)$ to be the (unique) antiderivative of h such that $k(0) = 0$. Then $k'(x) = h(x) = 0$ for $x \leq 0$ implies that $k(x) = 0$ for $x \geq 0$, and $k'(x) = h(x) = 0$ for $x \leq 1$ implies that

$$k(x) = k(1) = \int_0^1 h(t)\, dt$$

for $x \geq 1$. Since h is positive on (0,1), the antiderivative k must be strictly increasing on that interval, and the integral of h over the unit interval must be positive. Finally, take $\varphi(x) = k(x)/k(1)$.

4. Let M and N be smooth manifolds, suppose that f and g are diffeomorphisms from M to N, and let $f \times g : M \times M \to N \times N$ be the product map. Given a smooth manifold Y, let T_Y be the twist map on $Y \times Y$ defined by $T_Y(u, v) = (v, u)$. Prove that the composite $T_N \circ (f \times g) : M \times M \to N \times N$ is also a diffeomorphism. [*Hints:* Is T_N a diffeomorphism? If so, what is its inverse? Consider $f^{-1} \times g^{-1}$ and the inverse function identity $(h \circ k)^{-1} = k^{-1} \circ h^{-1}$.]

SOLUTION:

The map T_N is a diffeomorphism because it is its own inverse (*Proof:* $T_n(T_N(x, y)) = T_N(y, x) = (x, y)$). The map $f^{-1} \times g^{-1}$ is seen to be an inverse to $f \times g$ by direct calculation. In particular,

$$f^{-1} \times g^{-1}\,(f \times g(x, y)) = f^{-1} \times g^{-1}\,(f(x), g(y)) = \times g^{-1}\left(f^{-1}\,(f(x))\,, g^{-1}\,(g(y))\right) = (x, y)$$

for all x and y, and similarly if we interchange f and g with f^{-1} and g^{-1} respectively.

Finally, the inverse formula shows that the composite of two diffeomorphisms is a diffeomorphism, and since $T_N \circ (f \times g)$ has been shown to be a composite of two diffeomorphisms, it must also be a diffeomorphism.