Vector Fields, Tensor Fields and Exterior Forms

Mathematics 205C, Spring 2003

1. Vector fields

Intuitively, a (tangent) vector field on a manifold is supposed to specify a tanget vector for each point of the manifold. Here is the formal definition.

Definition. Let M be a smooth manifold and let $\tau_M : T(M) \to M$ be its tangent bundle. A **(tangent) vector field** on M is a continuous map $X : M \to T(M)$ such that $\tau_M \circ X = \mathbf{1}_M$. Unless specfically stated otherwise, we shall assume that all vector fields under consideration are smooth.

If U is open in $\mathbf{R}^n$, so that $T(U) \cong U \times \mathbf{R}^n$, then a (smooth!) tangent vector field has the form

$$X(u) = (u, \mathbf{F}(u))$$

where $\mathbf{F} : U \to \mathbf{R}^n$ is smooth mapping. This is clearly equivalent to the notion of vector field that one sees in undergraduate physics and multivariable calculus courses.

Examples. 1. The preceding paragraph gives a vast collection of examples that is exhaustive if M is an open subset of $\mathbf{R}^n$. On an arbitrary smooth manifold M the zero map from M to $T(M)$ is a smooth vector field.

2. If X and Y are vector fields on M, then their sum $X + Y$ can be defined and it is also a vector field. Likewise, if $g : M \to \mathbf{R}$ is smooth then the pointwise scalar product $g \cdot X$ is also a vector field. The operations of addition and multiplication by functions in $C^\infty(M)$ make the set of vector fields $\mathcal{X}(M)$ into a *module* over $C^\infty(M)$; *i.e.*, the operations satisfy all the rules for vector addition and scalar multiplication that one has for a vector space (although $C^\infty(M)$ is merely a commutative ring with unit rather than a field).

3. If M is a smooth manifold with tangent bundle $\tau_M : T(M) \to M$ and U is open in M with inclusion map $i : U \subset M$, then $T(U)$ is isomorphic to $\tau_M^{-1}(U)$ and the accordingly restriction $X|U$ determines a smooth vector field i^*X on U. The construction $i^* : \mathcal{X}(M) \to \mathcal{X}(U)$ is a linear transformation of real vector spaces (verify this!), and the product with smooth functions satisfies the identity

$$i^*(g \cdot X) = (g|U) \cdot X.$$

4. If M is a smooth manifold and $f : M \to N$ is a diffeomorphism, then an isomorphism of real vector spaces $f_* : \mathcal{X}(M) \to \mathcal{X}(N)$ is defined by $f_*Y(p) = T(f) \circ Y \circ f^{-1}$. This is a direct generalization of the previously defined construction for open subsets of Euclidean spaces. One natural question is what can be said about $f_*(g \cdot Y)$ if $g \in C^\infty(M)$; this is left to the reader as an exercise. One important property of the f_* construction is that it is covariantly functorial: $(h \circ f)_* = h_* \circ f_*$ and $(\mathbf{1}_M)_* = \mathbf{1}_{\mathcal{X}(M)}$.

In the classical approach to tensor analysis, tangent vector fields for a smooth manifold M are described in terms of a smooth atlas $\mathcal{A} = \{(U_\alpha, h_\alpha)\}$ for M. It will be convenient for us to write $\psi_{\beta\alpha}$ for the transition maps "$h_\beta^{-1} \circ h_\alpha$" in this note, both for the sake of notational conciseness and for its consistency with the standard notation of tensor analysis. The following result is essentially the classical characterization of vector fields (or, in the terminology of tensor analysis, a *contravariant tensor field of rank* 1.

Proposition. *Given a smooth n-manifold M with smooth atlas $\mathcal{A} = \{(U_\alpha, h_\alpha)\}$, suppose that for each α we are given a smooth map $\mathbf{F}_\alpha : U_\alpha \to \mathbf{R}^n$ and that these maps satisfy the condition*

$$\mathbf{F}_\beta\left(\psi_{\beta\alpha}(u)\right) = \left[D\psi_{\beta\alpha}(u)\right]\left(\mathbf{F}_\alpha(u)\right)$$

for all α and β. Let $i_\alpha : h_\alpha(U_\alpha) \to M$ be inclusion. Then there is a unique vector field Y on M such that $i_\alpha^* Y = (h_\alpha)_* X_\alpha$, where $X_\alpha(u) = (u, \mathbf{F}_\alpha(u))$ for all u.

The displayed condition is exactly what is needed to verify that the maps X_α piece together to form a well defined continuous map from M to $T(M)$ that is a continuous vector field, and the smoothness of the functions $\mathbf{F}_\alpha$ implies that the globally constructed map Y is smooth.

Alternate formulation. Another way of expressing the consistency relationship is that one has vector fields X_α over each U_α and they satisfy

$$(\psi_{\beta\alpha})_* X_\alpha = X_\beta$$

on the appropriate open subsets of U_α and U_β respectively.

Integral flows of vector fields

Most of the basic theory is the same as in the special case of open subsets of Euclidean spaces. Given a smooth vector field X on M, the integral flow is a smooth map

$$\Phi = \Phi_X : \mathcal{D}(X) \to M$$

where $\mathcal{D}(X)$ is an open neighborhood of $M \times \{0\}$ in $M \times \mathbf{R}$ such that

(i) $\Phi(y, 0) = y$ for all $y \in M$,

(ii) For each $y \in M$, the restriction of Φ to $\mathcal{D}(X) \cap \{y\} \times \mathbf{R}$ is the unique maximal integral curve for X with initial condition y.

The construction of Φ in terms of maximal integral curves implies that Φ satisfies the conditions for a local 1-parameter group. More generally, given a space Y and a continuous map $\Phi : \mathcal{D} \to Y$ defined on an open neighborhood $\mathcal{D}$ of $Y \times \{0\}$ in $Y \times \mathbf{R}$, we say that Φ is a *local 1-parameter group* if it satisfies the following conditions:

- $\Phi(y, 0) = y$ for all $y \in Y$.
- If U is an open subset of Y such that $U \times \{t\} \subset \mathcal{D}$ for some real number t, then $\varphi_t(u) = \Phi(u, t)$ maps U homeomorphically onto an open subset of Y.
- If $v = \Phi(u, t) = \varphi_t(u)$ and $w = \Phi(v, s) = \varphi_s(v)$ are defined, then $(u, t + s) \in \mathcal{D}$ and $w = \varphi_{t+s}(u)$.

The first condition can be rewritten $\varphi_0 = \mathbf{1}_Y$, and the third can be rewritten informally as $\varphi_{t+s} = \varphi_s \circ \varphi_t$. For every point $y \in Y$ there is an open neighborhood U of y and an interval $(-\varepsilon, \varepsilon)$ such that $U \times (-\varepsilon, \varepsilon) \subset \mathcal{D}$, and on this neighborhood one can informally write $\varphi_t^{-1} = \varphi_{-t}$.

The following is a straightforward generalization of Lemma 4.1.10 on page 133 of Conlon:

Proposition. *If $\mathcal{D}$ contains $Y \times (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, then $\mathcal{D} = Y \times \mathbf{R}$.*

Corollary. *If Y is compact then $\mathcal{D} = Y \times \mathbf{R}$.*

Compare this with Corollary 4.1.12 on page 134 of Conlon. If Y is a smooth manifold and Φ is the flow associated to a vector field X, then x is said to be *complete*.

