

## LEVEL SETS OF REGULAR VALUES

Aside from open subsets in Euclidean spaces, the most basic examples of topological manifolds include the unit sphere  $S^n$  in Euclidean  $(n + 1)$ -space. If  $f(x)$  is the real valued function given by the sum of the squares of the coordinates of  $x$ , it follows from the definition that  $f(x) = 1$  if  $x \in S^n$  and it is straightforward to check that  $Df(x) \neq 0$  if  $x \in S^n$ . Our objective here is to describe a large class of further examples along these lines and to show that they have smooth structures.

More generally, if  $n > m$  and  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$  is a smooth map such that  $Df(x)$  has rank  $m$  whenever  $f(x) = y$ , then we say that  $y$  is a *regular value* of  $f$ . The inverse image  $f^{-1}(\{y\})$  is said to be the *level set* for the value  $y$ .

Our definition of regular value implies that  $y$  is a regular value if the level set is empty; of course, we are primarily interested in situations where this is not the case.

**THEOREM.** *Let  $f$  be as above, and assume that  $y$  is a regular value of  $f$ . Then the level set  $f^{-1}(\{y\})$  is a second countable topological  $(n - m)$ -manifold and has a smooth atlas.*

**Proof.** The first step is to prove that the level set is a topological manifold; for convenience of notation we shall call this set  $L$ .

By hypothesis  $Df(x)$  has rank  $m$  for all  $x \in L$ ; in fact, we claim this also holds for all points in some open neighborhood of  $L$ . One simple but inelegant way of seeing this is as follows: If we let  $x \in L$  and consider the  $n \times m$  matrix for  $Df(x)$  formed by the partial derivatives of the coordinate functions then the assumption on its rank implies that one can form an  $m \times m$  submatrix of columns  $A(x)$  that is invertible. Consider the square matrices  $A(z)$  formed by taking the corresponding columns of  $Df(z)$  for other choices of  $z \in \mathbf{R}^n$ . Since the partial derivatives are continuous and invertibility corresponds to the nonvanishing of the Jacobian  $\det A(z)$ , it follows that  $A(z)$  is invertible for all  $z$  sufficiently close to  $x$  and consequently that the rank of  $A(z)$  is  $m$ . Since  $x$  is arbitrary this means there is an open neighborhood  $G$  of  $L$  on which  $Df$  always has rank  $m$ ; *i.e.*, the map  $f|G$  is a smooth submersion.

We can now apply Proposition 2 from the note, *Smooth Mappings of Maximum Rank*, to analyze the behavior of  $f$  at points of  $L$ . Given a point  $x \in L$ , it follows that we can find open neighborhoods  $U$  of  $x$  and  $V$  of  $y$ , an open set  $W \subset \mathbf{R}^{n-m}$ , and a diffeomorphism  $k : V \times W \rightarrow U$  so that  $f(k(v, w)) = v$  for all  $(v, w) \in V \times W$ . It then follows that  $k$  maps  $\{y\} \times W$  homeomorphically to  $L \cap U$ , which is an open neighborhood of  $x$  in  $L$ . Since  $\{y\} \times W$  is homeomorphic to  $W$ , this means that the neighborhood  $U \cap L$  is homeomorphic to an open subset  $W \subset \mathbf{R}^{n-m}$ . To complete the proof that  $L$  is a topological manifold, one must check that it is Hausdorff, but this follows quickly because it is a subset of a Euclidean space; the same sorts of considerations show that  $L$  is also second countable.

We now need to construct a smooth atlas for  $L$ , and this will be done using the diffeomorphisms constructed in the first part of the proof. As in the note, *Smooth Mappings of Maximum Rank*, if we are given a function  $B$  we shall frequently use “ $B$ ” to denote a function defined by the same rules as  $B$  but possibly defined on a subset of the domain of  $B$  with a codomain that is possibly a subset of the codomain of  $B$ .

Suppose that we are given two diffeomorphisms  $k_i : V_i \times W_i \rightarrow U_i$  as in the third paragraph of this proof, where  $i = 1, 2$  and  $U_i$  is open in  $G$ , and suppose that the images of  $k_1$  and  $k_2$  have a nonempty intersection  $U_1 \cap U_2 = k_1(V_1 \times W_1) \cap k_2(V_2 \times W_2)$ .

It then follows that there is a diffeomorphism

$$\Psi : (V_1 \times W_1) \cap k_1^{-1}(k_2(V_2 \times W_2)) \rightarrow (V_2 \times W_2) \cap k_2^{-1}(k_1(V_1 \times W_1))$$

such that  $\Psi$  is given by “ $k_2^{-1} \circ k_1$ ” in the sense of the previous paragraph.

Since  $f(k_i(v, w)) = v$  for  $i = 1, 2$  it follows that  $\Psi$  maps

$$(\{y\} \times W_1) \cap k_1^{-1}(k_2(\{y\} \times W_2))$$

to the corresponding set

$$(\{y\} \times W_2) \cap k_2^{-1}(k_1(\{y\} \times W_1)).$$

Therefore we may write  $\Psi(y, w) = (y, \psi_0(w))$ , and it will follow immediately that  $\psi_0$  is smooth (the coordinate functions of  $\psi_0$  are given by those of  $\Psi$  and the latter have smooth partials). Applying the same considerations to the inverse map “ $k_1^{-1} \circ k_2$ ” =  $\Psi^{-1}$  we see that  $\psi_0$  also has a smooth inverse so that  $\psi_0$  is a diffeomorphism. Therefore if we choose a family of smooth diffeomorphisms  $k_\alpha$  as above so that the image sets  $k_\alpha(V_\alpha \times W_\alpha)$  form an open covering of (a neighborhood of)  $L$ , it will follow that the corresponding charts  $h_\alpha : W_\alpha \rightarrow L$  defined by  $h_\alpha(w) = k_\alpha(y, w)$  define a smooth atlas for  $L$ . This completes the proof.

**VARIANT OF THE THEOREM.** Frequently textbooks define a smooth  $n$ -manifold in a manner equivalent to the following: *A subset  $X$  of some Euclidean space  $\mathbf{R}^K$  such that every point  $x \in X$  has an open neighborhood  $U$  in  $\mathbf{R}^K$  such that  $X \cap U$  is the level set of a regular value for some smooth function  $g_U : U \rightarrow \mathbf{R}^{K-n}$ .* The proof of the theorem extends with only a few changes to show that such a set  $X$  is a second countable topological  $n$ -manifold and has a smooth atlas. In fact, since every smooth (second countable)  $n$ -manifold embeds smoothly in  $\mathbf{R}^{2n+1}$  by the (relatively easy version of the) Whitney Embedding Theorem, it follows that smooth manifolds in the sense of the given definition are equivalent to smooth manifolds in the sense of the text for this course.