Level Sets of Regular Values — II

This is an addendum to the previous note, *Level sets of regular values*, in which an important additional result is established.

THEOREM. *Let $n > m$ and let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a smooth map such that y is a nontrivial regular value f (i.e., there is some x so that $f(x) = y$), and let $M = f^{-1}(\{y\})$. Let $j : M \to \mathbf{R}^n$ be the inclusion map, let U be an open subset of $\mathbf{R}^k$, and suppose that $g : U \to M$ is a map of sets. Then g is continuous if and only if $j \circ g$ is continuous, and g is smooth if and only if $j \circ g$ is smooth.*

Proof. The statement about continuity is just a general statement about the subspace topology and works if $\mathbf{R}^n$ and M are replaced by an arbitrary topological set X and a subspace A with the subspace topology.

Suppose now that g is continuous; we want to show that g is smooth if and only if $j \circ g$ is smooth. To prove the $(\Rightarrow)$ implication it suffices to show that j is smooth (composites of smooth maps are smooth). Recall that smooth charts for M are given as follows: For each point $x \in M$, we have open neighborhoods U of x and V of y, an open set $W \subset \mathbf{R}^{n-m}$, and a diffeomorphism $k : V \times W \to U$ so that $f(k(v,w)) = v$ for all $(v,w) \in V \times W$. It then follows that k maps $\{y\} \times W$ homeomorphically to $M \cap U$, which is an open neighborhood of x in M. A smooth chart for M is then defined by (ℓ, W), where ℓ is essentially given by the composition of $k|\{y\} \times W$ with the standard identification $\{y\} \times W \cong W$. By construction we then have "$k^{-1} \circ j \circ \ell$"$(w) = (y, w)$ which is clearly smooth, and therefore j itself is smooth.

To prove the $(\Leftarrow)$ implication, let $x \in U$, let k be defined as above with respect to x, and choose a subneighborhood $U_0 \subset U$ containing x such that $g(U_0) \subset k(V \times W)$ and $M \cap k(V \times W) = k(\{y\} \times W)$. Since $j \circ g$ is smooth we know that "$k^{-1} \circ j \circ g$" is smooth. Since the images of g and $j \circ g$ lie in M we know that "$k^{-1} \circ j \circ g$"$(z) = (y, h(z))$ for some function h. By the smoothness of $j \circ g$ we know that h is smooth.

On the other hand, by construction we know that that "$k^{-1} \circ j \circ g$" is the composite of "$k^{-1} \circ j \circ \ell$" and "$\ell^{-1} \circ g$"; the latter implies that "$\ell^{-1} \circ g(z)$"$= h(z)$, where h is the smooth function in the previous paragraph. Therefore g is smooth at x, and since x was arbitrary this shows that g is smooth everywhere.