The preceding corollary illustrates one feature of the global situation that does not arise for open sets in Euclidean space. All vector fields on a compact manifold are complete, but this is never true for open subsets of Euclidean spaces (we have already given examples of noncomplete vector fields for $\mathbf{R}^n$; in other cases the constructions require more work).

Physically speaking, a noncomplete vector field (one where $\mathcal{D}(X) \neq M \times \mathbf{R}$) corresponds to a dynamical system that breaks down or blows up in a finite amount of time. The next proposition can be viewed as a mathematical formulation of this principle:

Proposition. *Suppose that X is a vector field on M and that the maximal integral curve of X with initial condition p is only definable for $t \in (a^-, a^+)$ where $-\infty \leq a^- < a^+ < +\infty$. If Γ is the image of this curve, then Γ is not contained in any compact subset of M.*

Proof. Let K be the closure of Γ. By construction we know that Φ maps $\mathcal{D} \cap \Gamma \times \mathbf{R}$ to itself, so by continuity it must also map $\mathcal{D} \cap K \times \mathbf{R}$ to itself. It follows that $\Phi|\mathcal{D} \cap K \times \mathbf{R}$ is a local 1-parameter group of transformations on K. If Γ were contained in a compact set, then K would also be compact, and by the preceding corollary it would follow that $\mathcal{D} \cap K \times \mathbf{R}$ would be all of $K \times \mathbf{R}$. The same would also hold if K were replace by Γ. But this contradicts the hypothesis on Γ, and therefore Γ cannot lie in any compact subset of M.

A similar result holds if where $-\infty < a^- < a^+ \leq +\infty$ (one can retrieve this by considering the reverse vector field $-X$, whose flow is given by $\Psi(u,t) = \Phi(u,-t)$).

Lie brackets

Note. The name "Lie" is pronounced "lee" (named after the Norwegian mathematician Sophus Lie).

The set of vector fields $\mathcal{X}(M)$ has another important piece of structure aside from being a module over the ring $C^\infty(M)$. This can be motivated in a purely algebraic fashion:

Definition. Let A be an associative algebra with unit over the real numbers; *i.e.*, A is a real vector space and an associative ring with unit satisfying the compatibility condition $(t{\cdot}1){\cdot}a = t{\cdot}a$ for $t \in \mathbf{R}$ and $a \in A$. A *derivation* on A is a linear transformation $D : A \to A$ satisfying the Leibniz rule $D(ab) = (Da)b + a(Db)$.

Ordinary differentiation is an obvious example of a derivation on $C^\infty(\mathbf{R})$, and partial derivatives are examples on $C^\infty(\mathbf{R}^n)$ for $n \geq 2$. In fact, if U is open in $\mathbf{R}^n$ and Y is a vector field on U defined by $Y(u) = (u, \mathbf{F}(u))$ for some smooth function $\mathbf{F} : U \to \mathbf{R}^n$, with coordinate functions f^i, then a derivation may be defined on $C^\infty(M)$ by the formula

$$\mathcal{L}_Y(g) = Yg = \sum f^i \frac{\partial g}{\partial x^i}.$$

This is a sort of paramaterized directional derivative in the direction of the vector field, and it is called the *Lie derivative* of g with respect to Y. One can globalize this to arbitrary manifolds by taking Yg to be the composite

$$\pi_2 \circ \Gamma \circ T(g) \circ Y : M \to T(M) \to T(\mathbf{R}) \cong \mathbf{R} \times \mathbf{R} \to \mathbf{R}$$

where $\Gamma : T(\mathbf{R}) \cong \mathbf{R} \times \mathbf{R}$ by the usual vector bundle isomorphism (τ corresponds to projection onto the first factor) and π_2 is projection onto the second factor.

In the note, *Derivations and vector fields*, there is an outline of a proof that *every derivation on $C^\infty(M)$ is given by a Lie derivative* that is close to the analogous proof in Conlon. The following purely algebraic result then yields an important nonassociative multiplication on $\mathcal{X}(M)$:

Proposition. *Let A be an associative algebra with unit over the reals. Then the set of derivations is a vector subspace of the space of all linear transformations from A to itself, and if D and E are derivations, then so is their* **commutator** $[D, E] = DE - ED$.

The verification of this is a routine exercise.

It follows that if X and Y are vector fields on a smooth manifold M, then there a **Lie bracket** vector field $[X, Y]$ that is completely characterized by the Lie derivative identity

$$[X, Y]g = X(Yg) - Y(Xg).$$

Local Formula. *Let X and Y be vector fields on the open set $U \subset \mathbf{R}^n$ given by the formulas $X(u) = (u, \mathbf{F}(u))$ and $Y(u) = (u, \mathbf{G}(u))$ respectively. Then $[X, Y]$ is given by the formula $[X, Y](u) = (u, \mathbf{H}(u))$ where*

$$\mathbf{H}(u) = [D\mathbf{G}(u)]\mathbf{F}(u) - [D\mathbf{F}(u)]\mathbf{G}(u).$$

This reduces to a tedious but entirely routine calculation.

Here is another useful relation:

Proposition. *If X and Y are vector fields on M and $f : M \to N$ is a diffeomorphism, then $f_*[X, Y] = [f_*X, f_*Y]$.*

This is a direct consequence of the Lie derivative identity for the bracket and the formula for f_* of a vector field.

Finally, we have the following important relationship which shows that the Lie bracket's behavior with respect to multiplication by functions is relatively complicated:

Formula. *If X and Y are vector fields on M and a and b are smooth functions on M. then*

$$[aX, bY] = ab[X, Y] + a(Xb)Y - b(Ya)X.$$

The multiplication on $\mathcal{X}(M)$ defined by the Lie bracket is an important special case of a fundamentally important class of mathematical objects,

Definition. A *Lie algebra* over the real numbers consists of a real vector space L together with a distributive binary operation [,] (often called the Lie bracket or commutator product) such that the operation is distributive in both variables, it is anticommutative with $[a, b] = -[b, a]$, it is homogeneous in that $k[a, b] = [ka, b] = [a, kb]$ for all scalars k, and it satisfies the *Jacobi identity*

$$[a, [b.c]] + [b, [c.a]] + [c, [a.b]] = 0.$$

These systems are almost never associative. Perhaps the most well known nontrivial example is $\mathbf{R}^3$ with the usual cross product.

Theorem. *Let A be an associative algebra over the reals, and let $[a, b] = ab - ba$ be the commutator product on A. Then A is a Lie algebra with respect to the commutator product.*

Verification of this is an elementary exercise.

Corollary. *The Lie bracket makes $\mathcal{X}(M)$ into a Lie algebra over the real numbers.*

Lie groups

Lie algebras are closely related to another class of objects in smooth manifold theory known as *Lie groups*. The latter are smooth manifolds G such that the multiplication map $m : G \times G \to G$ and the inverse map inv : $G \to G$ are smooth. One basic example is the group $GL(n, \mathbf{R})$ of all invertible $n \times n$ matrices over the reals (the notation stands for "general linear group").

