

Spring 2003

I. Topological background

(Conlon, §§ 1.1–1.4)

1. Let X be a Hausdorff space, and suppose that X has an open covering $\{U_\alpha\}$ such that each U_α is a topological n -manifold for some fixed n . Prove that X is a topological manifold. Give a counterexample to this statement if the Hausdorff condition is removed.

II. Local theory of smooth functions

(Conlon, §§ 2.1–2.4, 2.6–2.8)

Problems from Conlon: 2.2.24, p. 50/ 2.7.19, p. 70.

1. Consider the vector field Y on the plane defined by the vector-valued function (y, y^2) . Find the integral curve φ_p of Y such that $\varphi(0) = p$, and specify the maximal interval for which this curve is defined. For which points in the plane does $\varphi_p(1)$ exist?

2. Let X be the vector field on \mathbf{R}^3 such that $X(p) = (p : (1, 1, 1))$, and let

$$S : (0, \infty) \times (0, 2\pi) \times (0, \pi) \rightarrow \mathbf{R}^3 - \{(x, y, z) \mid x \geq 0, y = 0\}$$

be the spherical coordinate diffeomorphism

$$(x, y, z) = (\rho \cos \theta \cos \phi, \rho \sin \theta \sin \phi, \rho \cos \phi).$$

What are the formulas for the coordinates of the vector field $S_*^{-1}(X)$?

3. Is every vector field on the real line complete?

4. Find the flow associated to the vector field on \mathbf{R}^2 given by

$$y \frac{\partial}{\partial x} - y^3 \frac{\partial}{\partial y}.$$

5. Find the flow associated to the vector field on \mathbf{R}^3 given by

$$ay \frac{\partial}{\partial x} - ax \frac{\partial}{\partial y} + a^2 \frac{\partial}{\partial z}.$$

6. Find the flow associated to the vector field on \mathbf{R}^3 given by

$$y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}.$$

7. Prove that $F(x, y) = (xe^y + y, ye^y - y)$ defines a diffeomorphism of \mathbf{R}^2 .

8. Prove that

$$F(x, y, z) = \left(\frac{x}{2+y^2} + ye^z, \frac{x}{2+y^2} - ye^z, 2ye^z + z \right)$$

defines a diffeomorphism of \mathbf{R}^3 .

III. Global theory of smooth manifolds and mappings

(Conlon, §§ 2.5, 3.1–3.5, 3.7–3.8)

Problems from Conlon: 2.5.8–2.5.9, p. 61/ 2.5.15, p. 62/ 3.4.18, p. 102/ 3.5.11–3.5.12, p. 107/ 3.8.5, p. 117.

1. Let $f(x, y) = x^3 + xy - x^3$. Show that the level set for the value 1 is a smooth submanifold but the level set for the value 0 is not. [*Hint:* In the second case, prove that otherwise there would be a pair of C^∞ functions x and y satisfying $f(x, y) = 0$ where $x(0) = y(0) = 0$ and $(x'(0), y'(0)) \neq (0, 0)$. Then consider the existence of the limits of $x(t)/y(t)$ and $y(t)/x(t)$ as $t \rightarrow 0$.]

2. For which real numbers c is the set $y^2 - x(x-1)(x-c) = 0$ a submanifold of \mathbf{R}^2 ?

3. Let $f(x, y) = y^2 + x^4 - x^2$. Find all real numbers c such that the level set $f^{-1}(\{c\})$ is a smooth submanifold. Give reasons for your answer.

4. Let $Q \subset \mathbf{R}^{n+1}$ be the unit cube consisting of all (x_0, \dots, x_n) such that $\max_i |x_i| = 1$. Prove that Q is homeomorphic to S^n , that Q has a smooth atlas for which Q is diffeomorphic to S^n , but Q is not a smooth submanifold of \mathbf{R}^{n+1} .

5. Prove that a smooth map from the 2-sphere to the unit circle cannot be 1–1.

6. Let M be a *connected* topological manifold, and let x and y be distinct points of M . Prove that there is a homeomorphism $h : M \rightarrow M$ such that $h(x) = y$. If M has a smooth structure prove that one can in fact find a diffeomorphism with this property.

7. Let X be the y -axis in the Cartesian plane, and let Y be the graph of $\sin \frac{1}{x}$ for $x > 0$. Prove that $X \cup Y$ is an immersed but not embedded submanifold but that each of X and Y taken separately is an embedded submanifold.

