Smooth Mappings of Maximum Rank

If U and V are open subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, then U and V cannot be diffeomorphic but one can still discuss smooth maps for which the rank of the derivative is as large as possible. If $n \leq M$ this means that the derivative is always 1–1, while if $n \geq M$ this means that the derivative is always onto. In these cases we say that f is an *immersion* or *submersion* respectively.

The Inverse Function Theorem yields important information on the local behavior of immersions and submersions.

Proposition 1. *Let $f : U \to V$ be a smooth map on open subsets in Euclidean spaces. Then f is an immersion if and only if for each $x \in U$ on can find open neighborhoods U_0 and V_0 of x and $f(x)$ and a diffeomoprhism $h : V_0 \to U_0 \times \operatorname{Int} D^{m-n}$ so that*

$$h(f(y)) = (y, 0)$$

for all $y \in U_0$.

Proof. Given a function G we shall repeatedly use "G" to denote a function defined by the same rules as G but possibly defined an a subset of the domain of G with a codomain that is possibly a subset of the codomain of G.

($\Leftarrow$) If h exists then D"$h \circ f$"(y) is 1–1 and $Df(y) = Dh(f(y))^{-1} D$"$h \circ f$"(y). ($\Rightarrow$) Let $T : \mathbf{R}^{m-n} \to \operatorname{Image} Df(x)^{\perp}$ be a linear isomorphism and define $g : U \times \mathbf{R}^{m-n} \to \mathbf{R}^m$ by $g(y,z) = f(y) + T(z)$. Then $Dg(x) = Df(x) + T$, which is an isomorphism, and hence it is a diffeomorphism on some open set of the form $U_0 \times \operatorname{Int} \varepsilon D^{m-n}$. By construction the image of f corresponds to points with vanishing second coordinate.

Proposition 2. *Let $f : U \to V$ be a smooth map on open subsets in Euclidean spaces. Then f is a submersion if and only if for each $x \in U$ on can find open neighborhoods U_0 and V_0 of x and $f(x)$ and a diffeomorphism $k : V_0 \times \operatorname{Int} D^{n-m} \to U_0$ so that*

$$f(k(y,z)) = y$$

for all $(y,z) \in V_0 \times \operatorname{Int} D^{n-m} \to U - 0$.

Proof. The argument is similar to the preceding one.

($\Leftarrow$) If k exists then D"$f \circ k$"(y) is onto and $Df = [D$"$f \circ k$"$] \circ Dk^{-1}$. ($\Rightarrow$) By hypothesis the kernel of $Df(x)$ is $(n-m)$-dimensional. Let $S_1 : \operatorname{Kernel} Df(x) \to \mathbf{R}^{n-m}$ be a linear isomorphism, let $S_2 : \mathbf{R}^m \to \operatorname{Kernel} Df(x)$ be perpendicular projection, and define

$$g : U \to V \times \mathbf{R}^{n-m}$$

by the formula

$$g(u) = (f(u), S_1 S - 2(u)) .$$

By construction $Dg(x)$ is invertible, and therefore by the Inverse Function Theorem there is a local inverse $k : V_0 \times \operatorname{Int} \varepsilon D^{n-m} \to U_0$. If P denotes projection onto V then $f = P \circ g$, so that $f(k(y,z)) = P(g(k(y,z)))$. Since g is inverse to k, the latter reduces to $P(y,z) = y$ as required.

The Inverse Function Theorem also implies the following *Implicit Function Theorem*; the version stated here is stronger than that of Conlon because the codomain is a Euclidean space of arbitrary finite dimension (not just the real line).

Theorem. *Let U and V be open in $\mathbf{R}^n$ and $\mathbf{R}^m$ respecitvely, and let $f : U \times V \to \mathbf{R}^m$ be a smooth function such that for some $(x, y) \in U \times V$ we have $f(x, y) = 0$ and the partial derivative of f with respect to the last m coordinates is invertible. Then there is an open neighborhood U_0 of x and a smooth function $g : U_0 \to V$ such that $g(x) = y$ and for all $u \in U_0$ we have $f(u, v) = 0$ if and only if $v = g(u)$.*

Proof. Define $h : U \times V \to \mathbf{R}^m \times \mathbf{R}^n$ by $h(u, v) = (f(u, v), u)$. Then the hypotheses imply that $Dh(x, y)$ is invertible, and therefore there is a local inverse $k : \mathrm{Int}\, \varepsilon D^m \times U_0 \to U \times V$. Since the second coordinate of $h(u, v)$ is u, it follows that the first coordinate of the inverse $k(z, w)$ is w so that we may write $k(z, w) = (w, Q(z, w))$ for some smooth function Q. On one hand we have $g(k(z, w)) = (z, w)$ but on the other hand we also have

$$g(k(z, w)) = g\left(w, Q(z, w)\right) = \left(f\left(w, Q(z, w)\right), w\right).$$

In particular, this means that

$$z = f\left(u, Q(z, u)\right)$$

for all z and u. If we take $g(u) = Q(0, u)$ it follows that $y = g(x)$ and $f(u, v) = 0$ if and only if $v = g(u)$.