There is a very rich theory of Lie groups that is beyond the scope of these notes. However, we shall make some observations to show the connection with the theory of finite-dimensional Lie algebras over the real numbers.

Let G bea Lie goup and for each $a \in G$ let $L_a : G \to G$ be the map sending x to ax (left multiplication by a).

Theorem. *The set $\mathbf{g}$ of all left invariant vector fields (that satisfy $L_{a*}Y = Y$ for all $a \in G$) is a vector subspace whose dimension is $\dim G$, and it is closed under Lie bracket.*

Therefore it follows that $\mathbf{g}$ is a finite-dimensional Lie algebra, and this is one definition of the Lie algebra associated to G. It turns out that Lie group homomorphisms determine homomorphisms of the associated Lie algebras, and for *simply connected* Lie groups there is 1–1 correspondence between such objects and their homomorphisms and the associated Lie algebras and *their* homomorphisms. A detailed study of Lie algebras in fact yields a very striking classification of compact connected Lie groups. Such groups are all finitely covered by groups that are products of standard building blocks given by the circle group, finite coverings of standard matrix groups such as the orthogonal and unitary matrices of determinant 1 and an analog (the symplectic) group for the quaternions. and five exceptional groups.

2. Cotangent spaces and 1-forms

We begin with some abstract concepts from linear algebra. Given a vector space V over a field $\mathbf{k}$, the *dual space* V^* is the vector space of all linear functionals $f : V \to \mathbf{k}$. If $v_1, \cdots, v_n$ is a basis for V, then there is a dual basis $f^1, \cdots, f^n$ for V^* uniquely determined by the equations $f^i(v_j) = 0$ if $i \neq j$ and 1 if $i = j$.

If V is finite dimensional then V and V^* are isomorphic as vector spaces, but there are numerous contexts where it is still useful to have both available. One way of distinguishing between the n-dimensional vector space $\mathbf{k}^n$ and its dual is to view the former as the set of all $n \times 1$ column vectors and the latter as the set of all $1 \times n$ row vectors. With this convention the evaluation of a linear functional $\mathbf{f}$ in the dual space of $\mathbf{k}^n$ at a vector $\mathbf{v} \in \mathbf{k}^n$ is given by the matrix product $\mathbf{f} \cdot \mathbf{v}$ (note that this is a 1×1 matrix).

Our next order of business is to construct an analog of the tangent bundle in which the vector space above each point can be viewed as the **dual space to the tangent space**. An obvious question is why anyone would be interested in doing such a thing. The reason is that some constructions are much easier using the duals of tangent spaces rather than the tangent spaces themselves. The extra work needed to construct and analyze the cotangent bundle is less than the grief one would experience in trying to express everything entirely in terms of tangent bundles.

We begin by constructing the cotangent bundle $\tau_M^* : T^*(M) \to M$.

Given a smooth atlas $\mathcal{A}$ for M, we have the "cocycle data" for constructing the tangent bundle

$$g_{\beta\alpha} : V_{\beta\alpha} \to GL(n, \mathbf{R})$$

that is given by $g_{\beta\alpha} = D\psi_{\beta\alpha}$.

In order to construct the cotangent bundle we take the "contragredient data" defined by

$$^{\mathbf{T}}(g_{\beta\alpha})^{-1} = {}^{\mathbf{T}}[D\psi_{\beta\alpha}]^{-1}$$

where ${}^{\mathbf{T}}P^{-1}$ denotes the transposed inverse of P (the matrix obtained does not depend upon which operation is performed first). The transposed inverse construction preserves multiplication and inverses, and therefore the contragredient of a set of cocycle data is again a set of cocycle data. Using these contragredient data for the tangent bundle of M we may construct a smooth vector bundle over M that will be called the cotangent bundle, and as for the tangent bundle if $\mathcal{A}$ is a subatlas of $\mathcal{B}$ then the associated atlases for the cotangent bundle will be equivalent.

For vector spaces the evaluation map $e : V^* \times V \to \mathbf{k}$ defined by $e(f, v) = f(v)$ is a bilinear map. We claim that there is a well behaved global version of this map for the tangent and cotangent bundles. More precisely, there is a smooth map $e : T^*(M) \times_M T(M) \to \mathbf{R}$ such that for each $p \in M$ the restriction of e to $T_p^*(M) \times T_p(M)$ is just this canonical evaluation map.

Locally this is easy to do. If we are given an open subset of Euclidean space, then the manifold $T^*(M) \times_M T(M)$ is simply $U \times \mathbf{R}^n \times \mathbf{R}^n$, and the map we want is simply the one sending $(u,\ \mathbf{v},\ \mathbf{w})$ to ${}^{\mathbf{T}}\mathbf{v} \cdot \mathbf{w}$.

We must now verify that this definition is compatible with the transition maps defining $T^*(M) \times_M T(M)$. If $\mathcal{A} = \{(U_\alpha, h_\alpha)\}$ is a smooth atlas for M, then charts for $T^*(M) \times_M T(M)$ have the form $(U_\alpha \times \mathbf{R}^n \times \mathbf{R}^n,\ etc.)$ and the associated transition maps send $(u,\ \mathbf{v},\ \mathbf{w})$ to

$$\left(\psi_{\beta\alpha}(u), {}^{\mathbf{T}}[D\psi_{\beta\alpha}(u)]^{-1}\mathbf{v}, [D\psi_{\beta\alpha}(u)]\mathbf{w}\right).$$

Therefore everything reduces to checking whether ${}^{\mathbf{T}}\mathbf{v} \cdot \mathbf{w}$ is equal to

$$^{\mathbf{T}}\left({}^{\mathbf{T}}[D\psi_{\beta\alpha}(u)]^{-1}\mathbf{v}\right) \cdot \left([D\psi_{\beta\alpha}(u)]\mathbf{w}\right).$$

But the latter simplifies to

$$\left({}^{\mathbf{T}}\mathbf{v}[D\psi_{\beta\alpha}(u)]^{-1}\right) \cdot \left([D\psi_{\beta\alpha}(u)]\mathbf{w}\right)$$

which further simplifies to ${}^{\mathbf{T}}\mathbf{v} \cdot \mathbf{w}$ as desired.

We shall now prove a result that raises serious questions about the reasons for carrying out the entire construction of the cotangent bundle.

Theorem. *If M is a smooth manifold, then the tangent and cotangent bundles are isomorphic vector bundles.*

Proof. Let g be a riemannian metric on (the tangent bundle of) M. We can define a map Γ of sets from $T(M)$ to $T^*(M)$ such that for each p the map sends $T_p(M)$ to $T_p(M)^* \cong T_p^*(M)$ by the formula $[\Gamma_p(\mathbf{v})]\mathbf{w} = g(\mathbf{v},\ \mathbf{w})$. For each p the map Γ_p is well defined, it maps $T_p(M)$ top a subspace of the same dimension, and it is 1–1 because $0 = \Gamma_p(v) \Rightarrow 0 = [\Gamma_p(\mathbf{v})]\mathbf{v} = g_p(\mathbf{v}, \mathbf{v})$ and the latter is zero if and only if $\mathbf{v} = 0$. Thus Γ_p is an isomorphism for each p.

