Connected Metrizable Topological Manifolds Are Second Countable

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Since second countable topological manifolds are paracompact, Smirnov's Theorem implies that a a second countable topological manifold is metrizable. More generally, one can ask which topological manifolds are metrizable. The first observation is that this reduces quickly to the connected case because an arbitrary topological manifold is homeomorphic to the disjoint union of its components and a disjoint union of topological spaces is metrizable if and only if each of the summands is metrizable (*Proof:* The $\Rightarrow$ implication is obvious; to prove the other direction, note that the one can find metrics on the pieces with diameter ≤ 1 that define the same topologies as the original metrics, and then one can define a metric on the disjoint union by taking the bounded metric for points in the same piece and defining the distance between two points in different pieces to be 2). Therefore the question reduces to determining which connected topological manifolds are metrizable. Here is the main result.

Theorem. *If X is a connected topological n-manifold that is metrizable, then X is second countable.*

Corollary. *If X is a topological n-manifold, then X is metrizable if and only if each component of X is second countable.*

This follows immediately because the components of a topological n-manifold are also topological n-manifolds. Note that a disoint union of uncountably many copies of some (nonempty) topological n-manifold is metrizable but not second countable.

Proof of Theorem. By A. H. Stone's Theorem X is paracompact. There is a general result in point set topology stating that a paracompact, locally compact, connected Hausdorff space is a countable union of compact subspaces. In particular this follows directly from Theorem XI.7.3 on page 241 of Dugundji's topology textbook [D]. The result in [D] states that a paracompact, locally compact Hausdorff space can be written as a "free union" of pairwise disjoint open subspaces, each of which is a countable union of compact subspaces; however, if a space is connected, it cannot be expressed as a free union of more than one nonempty subspace (see [D] for more information).

Therefore we know that $X = \cup K_n$ where each K_n is compact, and since X is metrizable each K_n is also metrizable. The latter implies that each K_n has a countable dense subset D_n, and therefore the countable subset $\cup D_n$ is dense in X. Since X is metric, this means that it is also second countable.

References

[D] J. Dugundji, *Topology.* Allyn and Bacon, Boston, 1966.

[K] J. L. Kelley, *General Topology.* Van Nostrand, New York, 1955.