8. Prove that there is no immersion from a compact n -manifold into \mathbf{R}^n .

9. Given an immersion from a 1-connected compact smooth manifold to a smooth manifold of the same dimension, prove that it is a covering projection. Does the statement remain true if the manifolds are not necessarily compact? Prove this or give a counterexample.

10. Let A be a real nonsingular symmetric $n \times n$ matrix and let c be a nonzero real number. Show that the quadric hypersurface defined by the equation $\langle Ax, x \rangle = c$ is a smooth $n - 1$ -dimensional submanifold of \mathbf{R}^n .

11. Let M be a noncompact smooth manifold. Prove that there is a smooth embedding $f : [0, \infty) \rightarrow M$ such that the image of F is a closed subset.

12. Suppose that M is a noncompact smooth manifold and there is a smooth 1–1 immersion $f : M \rightarrow \mathbf{R}^N$. Prove that there is a smooth embedding $g : M \rightarrow \mathbf{R}^{N+1}$ such that $g(M)$ is a closed subset.

13. Let M be a smooth submanifold of N that is a closed subset of N , and let f and X be a smooth real valued function and a smooth vector field on M respectively. Prove that f and X extend to smooth function and vector field (respectively) on N . [*Hint:* Use submanifold charts.]

14. Let $(\pi : E \rightarrow B, \text{ etc.})$ be a smooth vector bundle projection, and let $z : B \rightarrow E$ be the zero section. Prove that the identity map of E is smoothly homotopic to $z \circ \pi$. [*Hint:* How can you prove this if B consists of a single point?]

15. Suppose that two vector bundles $(\pi : E \rightarrow B, \text{ etc.})$ and $(\pi' : E' \rightarrow B, \text{ etc.})$ are isomorphic, and let $z : B \rightarrow E$ $z' : B \rightarrow E'$ be the respective zero sections. Prove that $E - z(B)$ is homeomorphic to $E' - z'(B)$.

16. Prove that a continuous real valued function on a compact smooth manifold M can be approximated arbitrarily closely (in the uniform metric) by smooth functions and that the Banach space $C(M)$ of continuous functions is a separable metric space.

17. Let U be an open subset in Euclidean n -space, and let g be a riemannian metric on M . Prove that there is a set of n vector fields over U that are orthonormal with respect to g . Prove also that there is a vector bundle automorphism of U (i.e., a homeomorphism Ψ from $U \times \mathbf{R}^n$ to itself such that for each $u \in U$ Ψ maps $\{u\} \times \mathbf{R}^n$ to itself by an invertible linear transformation) such that Ψ sends g to the trivial metric. In other words

$$g(\Psi((u, \mathbf{v})), (\Psi((u, \mathbf{w}))) = \langle \mathbf{v}, \mathbf{w} \rangle.$$

18. The Poincaré metric on the upper half plane $\mathbf{H}_+ = \{x + iy \mid y > 0\}$ is defined by the formula

$$g_{\mathbf{H}} = \frac{dx^2 + dy^2}{y^2}.$$

Given a 2×2 real matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with determinant $ad - bc = 1$, let F_A be defined by the formula

$$F_A(z) = \frac{az + b}{cz + d}$$

where the point z lies in the upper half plane and the right hand side is interpreted using complex numbers. Prove that F_A defines a diffeomorphism of the upper half plane to itself and that the associated map of tangent spaces is an isometry with respect to the Poincaré metric.

19. The Poincaré metric on the open disk $\mathbf{D} = \{x + iy \mid x^2 + y^2 < 1\}$ is given by the formula

$$g_{\mathbf{D}} = \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}.$$

Let $f : \mathbf{D} \rightarrow \mathbf{H}_+$ be the complex analytic map

$$f(z) = i \frac{1 - z}{1 + z}.$$

Prove that f is a diffeomorphism and sends the Poincaré metric on \mathbf{D} to the Poincaré metric on \mathbf{H}_+ ; *i.e.*, if \mathbf{v} and \mathbf{w} are tangent vectors over the same point x then

$$g_{\mathbf{D}}(\mathbf{v}, \mathbf{w}) = g_{\mathbf{H}}(T(f)\mathbf{v}, T(f)\mathbf{w}).$$

20. Given two smooth manifolds M and N , outline a proof that the tangent bundle of $M \times N$ is diffeomorphic to the product of the tangent bundle of M and the tangent bundle of N . [*Hint:* Look at the “product” of two sets of amalgamation data.]