We need to show that this map is a diffeomorphism. It suffices to work locally. Suppose that U is open in $\mathbf{R}^n$ and for each $u \in U$ let $G(u)$ be the Gram matrix of the riemannian metric g with respect to the standard unit basis (hence the (i,j) entry is the value of g at $(u,\ \mathbf{e}_i,\ \mathbf{e}_j)$ where $\{\mathbf{e}_k\}$ denotes the standard unit vectors in $\mathbf{R}^n$). Then Γ takes the form

$$\Gamma(u,\ \mathbf{v}) = \left(u, {}^{\mathbf{T}}\mathbf{v} \cdot G(u)\right)$$

and the smoothness of this follows because G is smooth. Therefore we have shown that we have a smooth map $\Gamma : T(M) \to T^*(M)$ with the desired properties. Since $G(u)$ is an invertible symmetric matrix for all u one can show directly that Γ^{-1} is given locally by

$$\Gamma^{-1}(u,\ \mathbf{w}) = \left(u,\ G(u)^{-1}({}^{\mathbf{T}}\mathbf{w})\right)$$

and therefore Γ is a diffeomorphism.

We return to the question: Why do we need both the tangent and the cotangent spaces? The reason is that each is better for some purposes and each has different uses. Cross sections of the tangent bundle provide the right way to look at ordinary differential equations and the Lie bracket. Cross sections of the tangent bundle provide the right way to look at line integrals, pullback constructions and certain differentiation constructions.

On a more abstract note, the crucial point is that the isomorphism between a finite dimensional vector space and its dual space is unnatural in the sense that it requires one to pick some extra structure in order to construct an isomorphism; the structure may be a basis or an inner product or certain generalizations of either, and it is often very clumsy to manipulate objects using these isomorphisms that depend on extrinsic data.

Given an arbitrary vector bundle $\pi : E \to B$ one can imitate the construction above to construct a dual vector bundle $\pi^* : E^* \to B$, and there is also a good evaluation map $E^* \times_B E \to \mathbf{R}$ in analogy with the tangent bundle. If one has a riemannian metric on E it is also possible to construct a vector bundle isomorphism $E \to E^*$ like the isomorphism Γ that we constructed for the tangent bundle.

3. Differential 1-forms

We define a *differential* (or *exterior*) 1-*form* on M to be a (smooth) cross section of the cotangent bundle; *i.e.*, a smooth map $\omega : M \to T^*(M)$ such that $\tau_M^* \circ \omega = \mathbf{1}_M$.

Following standard practice, if U is open in $\mathbf{R}^n$ we write dx^i to denote the 1-form sending u to $(u, \mathbf{f}^i)$ where $\mathbf{f}^i$ is the i^{th} vector in the dual basis to the standard unit vectors. Clearly we then have

$$e\left(dx^i, \frac{\partial}{\partial x^j}\right) = \delta_j^i$$

where δ_j^i is 0 if $i \neq j$ and 1 if $i = j$. On U every form can be expressed as a $C^\infty(M)$ linear combination

$$\omega(p) = \sum_i g^i(p)\, dx^i$$

for suitable smooth functions $g^i : U \to \mathbf{R}$.

In tensor analysis, differential 1-forms correspond to *covariant vector fields of rank* 1, and the standard local description of such objects can be formulated as follows from our perspective: Once again, we assume that we are given a smooth atlas $\mathcal{A} = \{(U_\alpha, h_\alpha)\}$ for M and smooth 1-forms ω_α on the corresponding open sets U_α; we shall write these local forms in coordinates as

$$\omega_\alpha(u) = (u, \widetilde{\omega_\alpha}(u))$$

where $\widetilde{\omega_\alpha}$ is a smooth function from U to $\mathbf{R}^n$. One can then push these forms over to $h_\alpha(U_\alpha)$ using the associated atlas for $T^*(M)$, and the compatibility condition needed to construct a global form on M is given by

$$\widetilde{\omega_\alpha}(u) = [{}^{\mathbf{T}}D\psi_{\beta\alpha}(u)]\widetilde{\omega_\beta}\left(\psi_{\beta\alpha}(u)\right).$$

The set of all differential 1-forms on a manifold M is a module over the algebra $C^\infty(M)$, and it is denoted by $\wedge^1(M)$. Since the tangent and cotangent bundles are isomorphic we know that $\mathcal{X}(M)$ and $\wedge^1(M)$ are isomorphic. The former has a natural extra structure given by the Lie bracket, and the latter has a much different property: *Given a smooth map $f : M \to N$, there is a natural way of pulling back a 1-form on N to a 1-form on M.*

One approach to proving this is given by the following extremely useful construction: Given a 1-form ω on M, one can define a $C^\infty(M)$-linear map $E\omega$ from $\mathcal{X}(M)$ to $C^\infty(M)$ by setting $E\omega(Y)$ equal to the function $e(\omega, Y)$, where $e : T^*(M) \times_M T(M) \to \mathbf{R}$ is the evaluation map described before.

Theorem. *The construction sending ω to $E\omega$ defines a $1-1$ correspondence between differential 1-forms on M and $C^\infty(M)$-linear maps from $\mathcal{X}(M)$ to $C^\infty(M)$.*

Proof. For the time being we shall only do the case where M is an open set in some Euclidean space and defer the general case until the end of the course (the details also appear in Conlon). For open sets in $\mathbf{R}^n$ one can write the form as $\Sigma g^i\, dx^i$. To show that different forms determine different maps, it suffices to show that the zero form is the only one that determines the zero map. But if $E\omega = 0$ then its evaluation at every standard tangent vector field $\frac{\partial}{\partial x^i}$ is zero; since the evaluation at such a tangent vector field is simply g^i, this means that $g^i = 0$ for all i, so that $\omega = 0$. On the other hand, given a $C^\infty(M)$-linear map $\theta : \mathcal{X}(M) \to C^\infty(M)$ it is a routine exercise to verify that $\theta = E\omega$ where

$$\omega = \sum \theta\left(\frac{\partial}{\partial x^i}\right) dx^i.$$

One can now define the *pullback construction*

$$f^\# : \wedge^1(N) \to \wedge^1(M)$$

by sending the form ω to the unique form $f^\#\omega$ such that

$$e_M(f^\#\omega, Y) = e_N(\omega, T(f)Y)$$

for all $Y \in \mathcal{X}(M)$; as usual, e_P denotes the evaluation map from $T^*(P) \times_P T(P)$ to $\mathbf{R}$.

We can also use the theorem to give a matheamtically rigorous definition of the *differential* or *exterior derivative* of a smooth function. Specifically, if $h \in C^\infty(M)$, then $dh \in \wedge^1(M)$ is the unique 1-form for which

$$e_M(dh, Y) = Yh.$$

What do these constructions look like in local coordinates? It is convenient to dispose of the second one before considering the first. In this case, one can evaluate at the standard unit vector fields $\frac{\partial}{\partial x^i}$ to conclude that

$$dh = \sum \frac{\partial h}{\partial x^i}\, dx^i.$$

Suppose now that ω is given over an open subset V in $\mathbf{R}^n$ by the expression $\Sigma g^i\, dx^i$ and that $f : U \to V$ is smooth, where U is open in $\mathbf{R}^m$. Let f^i be the i^{th} coordinate function for f. Then the local expression is

$$f^{\#}\omega = \sum (g^i \circ f)\, df^i$$

where df^i is defined as above.

