DERIVATIONS AND VECTOR FIELDS

Let U be an open subset of $\mathbb{R}^n$ and let $\mathcal{C}^\infty(U)$ denote the ring of smooth $\mathbf{C}^\infty$ functions on U.

Given a smooth vector field X on U, the directional (or Lie) derivative along X defines a derivation on $\mathcal{C}^\infty(U)$. It is a routine exercise to show that distinct vector fields determine distinct derivations, and a fundamental result states that ***all*** of the derivations on $\mathcal{C}^\infty(U)$ come from such directional derivatives. One immediate consequence of this and some elementary algebra is a simple definition of the Lie bracket of two vector fields: If D and E are derivations on an algebra A, then a routine calculation shows that the commutator $[D,E] = DE - ED$ is also a derivation, and therefore there if X and Y are smooth vector fields on U there is a unique vector field $[X,Y]$ such that the associated directional derivatives are given by the formula

$$[X,Y]f = X(Yf) - Y(Xf).$$

Since the treatment in the course is not quite the same as in Conlon, we shall indicate how one can use the results of Conlon in the spirit of the lectures.

Leibniz functionals

Given an algebra A over the real numbers, define a ***Leibniz functional*** to be a linear functional $D: A \to \mathbb{R}$ satisfying the Leibniz rule for products:

$$D(ab) = (Da)b + a(Db)$$

It follows immediately that the set of all Leibniz functionals is a vector subspace of the space of all linear functionals on A.

Let $x \in \mathbb{R}^n$, and let $\mathcal{C}^\infty[x]$ denote the local ring of germs of smooth functions defined on neighborhoods of x; specifically the elements of the ring are represented by smooth functions defined on neighborhoods of x and two functions define the same germ if there is a common subneighborhood of x on which the two functions are equal (this is denoted by $\mathcal{G}_x$ in Conlon).

Given a vector $\mathbf{V} \in \mathbb{R}^n$ one can define a Leibniz functional $D_{\mathbf{V}}$ on $\mathcal{C}^\infty[x]$ by taking the directional derivative of a representative function f at x in the direction of $\mathbf{V}$. Since the directional derivative only depends upon the behavior of a function on an arbitrarily small open neighborhood of x it follows that this directional derivative is independent of the choice of representative. It is a routine exercise to verify that the map sending a vector $\mathbf{V}$ to the associated Leibniz functional $D_{\mathbf{V}}$ is a linear map of vector spaces and that the kernel of this map is trivial.

The key point is that ***every Leibniz functional has the form*** $D_{\mathbf{V}}$ ***for some*** $\mathbf{V}$. The proof of this is implicit in Lemma 2.2.19 to Corollary 2.22 on pages 47 to 49 of Conlon.

Application to derivations

We shall indicate how the discussion in Section 2.7 of Conlon is related to our setting. Suppose that Δ is a derivation on $\mathcal{C}^\infty(U)$. Lemmas 2.7.12 and 2.7.13 as well as Corollary 2.7.14 go through as stated. For each $x \in U$, Definition 2.7.15 and Proposition 2.7.16 define a Leibniz functional Δ_x on $\mathcal{C}^\infty[x]$. By the results on Leibniz functionals, it follows that $\Delta_x = D_{\mathbf{V}(x)}$ for some unique vector $\mathbf{V}(x) \in \mathbb{R}^n$. Clearly the map sending x to $\mathbf{V}(x)$ determines a set-theoretic function from U to $\mathbb{R}^n$; in order to establish the one-to-one correspondence between vector fields and derivations, it is necessary to show that $\mathbf{V}(x)$ defines a ***smooth*** function from U to $\mathbb{R}^n$. This is essentially the content of Proposition 2.7.17, but because of the importance of this result we shall sketch the argument here. If $\mathbf{V}$ is the i-th unit vector $\mathbf{e}_i$, then $D_{\mathbf{V}}$ is given by partial differentiation with respect to the variable x_i, and we shall write $D_{\mathbf{V}} = D_i$ in this case. The results on Leibniz differentials show that the derivations D_i form a basis for the space of all such objects, and therefore we can write

$$\mathbf{V}(x) = \sum g_i(x) D_i$$

for suitable functions g_i; the function $\mathbf{V}(x)$ is smooth if and only if each $g_i(x)$ is smooth, so we need to show that each of these real valued functions is smooth.

Let i be an arbitrary integer between 1 and n, and let x_i be the corresponding coordinate function on $\mathbb{R}^n$. By our assumptions $\Delta(x_i)$ is a smooth function on U. On the other hand, since $\Delta = \sum g_j(x) D_j$ it follows that $\Delta(x_i) = g_i(x)$. Therefore the latter is smooth, and by our previous remarks it follows that $\Delta(f) = \mathbf{V}f$ for the vector field $\mathbf{V}$ whose coordinate functions are given by the smooth functions $\Delta(x_i)$.