21. Construct a smooth vector field on S^2 that is zero at exactly one point. [*Hint:* Use stereographic projection.]

22. Let $f : S^2 \rightarrow \mathbf{R}^4$ be the smooth map sending (x, y, z) to $(x^2 - y^2, xy, xz, yz)$. Show that $f(x, y, z) = f(-x, -y, -z)$ for all (x, y, z) and that the associated map $g : \mathbf{RP}^2 \rightarrow \mathbf{R}^4$ on the quotient manifold is a smooth embedding.

23. Show that it is possible to make the subset of the plane defined by the equation $x^3 - y^2 = 0$ into a smooth manifold but that the set in question is not a smooth submanifold of \mathbf{R}^2 . What happens for the set $x^4 - y^2 = 0$?

24. Let $F : \mathbf{R}^4 \rightarrow \mathbf{R}^2$ be the smooth map defined by

$$F(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + 2 + y).$$

Show that $(0, 1)$ is a regular value and that the level set is diffeomorphic to S^2 .

25. Show that the functions $g_{1,1} = y^4 + y^2 + 2xy + x^3 + 1$, $g_{1,2} = g_{2,1} = y + xy^2 + 2x$, $g_{2,2} = 2x^2 + 1$ define a riemannian metric on \mathbf{R}^2 .

26. Show that the functions $g_{1,1} = 2$, $g_{1,2} = g_{2,1} = x$, $g_{2,2} = x^2 + 1$, $g_{2,3} = g_{3,2} = x$, $g_{1,3} = g_{3,1} = y$, $g_{3,3} = x^2 + y^2 + 1$ define a riemannian metric on \mathbf{R}^3 .

IV. Vector fields and Lie groups

(Conlon, §§ 4.1–4.4, 5.1–5.2, parts of §§ 2.2, 2.7, 2.8)

Problems from Conlon: 4.1.9, p. 133/ 4.1.13, p. 134/ 5.1.28, p. 167/ 5.1.38, p. 170.

1. Let X and Y be the vector fields in the plane defined by the vector-valued smooth functions (x, xy) and (y^2, xy) respectively. Compute the Lie bracket $[X, Y]$.

2. Let X be a smooth vector field on a manifold M , and let $f : M \rightarrow (0, \infty)$ be smooth. Show that the maximal integral curves $\varphi_p(t)$ for X and $\psi_p(t)$ for $f \cdot X$ with initial point p are reparametrizations of each other. If $(-a^-, a^+)$ and $(-b^-, b^+)$ are the maximal intervals on which φ_p and ψ_p are defined, show that

$$b^+(p) = \int_0^{a^+(p)} \frac{dt}{f(\phi_p(t))}$$

and

$$b^-(p) = \int_{-a^-(p)}^0 \frac{dt}{f(\phi_p(t))}.$$

3. Given a smooth vector field on a noncompact connected (second countable) manifold M , show that there is a smooth function $f : M \rightarrow (0, \infty)$ such that $f \cdot X$ is complete. [*Hint:* Take an increasing family of compact subspaces K_i such that $K_i \subset \text{Int}(K_{i+1})$ and $M = \cup_i K_i$. Note that

$$K_i - K_{i-1} \subset \text{Int}(K_{i+1} - K_{i-1})$$

and therefore there exists a nonnegative smooth function ρ_i on M that is 1 on $K_i - K_{i-1}$ and whose support (= closure of the set where the function is nonzero) is contained in $\text{Int}(K_{i+1} - K_{i-1})$. By convention K_0 and K_{-1} are empty. For each i show that there is an $\varepsilon_i > 0$ such that $|t| < \varepsilon_i \Rightarrow \Phi_t(K_i) \subset K_{i+1}$, where Φ denotes the flow of X . If

$$f = \sum_{i \geq 1} \varepsilon_i \cdot \rho_i$$

verify that f is a smooth positive function on M and use the preceding exercise to show that

$$b^+(p) = \int_0^{a^+(p)} \frac{dt}{f(\phi_p(t))} \geq \int_0^{\varepsilon_i} \frac{dt}{\varepsilon_i} = 1$$

$$b^-(p) = \int_{-a^-(p)}^0 \frac{dt}{f(\phi_p(t))} \geq \int_{-\varepsilon_i}^0 \frac{dt}{\varepsilon_i} = 1$$

where a^\pm and b^\pm are defined as in the preceding exercise. This means that the domain of the flow for $f \cdot X$ contains $(-1, 1) \times M$.