NOTE. The form that is traditionally called dx^i is in fact the exterior derivative of the i^{th} coordinate function $x^i : \mathbf{R}^n \to \mathbf{R}$.

The pullback construction is contravariantly functorial; *i.e.*, the pullback of a form ω under the identity map is simply ω, and for composite functions one has $(f_2 \circ f_1)^{\#} = f_1^{\#} \circ f_2^{\#}$. Furthermore, if $f : U \to V$ and $g : V \to \mathbf{R}$ are smooth and we set $f^{\#}g = g \circ f$, then we have $d_U \circ f^{\#}(g) = f^{\#}(d_V g)$. These are all special cases of more general and far-reaching phenomena (see the subheading "Differential forms" below).

4. Tensor fields and k-forms

We begin with a very rapid introduction to tensor products. If V and W are finite-dimensional vector spaces over the field $\mathbf{k}$ with bases $\{v_1, \cdots, v_m\}$ and $\{w_1, \cdots w_n\}$ respectively, then their **tensor product** $V \otimes W$ is a finite-dimesnional vector space with basis given by $v_i \otimes w_j$ where $1 \leq i \leq m$ and $1 \leq j \leq n$. Given $v \in V$ and $w \in W$, one defines $v \otimes w \in V \otimes W$ by writing $v = \Sigma a_i v_i$ and $b = \Sigma b_j w_j$ for suitable scalars a_i and b_j, and sets $v \otimes w$ equal to $\Sigma(a_i b_j) v_i \otimes v_j$. By construction

$$\dim(V \otimes W) = (\dim V)(\dim W).$$

The following algebraic statement is elementary but tedious to verify.

Proposition. *Let $S : V_0 \to V_1$ and $T : W_0 \to W_1$ be linear transformations, where all the vector spaces that appear are finite dimensional over the field* $\mathbf{k}$. *Then there is a unique linear transformation*

$$S \otimes T : V_0 \otimes W_0 \to V_1 \otimes W_1$$

such that $[S \otimes T](v \otimes w) = S(v) \otimes T(w)$ for all $v \in V_0$ and $w \in W_0$. This constuction is functorial in S and T in the following sense:

(i) If S and T are the identity transformations, then $S \otimes T$ is also the identity.

(ii) If we are also given linear transformations $P : V_1 \to V_2$ and $Q : W_1 \to W_2$, then

$$(P \otimes Q) \circ (S \otimes T) = (P \circ S) \otimes (Q \circ T).$$

The following is an immediate consequence of (i) and (ii):

(iii) *If S and T are invertible then so is $S \otimes T$ and*

$$(S \otimes T)^{-1} = S^{-1} \otimes T^{-1}.$$

Suppose that we are given ordered bases $\mathcal{A}_i$ and $\mathcal{B}_j$ for V_i and W_j respectively, where $i = 0, 1$. If C and D are the matrices representing S and T with respect to the appropriate ordered bases, then the entries for the matrix of $S \otimes T$ with respect to the ordered bases

$$\mathcal{A}_0 \otimes \mathcal{B}_0 \qquad \text{and} \qquad \mathcal{A}_1 \otimes \mathcal{B}_1$$

(use the dictionary or *lexicographic* ordering on index pairs)

are just products of the entries of C and D. This and the preceding result yield the following important fact:

Theorem. *The tensor product construction defines a* **smooth** *homomorphism*

$$\text{Tensor}_{(m,n)} : GL(m, \mathbf{R}) \times GL(n, \mathbf{R}) \to GL(mn, \mathbf{R}).$$

Proof. Let $L_A : \mathbf{R}^m \to \mathbf{R}^m$ be the linear transformation on $n \times 1$ column vectors defined using left multiplication by the $n \times n$ matrix A, and define L_B similarly for $\mathbf{R}^n$. Then the matrix of L_A with respect to the standard ordered basis of unit column vectors is simply A itself, and likewise the matrix of L_B with respect to the standard ordered basis of unit column vectors is just B. If A and B are invertible matrices then L_A and L_B are invertible linear transformations. By property (iii) above the linear transformation $L_A \otimes L_B$ is also invertible. Therefore the tensor product construction maps $GL(m, \mathbf{R}) \times GL(n, \mathbf{R})$ to $GL(mn, \mathbf{R})$. The functoriality properties imply that the map is a group homomorphism.

If $\mathbf{e}_i^{(m)}$ and $\mathbf{e}_j^{(n)}$ are the standard unit vectors for the spaces of $m \times 1$ and $n \times 1$ column vectors, then the discussion preceding the statement of the theorem implies that the matrix of $L_A \otimes L_B$ with respect to the ordered basis $\{\mathbf{e}_i^{(m)} \otimes \mathbf{e}_j^{(n)}\}$ has entries given by the products of the entries of A and B. Therefore the entries of the matrix of $L_A \otimes L_B$ with respect to the ordered basis $\{\mathbf{e}_i^{(m)} \otimes \mathbf{e}_j^{(n)}\}$ are smooth functions of the entries of A and B, and this proves that the map from $GL(m, \mathbf{R}) \times GL(n, \mathbf{R})$ to $GL(mn, \mathbf{R})$ is smooth.

The tensor product construction can be iterated to form tensor products of an arbitrary finite ordered list of finite-dimensional vector spaces $V_1, \cdots V_q$, and the dimension of $V_1 \otimes \cdots \otimes V_q$ is just the product $\Pi_i \dim(V_i)$. If σ is a permutation of $\{1, \cdots, q\}$ and $\tau = \sigma^{-1}$, then there is a shuffle isomorphism

$$\text{Shuff}_\sigma : V_1 \otimes \cdots \otimes V_q \to V_{\tau(1)} \otimes \cdots \otimes V_{\tau(q)}$$

that sends each vector of the form $v_1 \otimes \cdots \otimes v_q$ to $v_{\tau(1)} \otimes \cdots \otimes v_{\tau(q)}$. This means that the factors of $v_1 \otimes \cdots \otimes v_q$ are permuted such that the i^{th} factor in the original expression becomes the $\sigma(i)^{\text{th}}$ factor in shuffled expression. If $q = 2$ and $\sigma = \tau$ is the unique nontrivial permutation on two letters, then this construction yields the twist map $\tau : V_1 \otimes V_2 \to V_2 \otimes V_1$ sending $v_1 \otimes v_2$ to $v_2 \otimes v_1$ for all v_1 and v_2.

Application to constructing vector bundles

If we combine the preceding with the discussion of vector bundles, we obtain an important method for constructing new vector bundles from old ones.

