Math 205C
Spring 2003
R. Schultz

Study sheet for Final Examination

There is a separate course outline that lists the course topics and keys them to the text (Conlon, *Differentiable Manifolds, Second Edition*).

1. Outline the construction of the tangent bundle in terms of smooth atlases, and using this explain why

(i) the tangent bundle of $M \times N$ is diffeomorphic to $T(M) \times T(N)$,

(ii) the tangent bundle of $\mathbf{RP}^2$ is double covered by the tangent bundle of S^2, and the nontrivial covering transformation is given by $T(A)$, where A is the antipodal map on S^2 taking x to $-x$,

(iii) if M can be written as a disjoint union of two open-closed subsets M_1 and M_2 that are both manifolds, then the tangent bundle of M is diffeomorphic to the the disjoint union of the tangent bundles of M_1 and M_2.

2. Explain why the standard atlases for a tangent space are smooth atlases.

3. Explain how to construct the direct sum of two vector bundles over the same base space, and explain why the fibered square $T(M) \times_M T(M)$ is diffeomorphic to the total space of the direct sum $\tau_M \oplus \tau_M$.

4. Let $p : E \to B$ be a fiber bundle projection with fiber F. Explain why E is second countable if F and B are, and explain why E is Hausdorff if F and B are.

5. Let G be a smooth manifold with a group structure given by multiplication $m : G \times G \to G$ and inverse $i : G \to G$. Suppose that the map $F : G \times G \to G$ given by $F(x, y) = xy^{-1}$ is smooth. Prove that m and i are smooth.

6. Outline the construction of a positive real valued smooth function f on the real line so that $f = 1$ except on (a) the interval $(\frac{3}{7}, \frac{4}{7})$ where $f > 1$ and (b) the interval $(\frac{5}{7}, \frac{6}{7})$ where $0 < f < 1$ and such that the integral of f on $[0, 1]$ is 1. [*Hint:* There is a function g that is positive on $(0, 1)$ and zero elsewhere. Rescale this function so that its maximum is less than $\frac{1}{2}$ and its support lies on one of the two intervals in question. Call the resulting functions g_1 and g_2, and consider $1 + g_1 - g_2$.]

7. Can a figure eight curve in the plane be the integral curve of a smooth vector field on $\mathbf{R}^2$? Give reasons for your answer.

8. Show that $\mathbf{R}^n$ has a complete vector field that is never zero.

9. Let X be a vector field on a manifold M and assume that X is not complete. Let $s > 0$ be arbitrary.

(i) Explain why some maximal integral curve is not definable on the interval $(-s, s)$.

(ii) Does it necessarily follow that some maximal integral curve is not defined at time s? Prove this or give a counterexample. [*Hint:* Look at some examples like those in Conlon or the lectures.]

10. Let X be a vector field on S^2, and let γ be an integral curve of X. Prove that the length of the curve from $t = a$ to $t = b$ is given by

$$\int_a^b |X(\gamma(t)|\, dt.$$

11. Suppose that we are given a map smooth L on a tangent space $T(M)$ such that for all $p \in M$, the map L sends $T_p(M)$ to itself linearly. Prove that L induces a map of 1-forms $L^\# : \wedge^1(M) \to \wedge^1(M)$ such that for all vector fields Y on M we have $L^\#(\omega)(Y) = \omega(L(Y))$. If K is another map of this type, express $(KL)^\#$ in terms of $K^\#$ and $L^\#$.

12. Explain why the tangent bundle for an open subset of $\mathbf{R}^n$ is a product bundle. [*Hint:* Find an atlas with only one chart.]

13. Prove that the composite of two smooth embeddings is a smooth embedding.

14. Given a vector field Z in $\mathcal{X}(M)$ define an $\mathbf{R}$-linear map $\mathbf{L}_Z$ from $\mathcal{X}(M)$ to itself by $\mathbf{L}_Z(Y) = [Z, Y]$. Prove that $\mathbf{L}_Z$ is a derivation on $\mathcal{X}(M)$ where the latter is viewed as a nonassociative algebra using the Lie bracket. [*Hint:* Use the Jacobi identity.]

15. Let ω and θ be differential forms on M, and let $f : M \to N$ be smooth.

(i) Explain why $f^\#\omega$ is closed if ω is closed and $f^\#\omega$ is exact if ω is exact.

(ii) Explain why $\omega \wedge \theta$ is closed if ω and θ are and why it is exact if either of the factors is exact (and the other is still assumed to be closed!).

(iii) If f is a constant map, and the degree of ω is positive, what can one say about $f^\#\omega$? What happens if the degree is zero?

16. If $f : M \to N$ is a smooth map, then the rank of f at $p \in M$ is the rank of the map of tangent spaces $T_p(f)$. Explain why the set of all points where f has the maximum possible rank $\mathbf{min}(\dim M, \dim N)$ is open in M. Is it necessarily nonempty?