4. Suppose that X and Y are smooth vector fields on an open set in some Euclidean space, and let D_X and D_Y be the corresponding derivations on the ring of smooth functions $C^\infty(M)$. Give an example to show that $D_X D_Y$ is not necessarily a derivation.

5. Consider the vector field on the 2-dimensional torus T^2 defined by $X(p) = (p, \mathbf{F}(p))$. Prove that any vector field Y that commutes with X (i.e., $[X, Y] = 0$) is collinear with X .

6. Let Φ and Ψ be the 1-parameter groups of diffeomorphisms of \mathbf{R}^3 defined by clockwise rotation about the x and y axes respectively, and let A and B be the associated vector fields. Compute the Lie bracket $[A, B]$.

7. Prove that the group of upper triangular $n \times n$ matrices with ones down the diagonal is a Lie group that is diffeomorphic to some Euclidean space and nonabelian if $n \geq 3$ (this is sometimes called the group of unipotent matrices).

8. Prove that the compact Lie groups $SO(3)$ is diffeomorphic to real projective 3-space. [*Hint:* Represent an orthogonal transformation as a rotation around an axis using quaternions. If q is a unit quaternion, then the conjugation $x \rightsquigarrow qxq^{-1}$ defines an orthogonal transformation on the 3-dimensional subspace of pure quaternions having the form $ai + bj + ck$ for suitable real numbers a, b, c .]

9. Let $E_0 = T(S^2) - z(S^2)$ where z denotes the zero section.

- (i) Prove that E_0 is homeomorphic to $SO(3) \times \mathbf{R} \cong \mathbf{RP}^3 \times \mathbf{R}$. [*Hint:* View $T(S^2)$ as the set of all (x, y) in $\mathbf{R}^3 \times \mathbf{R}^3$ such that $|x| = 1$ and $\langle x, y \rangle = 0$, and first check that $E_0 \subset T(M)$ is homeomorphic to the product of the real line with the set $V_2(\mathbf{R}^3)$ of all (x, y) in $S^2 \times S^2$ such that $\langle x, y \rangle = 0$ — there is a canonical map from E_0 to $V_2(\mathbf{R}^3)$ defined by sending y to the unit vector in the same direction, and $\log|y|$ defines the map onto the reals. Next note that $V_2(\mathbf{R}^3)$ is homeomorphic to $SO(3) \cong \mathbf{RP}^3$ by the map sending (x, y) to the matrix in $SO(3)$ whose columns are given by x, y and $x \times y$ (cross product).]
- (ii) Prove that $T(S^2)$ is not (topologically) isomorphic to the product vector bundle $S^2 \times \mathbf{R}^2$. [*Hint:* If two vector bundles over the same space are isomorphic, then the complements of their zero sets are homeomorphic. Compare the fundamental groups of E_0 and $S^2 \times (\mathbf{R}^2 - \{\mathbf{0}\})$.]

(iii) Prove that there is no continuous vector field X on S^2 such that $X(p) \neq 0$ for all $p \in S^2$. [*Hint:* Suppose such a vector field exists. Show that if there is another vector field Y such that Y is always perpendicular to X and Y is never zero, then X and Y define an isomorphism from the trivial bundle to $T(S^2)$. Define $X^\perp \subset T(M)$ to be the set of all tangent vectors \mathbf{w} such that \mathbf{w} is perpendicular to $X(\tau(\mathbf{w}))$. Then $X^\perp \cap T_p(M)$ is a 1-dimensional subspace for each p . Show that the set of all vectors in X^\perp of unit length is a 2-sheeted covering space over S^2 and thus splits into two pieces, each homeomorphic to S^2 , and that either of these determines a vector field of the desired type.]

10. Prove that a 1-dimensional real vector bundle over a simply connected manifold is a trivial vector bundle. [*Hint:* Look at Part (iii) of the preceding exercise.]

11. Find the Lie brackets of the following pairs of vector fields on \mathbf{R}^3 (we write ∂_u for $\frac{\partial}{\partial u}$ to save space):

- (i) $y \partial_z - 2xy^2 \partial_y$ and ∂_y .
- (ii) $-y \partial_x + x \partial_y$ and $y \partial_x + x \partial_y$.