Theorem. *Let $\pi^{(1)} : E^{(1)} \to B$ and $\pi^{(2)} : E^{(2)} \to B$ be topological vector bundles over B. Then there is a vector bundle $\pi^{(1)} \otimes \pi^{(2)} : E(\pi^{(1)} \otimes \pi^{(2)}) \to B$ and a map $P^{\otimes} : E^{(1)} \times_B E^{(2)} \to B$ such that for each $b \in B$ the map $P^{\otimes}$ corresponds to the tensor product map of fibers from $E_b^{(1)} \times E_b^{(2)}$ to from $E_b^{(1)} \otimes E_b^{(2)}$. If the original vector bundle is smooth, then the new bundle is also smooth.*

Sketch of proof. Take an open covering $\mathcal{U}$ of B for such that the restrictions to both vector bundles over open sets in the covering are isomorphic to product bundles. The vector bundles are then given by continuous cocycle data $f_{\beta\alpha} : V_{\beta\alpha} \to GL(m, \mathbf{R})$ and $g_{\beta\alpha} : V_{\beta\alpha} \to GL(n\mathbf{R})$ respectively. Since $\mathrm{Tensor}_{(m,n)}$ is a smooth homomorphism the maps

$$\mathrm{Tensor}_{(m,n)}\left(f_{\beta\alpha}, g_{\beta\alpha}\right)$$

satisfy the requirements for the continuous cocycle data of an mn-dimensional vector bundle. One can also verify that the obvious local constructions for the map $P^{\otimes}$ on the open sets of $\mathcal{U}$ fit together compatibly.

If we start out with smooth vector bundles, then one can choose atlases such that the analogs of all the maps $f_{\beta\alpha}$ and $g_{\beta\alpha}$ are smooth. Since $\mathrm{Tensor}_{(m,n)}$ is smooth the new cocycle data constructed above are also smooth.

The preceding construction can be iterated to form a tensor product of an arbitrary ordered finite list of vector bundles

$$\pi^{(j)} : E^{(j)} \to B, \qquad 1 \leq j \leq q$$

over the same base. Furthermore, if σ is a permutation of $\{1, \cdots, q\}$ with inverse τ then one has a shuffle isomorphism of vector bundles

$$\mathrm{Shuff}_{\sigma} : \left(\pi^{(1)} \otimes \cdots \otimes \pi^{(q)}\right) \to \left(\pi^{(\tau(1))} \otimes \cdots \otimes \pi^{(\tau(q))}\right)$$

that looks like the ordinary shuffle isomorphism of tensor product factors over each point $b \in B$. All of these objects and morphisms are smooth if the original vector bundles are smooth.

Definition. Let V be a finite-dimensional vector space over the field $\mathbf{k}$, and let r and s be nonnegative integers. The vector space $\mathbf{T}^r_s(V)$ of *tensors of contravariant rank r and covariant rank s* is simply the iterated tensor product

$$(\otimes^r(V)) \otimes (\otimes^s(V^*))$$

where $\otimes^q(W)$ denotes the q-fold tensor product of the vector space W with itself and $\otimes^0(W) = \mathbf{k}$ by convention (note that $U \otimes \mathbf{k} \cong \mathbf{k} \otimes U \cong U$ for all vector spaces U).

Given a smooth manifold M and r and s as above, the *tensor bundle of type (r, s) of M* is the bundle

$$\mathbf{t}^r_s(M) : \mathbf{T}^r_s(M) \to M$$

defined by the iterated tensor product

$$(\otimes^r \tau_M) \otimes (\otimes^r \tau^*_M)$$

with the same notational conventions as before (in this case the tensor product of an arbitrary vector bundle ξ with the trivial 1-dimensional vector bundle $\theta_1 := M \times \mathbf{R} \to M$ is always isomorphic to the original vector bundle). Elements of the total space $\mathbf{T}^r_s(M)$ are called *tensors of contravariant rank* r *and covariant rank* s on M, and a cross section of $\mathbf{t}^r_s(M)$ is called a *tensor field of contravariant rank* r *and covariant rank* s.

In classical tensor analysis such objects were again described locally by smooth maps $U_\alpha \to GL(n^{r+s}, \mathbf{R})$ satisfying analogs of the previously stated consistency conditions (Try writing these out in some simple cases, say for $r+s = 2, 3, 4$, which include important examples in the study of smooth manifolds). A brief glance at an old textbook of riemannian geometry shows that such descriptions quickly become hopelessly clumsy to manipulate. Eventually differential geometers developed a more convenient way of dealing with tensor fields based upon the following basic fact from (multi)linear algebra:.

Proposition. *Let* V *and* W *be finite-dimensional vector spaces over the field* $\mathbf{k}$*, and let* s *be a positive integer. Let* $\mathbf{M}^s(V, W)$ *be the set of functions from the s-fold product* $\Pi^s(V)$ *to* W *that are* $\mathbf{k}$*-linear in each variable (with the rest held constant), and make* $\mathbf{M}^s(V, W)$ *into a vector space by pointwise addition and scalar multiplication of functions. Then there is a natural isomorphism*

$$\mathbf{S} : \mathbf{M}^s(V, W) \to (\otimes^s V^*) \otimes W$$

defined as follows: Given a function φ *in the domain and an ordered basis* $\{v_1, \cdots, v_n\}$ *for* V *with dual basis* $\{f_1, \cdots, f_n\}$ *then*

$$\mathbf{S}(\varphi) = \sum_{i_1, \cdots, i_s} f_{i_1} \otimes \cdots \otimes f_{i_s} \otimes \varphi(e_{i_1} \otimes \cdots \otimes e_{i_s}).$$

The naturality property may be stated as follows: If $A : V \to V$ and $B : W \to W$ are invertible linear transformations, then the composite

$$B \circ \varphi \circ (\Pi^s A)$$

is also a multilinear function in $\mathbf{M}^s(V, W)$. Under the isomorphism S this corresponds to

$$\left[\left(\otimes^s (A^*)^{-1} \right) \otimes B \right] \circ (\mathbf{S}(\varphi))$$

where $A^* : V^* \to V^*$ is the (invertible) linear transformation sending $f \in V^*$ to the composite $f \circ A : V \to \mathbf{R}$ (note that $(A^*)^{-1} = (A^{-1})^*$).

Sketch of proof. Let $\{w_1, \cdots, w_n\}$ be an ordered basis for W. Then an ordered basis for $\mathbf{M}^s(V, W)$ is given by the unique multilinear functions $\varphi_{i_1, \cdots, i_s, j}$ such that

$$\varphi_{i_1, \cdots, i_s, j}\,(e_{i_1} \otimes \cdots \otimes e_{i_s}) = w_j.$$

The elements in $(\otimes^s V^*) \otimes W$ that are supposed to be images of these basis elements under $\mathbf{S}$ also form a basis for the vector space in which they lie. Therefore there is a unique vector space isomorphism taking the given basis in the first space to the given basis in the second. Verification of the naturality property is again a routine but somewhat tedious calculation.

In the proof of the 1–1 correspondence between smooth 1-forms on a manifold M and the set of $C^\infty(M)$-linear maps from $\mathcal{X}(M)$ to $C^\infty(M)$, and important step in the (not yet presented) argument is that given such a map L, a vector field Y on M and a point $p \in M$ the value of the function $L(Y)$ at p depends only on $Y(p)$. This can be generalized as indicated below; the proof is omitted for the sake of conciseness (see Conlon, Chapter 7, for details).

Proposition. *Given a smooth n-manifold M and a smooth vector bundle $\xi := (\pi : E \to M$, etc.) let $\Gamma(\xi)$ denote the $C^\infty(M)$-module of smooth cross sections of ξ. Let s be a positive integer let $Y_1, \cdots Y_s$ be vector fields on M, and let L be a $C^\infty(M)$-multilinear map*

$$\Pi^s \mathcal{X}(M) \to \Gamma(\xi).$$

Then the value of $L(Y_1, \cdots, Y_s)$ at p depends only on the valued of the vector fields Y_i at p.

In particular, the preceding implies that for each $p \in M$ the map L determines a multilinear map of real vector spaces L_p from $\Pi^s T_p(M)$ to $E_p = \pi^{-1}(\{p\})$.

These observations combine to yield has the following important charactrization for tensor fields:

Theorem. *Given a smooth n-manifold M and a smooth vector bundle $\xi := (\pi : E \to M$, etc.) let $\Gamma(\xi)$ denote the $C^\infty(M)$-module of smooth cross sections of ξ. For every pair of nonnegative integers (r, s) the tensor fields on M of contravariant rank r and covariant rank s are in $1-1$ correspondence with the $C^\infty(M)$-multilinear maps*

$$\Pi^s \mathcal{X}(M) \to \Gamma\left(\mathbf{t}_0^r(M)\right)$$

such that the following holds:
($\star$) If R is a multilinear map as above and $p \in M$, then the value of the tensor field at p is $\mathbf{S}(L_p)$.

Once again, it is fairly straightforward to check this locally using coordinates, and the globalization follows by the same sorts of methods used to characterize 1-forms on M and derivations on $C^\infty(M)$.

It is impossible to give a wide range of important tensor fields here; the discussions would lead us too far afield. However, it is worth noting how some objects that we have previously constructed can be viewed as tensor fields. In particular, if $\delta : T(M) \times T(M) \to R$ restricts to a bilinear function on each subset $T_p(M) \times T_p(M)$, then one can view Δ as a tensor field of **covariant** rank 2 by noting that the map

$$L_\Delta : (\mathcal{X}(M))^2 \to \Gamma\left(\mathbf{t}_0^0(M)\right) \cong C^\infty(M)$$

(by definition $\mathbf{t}_0^0$ is the trivial bundle θ_1)

sending the ordered pair of vector fields (Y_1, Y_2) to the function $\Delta(Y_1, Y_2)$ is $C^\infty(M)$-bilinear and that one can recover everything about Δ from L_Δ.

Differential forms

Since the local theory was covered in Fall 2002, the emphasis here will be on the globalization. We need to start by describing the *exterior powers* of a finite dimensional vector space V. Given a positive integer r, the r^{th} *exterior power* $\wedge^r(V)$ is obtained from the tensor product $\otimes^r(V)$ by factoring out the subspace $K_r(V)$ spanned by all vectors of the form $v_1 \otimes \cdots \otimes v_r$ where two of the factors are equal (*i.e.*, $v_i = v_j$ for some i, j with $i \neq j$). The image of a vector of the form

$u_1 \otimes \cdots \otimes u_r$ is denoted by $u_1 \wedge \cdots \wedge u_r$. By convention we write $\wedge^0(V) = \mathbf{k}$ (the underlying field). The exterior power construction has the following properties:

(1) *If $e_1, \cdots, e_n$ is a basis for V then a basis for $\wedge^r(V)$ is given by all vectors of the form $e_{i_1} \wedge \cdots \wedge e_{i_r}$ where $1 \leq i_1 < \cdots < i_r \leq n$. In particular the dimension of $\wedge^r(V)$ is equal to the binomial coefficient $\binom{n}{r}$.*

(2) *In particular, $\wedge^1(V)$ is canonically isomorphic to V and $\wedge^r(V) = \{0\}$ if $r > \dim V$.*

(3) *Given nonnegative integers p and q there is a bilinear map $\wedge^p(V) \times \wedge^q(V) \to \wedge^{p+q}(V)$ that is compatible with the tensor construction $\otimes^p(V) \times \otimes^q(V) \to \otimes^{p+q}(V)$ and these maps have the following alternating and properties:*

$$u \wedge u = 0, \qquad w \wedge u = (-1)^{pq} u \wedge w$$

(4) *If $T : V \to W$ is a linear transformation then for each $p \geq 0$ there is an associated linear transformation $\wedge^p(T) : \wedge^p(V) \to \wedge^p(W)$ that is covariantly functorial in T; in particular, if T is invertible then so is $\wedge^p(T)$. By convention $\wedge^0$ is the identity on the base field. If the latter is the real numbers and we choose ordered bases for V and W and take the associated bases for the exterior powers as in (1), then the entries of the matrix for $\wedge^p(T)$ are smooth functions of the entries of the matrix for T. Finally, if $V = W$ and $\dim V = n$, then $\wedge^n(V) \cong \mathbf{k}$ and $\wedge^n(T)$ is just multiplication by the determinant of T.*

The preceding observations allow us to construct exterior power bundles associated to a vector bundle ξ.

Definition. A **differential k-form** (also called an exterior k-form) on M is a cross section of $\wedge^k(\tau_M^*)$; note that we take the exterior powers of the *cotangent bundle* here. If $k = 0$ then we define 0-forms to be smooth functions. The integer k is called the *degree* of the form.

If U is open in $\mathbf{R}^n$ then every differential k-form in the sense described above can be written uniquely as a linear combination

$$\omega(u) \quad = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f^{1_1, \cdots, i_k}(u)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

for suitably chosen smooth functions $f^{1_1, \cdots, i_k} \in C^\infty(U)$. This is exactly the old definition for such forms. Also, if $k = 1$ this reduces to the previous gloabal definition of differential forms, and if $k = 0$ the objects we obtain are simply smooth functions on M.

The space of differential k-forms on M is denoted by $\wedge^k(M)$; for some notational purposes it is convenient to set this equal to zero for negative values of k.

Several important algebraic properties of differential forms in the local case can be globalized.

Pullbacks. If $f : M \to N$ is a smooth map then there are associated $C^\infty(M)$-linear maps from $\wedge^p(N) \to \wedge^p(M)$ for all $p \geq 0$. This construction is contravariantly functorial with respect to smooth functions.

Exterior derivatives. There are $\mathbf{R}$-linear maps $d_M^p : \wedge^p(M) \to \wedge^{p+1}(M)$ (the exterior derivative) such that $d_M^p \circ d_M^{p-1} = 0$ and the exterior derivative commutes with pullbacks: $d_M^p \circ f^\# = f^\# \circ d_N^p$.

Wedge products. There are $C^\infty(M)$-bilinear wedge product maps

$$\wedge : \wedge^p(M) \times \wedge^q(M) \to \wedge^{p+q}(M)$$

such that the following hold:

(i) Pullbacks are compatible with wedge products: $f^\#(\omega \wedge \theta) = (f^\#\omega) \wedge (f^\#\theta)$.

(ii) Exterior differentiation satisfies a **graded** Leibniz rule with respect to wedge products: In the notation of (i) we have

$$d_N(\omega \wedge \theta) = d_N(\omega) \wedge \theta + (1)^p \omega \wedge d_N(\theta)$$

where p is the degree of ω.