12. Suppose that a smooth function satisfies $[fX, Y] = f[X, Y]$ for all vector fields X and Y . What can one say about f ?

V. Cotangent spaces and tensor algebra

(Conlon, §§ 6.1–6.4, 7.1–7.2, 7.4–7.5)

Problems from Conlon: 7.2.21, p. 225/ 7.2.23, p. 226.

1. Let U, V, W be finite-dimensional real vector spaces. Prove the following relationships:

- (a) $(U \oplus V) \otimes W \cong (U \otimes W) \oplus (V \otimes W)$.
- (b) $U^* \otimes V^* \cong (U \otimes V)^*$.
- (c) $\text{Hom}(U, V) \cong U^* \otimes V$.

2. If U is an open subset of \mathbf{R}^n and $\mathcal{X}(U)$ is the space of smooth vector fields on U , then the general considerations about tensor fields show that the identity map on $\mathcal{X}(U)$ defines a tensor field \mathbf{K} of type $(1, 1)$ on U . The latter can be written in the form

$$\mathbf{K}(u) = \sum_{i,j} b_i^j(u) dx^i \otimes \frac{\partial}{\partial x^j}$$

for $b_i^j \in C^\infty(M)$. What are the functions b_i^j ? [*Hints:* Look at 1.(c) above; each function can be written down with a very small number of symbols.]

VI. Differential forms

(Conlon, §§ 8.1–8.6, parts of earlier sections)

Problems from Conlon: 8.4.7–8.4.8, p. 266.

1. What is the de Rham cohomology with compact supports for the real line?

2. Consider the 2-form on $\mathbf{R}^3 - \{0\}$

$$\omega = \frac{1}{r} x dy \wedge dz - y dx \wedge dz + z dx \wedge dy$$

where $r^2 = x^2 + y^2 + z^2$. Show that ω is not exact and that $dr \wedge \omega = dx \wedge dy \wedge dz$.

3. Let $f : S^2 \rightarrow \mathbf{R}^2$ be smooth, and let ω be a 2-form on \mathbf{R}^2 . Prove that the integral of the pulled back form over the sphere is zero, and prove that there must be a point on the sphere where the pullback vanishes.

4. Let $\alpha = x dx + y dy + z dz$ and $\Omega = dx \wedge dy \wedge dz$ be differential forms on \mathbf{R}^3 . Write down a differential form β on $\mathbf{R}^3 - \{0\}$ such that $\Omega = \alpha \wedge \beta$, and show that there is no differential form γ on \mathbf{R}^3 such that $\Omega = \alpha \wedge \gamma$.

5. Let $M = \mathbf{R}^3 - (X \cup Y)$, where X and Y denote the x - and y -axes respectively. Find closed 1-forms representing a basis for the first de Rham cohomology group $H_{\text{DR}}^1(M)$.

6. Let ω be a nowhere zero smooth 1-form on a smooth compact manifold M . Show that if $\omega \wedge d\omega = 0$ then there exists a 1-form α such that $d\omega = \alpha \wedge \omega$ [Hint: First do it locally then use a partition of unity.]

7. Consider a closed 2-form on $\mathbf{R}^3 - \{0\}$ defined by

$$\omega = P(x, y, z) dy \wedge dz + Q(x, y, z) dx \wedge dz + R(x, y, z) dx \wedge dy$$

where P, Q and R are all smooth functions. Let $r^2 = x^2 + y^2 + z^2$, and assume that

$$|P|, |Q|, |R| < \frac{1}{r}.$$

Show that ω is exact.

8. Let M be a second countable smooth manifold and let $f : M \rightarrow \mathbf{R}$ be a smooth function such that the exterior derivative df is nowhere zero. Prove that M is noncompact, and using partitions of unity show that there is a smooth vector field X such that $\langle df, X \rangle = 1$ at all points of M .

9. Let ω be the 1-form on \mathbf{R}^3 defined by $x dy - y dx + dz$. Show that for every nonzero real valued function f the form $f\omega$ is not closed.

10. Prove that if ω is a 1-form then $\omega \wedge \omega = 0$. Give an examples to show the analog is false for higher degree forms by exhibiting a 2-form ω on \mathbf{R}^{2n} such that the n -fold wedge $\wedge^n \omega = \omega \wedge \cdots \wedge \omega \neq 0$ at every point.