The global constructions and the basic identities are established in Section 8.1 of Conlon. Furthermore, one can extend the exterior form version of Stokes' Formula to differential forms on arbitrary manifolds (see Section 8.2 of Conlon).

One of the important features of the preceding constructions is that they yield the *de Rham cohomology groups* of M. Since $d^p_M \circ d^{p-1}_M = 0$ it follows that the kernel of d^p_M (the *closed p-forms*) contains the image of d^{p-1}_M (the *exact p-forms*). We define

$$H^p_{\mathrm{DR}}(M) := \mathrm{Kernel}(d^p_M)/\mathrm{Image}(d^{p-1}_M).$$

According to *de Rham's Theorem* these groups are isomorphic to the usual cohomology groups of M with real coefficients that one constructs in algebraic topology by any of several standard methods, and in fact the wedge product structure passes to bilinear maps

$$H^p_{\mathrm{DR}}(M) \times H^q_{\mathrm{DR}}(M) \to H^{p+q}_{\mathrm{DR}}(M)$$

that correspond to the topological cup product. Further information on this can be found in Conlon.

5. Orientations and volume forms

Once again we begin with some abstract comments about vector spaces, but this time we must assume that the scalars are the real numbers.

Definition. Let V be an n-dimensional vector space over the reals, so that $\dim \wedge^n(V) = 1$. An *orientation* Ω of V is a set of positive multiples of some nonzero element of $\wedge^n(V)$. Since any two nonzero vectors in the latter are nonzero multiples of each other, it follows that V has exactly two orientations. If $\mathcal{B} = \{v_1, \cdots, v_n\}$ is an ordered basis for V, then the orientation associated to $\mathcal{B}$ is the unique orientation containing the wedge product $v_1 \wedge \cdots \wedge v_n$ (note that this vector is always nonzero). The **standard orientation** of $\mathbf{R}^n$ is the orientation containing $\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n$.

The reason for defining orientations on vector spaces is to provide some groundwork for discussing orientations on manifolds. There are many ways of characterizing orientations; the motivation for our approach is that it yields many properties of orientations relatively quickly.

Definition. If M^n is a smooth manifold, then an *orientation* of M is a smooth n-form $\Omega \in \wedge^n(M)$ such that $\Omega(p) \neq 0$ for all $p \in M$. We say that M is *orientable* if it has an orientation.

We now need some examples of manifolds that are orientable as well as examples that are not orientable. It follows immediately that $\mathbf{R}^n$ is orientable (take the form $dx^1 \wedge \cdots \wedge dx^n$) and that open subsets of orientatable manifolds are orientable (the resriction of an orientation to an open subset is again an orientation); in particular, open subsets of $\mathbf{R}^n$ are always orientable.

Here is one fundamental suficient condition that is less immediate but still not difficult to prove:

Proposition. *If M is simply connected then it is orientable.*

Sketch of proof. Let $n = \dim M$. Then the vector bundle $\wedge^n(\tau_M^*)$ is 1-dimensional. If we put a riemannian metric on this bundle and let $S(\wedge^n(\tau_M^*))$ denote the set of all vectors of unit length, then one can show that the restriction of the bundle projection to $S(\wedge^n(\tau_M^*))$ is a 2-sheeted covering space projection. Since M is simply connected this splits into two copies of M. Either component determines an orientation of M. [Numerous details are omitted in this discussion.]

Here is another useful sufficient condition for orientability:

Proposition. *Suppose that there is a smooth atlas $\mathcal{A} = \{(U_\alpha, h_\alpha)$ for M such that the determinants of the derivatives of the transition maps D "$h_\beta^{-1} \circ h_\alpha$" are all positive. Then M is orientable.*

Sketch of proof. Let $\{\psi_{\beta\alpha}\}$ be the transition maps for $\mathcal{A}$. Under the associated transition maps defining $\wedge^n(\tau_M^*)$ the form $dx^1 \wedge \cdots \wedge dx^n$ is sent to $P_{\beta\alpha}^{-1}(u)\, dx^1 \wedge \cdots \wedge dx^n$ where

$$P_{\beta\alpha}(u) = \det D\psi_{\beta\alpha}(u) > 0.$$

For each α let ω_α be the form corresponding to the standard orientation on $\mathbf{R}^n$, where $n = \dim V$. If we glue together a collection of such forms over a suitably chosen subatlas of $\mathcal{A}$ using a smooth partition of unity, we obtain an orientation of M.

Example. One would expect that the Möbius strip is a nonorientable 2-manifold. Here is one way of showing this fact from our perspective. The Möbius strip M can be viewed as a smooth quotient of $\mathbf{R} \times \mathbf{R}$ obtained by factoring out an action of the infinite cyclic group $\mathbf{Z}$ by diffeomorphisms. Specifically, the generator of the latter group correspondes to the diffeomorphism $T(s,t) = (s+1, -t)$. Let $\pi : \mathbf{R}^2 \to M$ be the covering space projection.

To see that M is not orientable, assume the contrary; *i.e.*, assume M is orientable. Then there is a nowhere zero 2-form Ω on M. Consider the pullback $\pi^{\#}\Omega$, which is a nowhere zero 2-form on $\mathbf{R}^2$ and therefore can be written in coordinate form as

$$\pi^{\#}\Omega = F(s,t)\, ds \wedge dt$$

where $F(s,t)$ is nowhere zero. For the sake of convenience we shall assume that the function is positive (the proof in the other case follows by a parallel argument with only a few changes in wording). Since $\pi = \pi \circ T$ it follows that

$$T^{\#}\left(\pi^{\#}\Omega\right) = \pi^{\#}\Omega.$$

However,

$$T^{\#}\left(F(s,t)\, ds \wedge dt\right) = F(s+1,-t)\, ds \wedge (-dt) = -F(s+1,-t)\, ds \wedge dt$$

so that $F(s,t) = -F(s+1,-t)$, which in turn implies that F is not positive everywhere. This contradiction shows that Ω cannot exist and hence that M cannot be orientable.

A similar argument shows that $M \times \mathbf{R}^k$ is not orientable for all $k \geq 1$.

An orientation Ω of M is often called a *volume form* because one can use it to integrate real valued functions on M. If $f : M \to \mathbf{R}$ is a smooth function whose support is compact and lies in the image $h_\alpha(U_\alpha)$ of some connected coordinate chart, then one can integrate f by taking the ordinary integral of the expression $(f \circ h_\alpha) \cdot (h_\alpha^{\#}\Omega)$ over $\mathbf{R}^n$ (note that the pullback form is a nowhere zero multiple of the standard volume form on $\mathbf{R}^n$. In general one can paste together a global integral using smooth partitions of unity.

Since $\wedge^{n+1}(M^n) = \{0\}$ it follows that an orientation Ω is always closed. For conncected manifolds, an orientation Ω is exact if and only if M is NOT compact, and in the compact connected (orientable) case one has that $H^n_{\mathrm{DR}}(M^n) \cong \mathbf{R}$ and is spanned by the class containing the closed n-form Ω. A more detailed discussion appears in Conlon, Section 8.6.